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# *Brylawski's Decomposition of NBC Complexes of Abstract Convex Geometries and Their Associated Algebras*

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We introduce a notion of a *broken circuit* and an *NBC complex* for an (abstract) convex geometry. Based on these definitions, we shall show the analogues of the Whitney-Rota's formula and Brylawski's decomposition theorem for broken circuit complexes on matroids for convex geometries. We also present an Orlik-Solomon type algebra on a convex geometry, and show the NBC generating theorem. This note is on the same line as the studies in [10, 11, 12].

**Keywords:** broken circuit, characteristic polynomial, NBC basis theorem

## 1 Closure Systems, Matroids, and Convex Geometries

A collection $K \subseteq 2^E$ of subsets of a finite set $E$ is a *closure system* if

(1) $E \in K$,

(2) $X, Y \in K \Longrightarrow X \cap Y \in K$.

An element of $K$ is called a *closed set*. A closure system determines a closure operator

$$\sigma(A) = \bigcap_{X \in K, A \subseteq X} X \qquad (A \subseteq E). \tag{1.1}$$

An element in $\cap\{X : X \in K\} = \sigma(\emptyset)$ is a *loop*, and $K$ is *loop-free* if it has no loops.

A map $Ex : 2^E \to 2^E$ defined by $Ex(A) = \{e \in A : e \notin \sigma(A \setminus e)\}$ $(A \subseteq E)$ is an *extreme function*. We say that an element in $Ex(A)$ is an *extreme element* of $A$, and we call an extreme element of the entire set $E$ a *coloop*. A subset $A \subseteq E$ is an *independent set* if $Ex(A) = A$. A set which is not independent is *dependent*, and a minimal dependent set is called a *circuit*. It is easy to see that any subset of an independent set is independent.

When a closure operator satisfies the Steinitz-McLane exchange property below,

$$\text{if } x, y \notin \sigma(A) \text{ and } y \in \sigma(A \cup x), \text{ then } x \in \sigma(A \cup y) \qquad (x, y \in E,\ A \subseteq E), \tag{1.2}$$

then the corresponding closure system is the set of flats (closed sets) of a matroid $M$ on $E$, and vice versa. The notions of an independent set and a circuit introduced above as a closure system agree with the ordinary definitions of matroid theory.

Let $M$ be a matroid on $E$, and suppose we have a linear order $\omega$ on $E$. When $C$ is a circuit of $M$ and $e$ is the minimum element in $C$ with respect to $\omega$, we call $C \setminus e$ a *broken circuit*.

A subset of $E$ is *nbc-independent* if it contains no broken circuits of $M$. Evidently an nbc-independent set is an independent set of $M$. The collection of nbc-independent sets forms a simplicial complex $NBC(M, \omega)$, which is called a *broken circuit complex* of $M$ (with respect to $\omega$).

When the closure operator satisfies the anti-exchange property below

$$\text{if } x, y \notin \sigma(A) \text{ and } y \in \sigma(A \cup x), \text{ then } x \notin \sigma(A \cup y) \qquad (x, y \in E,\ A \subseteq E), \tag{1.3}$$

the closure system $K$ is called an *(abstract) convex geometry*. Convex geometries arise from various combinatorial objects such as affine point configurations, chordal graphs, posets, semi-lattices, searches on a rooted graph, and so on. (See [4, 9].)

Since a convex geometry itself is a closure system, we have the corresponding definitions of an independent set and a circuit for a convex geometry. In a circuit of a convex geometry there exists uniquely an element that is not extreme. (In a circuit of a matroid there is no element that is extreme.) That is, a circuit $C$ of a convex geometry contains a unique element $e$ such that $Ex(C) = C \setminus e$. We say that $e$ is the *root* of $C$, and $X = C \setminus e$ is a *broken circuit* with respect to the root $e$. And $(X, e)$ is a *rooted circuit*. Let us call a set *nbc-independent* if it contains no broken circuit. The collection of nbc-independent sets forms a simplicial complex, which is the *NBC complex* of $K$ denoted by $NBC(K)$.

Note that to determine a broken circuit for a matroid it is required to assume a linear order on the underlying set, while there is no need to suppose such an order when we define a broken circuit for a convex geometry.

