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Nonrepetitive colorings of graphs

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A vertex coloring of a graph $G$ is \textit{k}-nonrepetitive if one cannot find a periodic sequence with $k$ blocks on any simple path of $G$. The minimum number of colors needed for such coloring is denoted by $\pi_k(G)$. This idea combines graph colorings with Thue sequences introduced at the beginning of 20th century. In particular Thue proved that if $G$ is a simple path of any length greater than 4 then $\pi_2(G) = 3$ and $\pi_3(G) = 2$. We investigate $\pi_k(G)$ for other classes of graphs. Particularly interesting open problem is to decide if there is, possibly huge, $k$ such that $\pi_k(G)$ is bounded for planar graphs.

Let $k \geq 2$ be a fixed integer. A coloring $f$ of the vertices of a graph $G$ is \textit{k}-repetitive if there is $n \geq 1$ and a simple path $v_1v_2\ldots v_{kn}$ of $G$ such that $f(v_i) = f(v_j)$ whenever $i - j$ is divisible by $n$. Otherwise $f$ is called \textit{k}-nonrepetitive. The minimum number of colors needed for a $k$-nonrepetitive coloring of $G$ is denoted by $\pi_k(G)$. Notice that any 2-nonrepetitive coloring must be proper in the usual sense, while this is not necessarily the case for $k \geq 3$.

By the 1906 theorem of Thue\textsuperscript{[6]} $\pi_2(G) \leq 3$ and $\pi_3(G) \leq 2$ if $G$ is a simple path of any length. Let $\pi_k(d)$ denote the supremum of $\pi_k(G)$, where $G$ ranges over all graphs with $\Delta(G) \leq d$. A simple extension of probabilistic arguments from \cite{2} (for $k = 2$) shows that there are absolute positive constants $c_1$ and $c_2$ such that

$$c_1 \frac{d^{k/(k-1)}}{(\log d)^{1/(k-1)}} \leq \pi_k(d) \leq c_2 d^{k/(k-1)}.$$ 

Moreover, one can show that for each $d$ there exists a sufficiently large $k = k(d)$ such that $\pi_k(d) \leq d + 1$. On the other hand, any $\lfloor d/2 \rfloor$-coloring of a $d$-regular graph of girth at least $2k + 1$ is $k$-repetitive. The maximum number $t(d)$ such that for each $k$ there is a $d$-regular graph $G$ with $\pi_k(G) > t(d)$ is not known for $d \geq 3$.

Kündgen and Pelsmajer\textsuperscript{[4]} and Barát and Varjú\textsuperscript{[3]} proved independently that $\pi_2(G)$ is bounded for graphs of bounded treewidth. By the result of Robertson and Seymour\textsuperscript{[5]} it follows that if $H$ is any fixed planar graph then $\pi_k(G)$ is bounded for graphs not containing $H$ as a minor. However, it is still not known whether there are some constants $k$ and $c$ such that $\pi_k(G) \leq c$ for any planar graph $G$. The least possible constant $c$ for which this could hold (with possibly huge $k$) is $c = 4$.

In a weaker version of the problem we ask for nonrepetitive colorings of subdivided graphs. By the result of Thue every graph has a (sufficiently large) subdivision which is nonrepetitively 5-colorable (for any $k \geq 2$). Clearly this cannot happen for all graphs if we restrict the number of vertices added to an
edge. For instance, any \( c \)-coloring of the complete graph \( K_n \), with each edge subdivided by at most \( r \) vertices, is \( 2 \)-repetitive if \( c < \log_r \log_2(n/r) \). The question if there are constants \( c, k, r \) such that each planar graph \( G \) has an \( r \)-restricted subdivision \( S \) with \( \pi_k(S) \leq c \), is open.

There are many interesting connections of this area to other graph coloring topics. Let \( s(G) \) be the star chromatic number of a graph \( G \), that is, the least number of colors in a proper coloring of the vertices of \( G \), with additional property that every two color classes induce a star forest. It is not hard to see that \( \pi_2(G) \geq s(G) \) for any graph \( G \). Hence, by the results of Albertson et al. [1] it follows that there are planar graphs with \( \pi_2(G) \geq 10 \), and for each \( t \) there are graphs of treewidth \( t \) with \( \pi_2(G) \geq (t+1)\frac{1}{2} \).

References


[3] J. Barát, P. P. Varjú, Some results on square-free colorings of graphs, manuscript.

