Nonrepetitive colorings of graphs
Noga Alon, Jaroslaw Grytczuk

To cite this version:
hal-01184372

HAL Id: hal-01184372
https://hal.inria.fr/hal-01184372
Submitted on 14 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Nonrepetitive colorings of graphs

Noga Alon\(^1\) and Jarosław Grytczuk\(^2\)

\(^1\)Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel, noga@tau.ac.il
\(^2\)Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, 65-516 Zielona Góra, Poland, J.Grytczuk@wmie.uz.zgora.pl

A vertex coloring of a graph \(G\) is \(k\)-nonrepetitive if one cannot find a periodic sequence with \(k\) blocks on any simple path of \(G\). The minimum number of colors needed for such coloring is denoted by \(\pi_k(G)\). This idea combines graph colorings with Thue sequences introduced at the beginning of 20th century. In particular Thue proved that if \(G\) is a simple path of any length greater than 4 then \(\pi_2(G) = 3\) and \(\pi_3(G) = 2\). We investigate \(\pi_k(G)\) for other classes of graphs. Particularly interesting open problem is to decide if there is, possibly huge, \(k\) such that \(\pi_k(G)\) is bounded for planar graphs.

Let \(k \geq 2\) be a fixed integer. A coloring \(f\) of the vertices of a graph \(G\) is \(k\)-repetitive if there is \(n \geq 1\) and a simple path \(v_1v_2\ldots v_{kn}\) of \(G\) such that \(f(v_i) = f(v_j)\) whenever \(i - j\) is divisible by \(n\). Otherwise \(f\) is called \(k\)-nonrepetitive. The minimum number of colors needed for a \(k\)-nonrepetitive coloring of \(G\) is denoted by \(\pi_k(G)\). Notice that any 2-nonrepetitive coloring must be proper in the usual sense, while this is not necessarily the case for \(k \geq 3\).

By the 1906 theorem of Thue \([6]\) \(\pi_2(G) \leq 3\) and \(\pi_3(G) \leq 2\) if \(G\) is a simple path of any length. Let \(\pi_k(d)\) denote the supremum of \(\pi_k(G)\), where \(G\) ranges over all graphs with \(\Delta(G) \leq d\). A simple extension of probabilistic arguments from \([2]\) (for \(k = 2\)) shows that there are absolute positive constants \(c_1\) and \(c_2\) such that

\[
    c_1 \frac{d^{k/(k-1)}}{(\log d)^{1/(k-1)}} \leq \pi_k(d) \leq c_2 d^{k/(k-1)}.
\]

Moreover, one can show that for each \(d\) there exists a sufficiently large \(k = k(d)\) such that \(\pi_k(d) \leq d + 1\).

On the other hand, any \([d/2]\)-coloring of a \(d\)-regular graph of girth at least \(2k + 1\) is \(k\)-repetitive. The maximum number \(t(d)\) such that for each \(k\) there is a \(d\)-regular graph \(G\) with \(\pi_k(G) > t(d)\) is not known for \(d \geq 3\).

Kündgen and Pelsmajer \([4]\) and Barát and Varjú \([3]\) proved independently that \(\pi_2(G)\) is bounded for graphs of bounded treewidth. By the result of Robertson and Seymour \([5]\) it follows that if \(H\) is any fixed planar graph then \(\pi_k(G)\) is bounded for graphs not containing \(H\) as a minor. However, it is still not known whether there are some constants \(k\) and \(c\) such that \(\pi_k(G) \leq c\) for any planar graph \(G\). The least possible constant \(c\) for which this could hold (with possibly huge \(k\)) is \(c = 4\).

In a weaker version of the problem we ask for nonrepetitive colorings of subdivided graphs. By the result of Thue every graph has a (sufficiently large) subdivision which is nonrepetitively 5-colorable (for any \(k \geq 2\)). Clearly this cannot happen for all graphs if we restrict the number of vertices added to an
edge. For instance, any \( c \)-coloring of the complete graph \( K_n \), with each edge subdivided by at most \( r \) vertices, is 2-repetitive if \( c < \log^* \log_2(n/r) \). The question if there are constants \( c, k, \) and \( r \) such that each planar graph \( G \) has an \( r \)-restricted subdivision \( S \) with \( \pi_k(S) \leq c \), is open.

There are many interesting connections of this area to other graph coloring topics. Let \( s(G) \) be the star chromatic number of a graph \( G \), that is, the least number of colors in a proper coloring of the vertices of \( G \), with additional property that every two color classes induce a star forest. It is not hard to see that \( \pi_2(G) \geq s(G) \) for any graph \( G \). Hence, by the results of Albertson et al. [1] it follows that there are planar graphs with \( \pi_2(G) \geq 10 \), and for each \( t \) there are graphs of treewidth \( t \) with \( \pi_2(G) \geq (t+1)/2 \).

References

[3] J. Barát, P. P. Varjú, Some results on square-free colorings of graphs, manuscript.