# 2 Whitney-Rota's Formula and Its Analogue

## 2.1 Matroid

The NBC complexes of matroids appear in the Whitney-Rota's formula. Let $\mathcal{L}(M)$ be the lattice consisting of the closed sets (flats) of $M$. The characteristic polynomial $p(M; \lambda)$ of $M$ is defined by

$$p(\lambda; M) = \sum_{X \in \mathcal{L}(M)} \mu(\sigma(\emptyset), X) \lambda^{r(E) - r(X)}. \tag{2.1}$$

Then the Whitney-Rota's formula for matroids is described as

**Theorem 2.1 (Rota [14])** For an arbitrary linear order $\omega$ on $E$, we have

$$p(\lambda; M) = \sum_{X \in NBC(M, \omega)} (-1)^{|X|} \lambda^{r(E) - r(X)}. \tag{2.2}$$

## 2.2 Convex Geometry

Let $K$ be a loop-free convex geometry on a finite set $E$. The characteristic function of $K$ is

$$p(\lambda; K) = \sum_{X \in K} \mu_K(\emptyset, X) \lambda^{|E| - |X|} \tag{2.3}$$

where $\mu_K$ is the Möbius function of the lattice $K$. A set which is both closed and independent is a *free set*. The collection of the free sets constitutes a simplicial complex, called a free complex [3]. A free complex plays an important role in the counting formula of the interior points of an affine point configuration proved by Klain [8], and Edelman and Reiner [5]. A free complex of a convex geometry can be revealed to be equal to its NBC complex. That is,

**Theorem 2.2** A subset of $E$ is a free set if and only if it is nbc-independent. Equivalently, the free complex of a convex geometry coincides with its NBC complex.

Edelman [3] explicitly determined the values of $\mu_K$ as:

**Lemma 2.1 (Edelman [3])** For a closed set $X \in K$,

$$\mu_K(\emptyset, X) = \begin{cases} (-1)^{|X|} & \text{if } X \text{ is free,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Theorem 2.2 and Lemma 2.1 immediately give rise to the Whitney-Rota's formula for convex geometry:

**Theorem 2.3** For a convex geometry $K$ and the characteristic polynomial (2.3), it holds that

$$p(\lambda; K) = \sum_{X \in NBC(K)} (-1)^{|X|} \lambda^{|E|-|X|}. \tag{2.5}$$

# 3 Brylawskifs Decomposition and Its Analogue

## 3.1 Matroid

Brylawski [2] showed a direct-sum decomposition theorem of NBC complex of a matroid below.

**Theorem 3.1 (Brylawski [2])** Let $(M, \omega)$ be an ordered matroid, and $x$ be the maximum element with respect to $\omega$. Then

$$NBC(M, \omega) = NBC(M \setminus x, \omega) \uplus (NBC(M/x, \omega) * x) \tag{3.1}$$

where $NBC(M/x, \omega) * x = \{A \cup x \;:\; A \in NBC(M/x, \omega)\}$

## 3.2 Convex Geometry

Let $K$ be a convex geometry on $E$. For a coloop $e$, $K \setminus e = \{X : X \in K, e \notin X\}$ is a convex geometry on $E \setminus e$, which is a *deletion* of $e$ from $K$. For any element $e \in E$, $K/e = \{X \setminus e : X \in K, e \in X\}$ is a convex geometry on $E \setminus e$, which is a *contraction* of $e$ from $K$. We have Brylawski's decomposition theorem for convex geometries as

**Theorem 3.2** For a coloop $x \in E$ of a convex geometry $K$, we have

$$NBC(K) = NBC(K \setminus x) \uplus (NBC(K/x) * x) \tag{3.2}$$

# 4 Orlik-Solomon Algebra and Its Analogues

## 4.1 Matroid

An NBC complex is known to provide a linear basis of the Orlik-Solomon algebra, which we shall describe below. Suppose $E = \{e_1, \ldots, e_n\}$. Taking $e_1, \ldots, e_n$ as generators, we denote a graded external algebra over the free module $\oplus_{e\in E}\mathbb{Z}e$ by $\bigwedge E = \oplus_{i\in\mathbb{N}} \bigwedge^i E$. A linear map $\partial : \bigwedge E \longrightarrow \bigwedge E$ is defined by

(1) $\partial_0 : \mathbb{Z} \longrightarrow (0)$, (2) $\partial_1 : \bigwedge^1 E \longrightarrow \mathbb{Z}$ where $\partial(e) = 1 \quad (e \in E)$,

(3) for $k = 2, \ldots, n$:

$$\partial_k : \bigwedge^k E \longrightarrow \bigwedge^{k-1} E, \qquad \partial_k(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_k}$$

Although it is a little abuse of terminology, for the sake of simplicity, we associate a term $e_X = x_1 \wedge \cdots \wedge x_t$ in $\bigwedge E$ with each subset $X = \{x_1, \ldots, x_t\} \subseteq E$.

Suppose $I(M)$ to be an ideal generated by $\{\partial(e_C) : C \text{ is a circuit of } M \text{ with } |C| \geqslant 2\} \cup \{e : e \text{ is a loop od } M\}$. Then the *Orlik-Solomon algebra* of $M$ is defined as

$$OS(M) = \left(\bigwedge E\right) / I(M). \tag{4.1}$$

**Theorem 4.1 (NBC basis theorem for the Orlik-Solomon algebra [1], [13])** Let $M$ be a matroid on $E$, and $\omega$ be an arbitrary linear order on the underlying set. Then $\{e_X : X \in NBC(M, \omega)\}$ is a linear basis of module $OS(M)$.

## 4.2 Convex Geoemtry

Suppose $K$ to be a loop-free convex geometry on $E = \{e_1, \ldots, e_n\}$. The graded external algebra $\bigwedge E = \oplus_{i=0}^{n} \bigwedge^i E$ and a linear map $\partial : \bigwedge E \longrightarrow \bigwedge E$ are defined in the same way as before. And let $I(K)$ be the ideal in $\bigwedge E$ generated by $\{\partial(e_C) : C \text{ is a circuit of } K\}$, and let us define an *Orlik-Solomon type algebra of a convex geometry* $K$ by

$$OS(K) = \left(\bigwedge E\right) / I(K). \tag{4.2}$$

It can be shown that $\{e_X : X \in NBC(K)\}$ is a linear generating set of $OS(K)$. That is, although the NBC basis theorem (Theorem 4.1) does not hold for $OS(K)$, we have a weaker form, the NBC generating theorem, below.

**Theorem 4.2** An arbitrary element in $OS(K)$ can be represented as a linear combination of the terms in $\{e_X : X \in NBC(K)\}$.

There is an alternative definition of an Orlik-Solomon type algebra so that the NBC basis theorem would be satisfied. Let $J(K)$ be the ideal generated by $\{e_X : X \text{ is a broken circuit of } K\}$, and let us define an algebra

$$A(K) = \left(\bigwedge E\right) / J(K) \tag{4.3}$$

By definition $\{e_X : X \in NBC(K)\}$ is necessarily a linear basis of module $A(K)$.

Hence the decomposition of Theorem 3.2 readily implies the short exact split sequence theorem for $A(K)$.

**Theorem 4.3** For a coloop $x$ of a convex geometry $K$,

$$0 \rightarrow A(K \setminus x) \xrightarrow{i_x} A(K) \xrightarrow{p_x} A(K/x) \rightarrow 0 \tag{4.4}$$

is an exact short split sequence.

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# References

[1] A. Björner, "The homology and shellability of matroids and geometric lattices." in: N. White ed., *Matroid Applications*, Cambridge Univ. Press, 1992, 226–283.

[2] T. Brylawski, "The broken-circuit complex," *Trans. Amer. Math. Soc.* 234 (1977) 417–433.

[3] P. H. Edelman, "Abstract convexity and meet-distributive lattices," *Contemporary Math.* 57 (1986) 127-150.

[4] P. H. Edelman and R. E. Jamison, "The theory of convex-geometries," *Geometriae dedicata* 19 (1985) 247–270.

[5] P.H. Edelman and V. Reiner, "Counting interior points of a point configuration," *Disc. Math.* 23 (2000) 1–13.

[6] M. Falk, "Combinatorial and algebraic structure in Orlik-Solomon algebras," *Europ. J. Combi.* 22 (2001) 687–698.

[7] Y. Kawahara, "On matroids and Orlik-Solomon algebras," *Annals of Combinatorics* 8 (2004) 63–80.

[8] D. A. Klain, "An Euler relation for valuations on polytopes," *Adv. in Math.* 147 (1999) 1–34.

[9] B. Korte, L. Lovász and R. Schrader, *Greedoids*, Springer-Verlag, 1991.

[10] M. Nakamura, "Excluded-minor characterizations of antimatroids arisen from posets and graph searches," *Discrete Applied Mathematics* 129/2-3 (2003) 487–498.

[11] Y. Okamoto and M. Nakamura, "The forbidden minor characterization of line-search antimatroids of rooted digraphs," *Discrete Applied Mathematics* 131 (2003) 523–533.

[12] M. Nakamura, "The deletion-contraction rule and Whitney-Rota's formula for the characteristic polynomials of convex geometries," preprint.

[13] P. Orlik and H. Terao, *Arrangements of Hyperplanes,* Springer, 1991.

[14] G.-C. Rota, "On the Foundations of Combinatorial Theory I. Theory of Möbius Functions," *Z. Wahrscheinlichkeitstheoreie* 2 (1964) 340–368.

[15] G. Ziegler, "Matroid shellability, $\beta$-Systems and Affine Hyperplane Arrangements," *J. of Algebraic Combinatorics* 1 (1992) 283–300.