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Fast separation in a graph with an excluded minor $^{†}$

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Let $G$ be an $n$-vertex $m$-edge graph with weighted vertices. A pair of vertex sets $A, B \subseteq V(G)$ is a $\frac{3}{2}$-separation of order $|A \cap B|$ if $A \cup B = V(G)$, there is no edge between $A \setminus B$ and $B \setminus A$, and both $A \setminus B$ and $B \setminus A$ have weight at most $\frac{3}{2}$ the total weight of $G$. Let $\ell \in \mathbb{Z}^+$ be fixed. Alon, Seymour and Thomas [J. Amer. Math. Soc. 1990] presented an algorithm that in $O(n^{1/2}m)$ time, either outputs a $K_{\ell}$-minor of $G$, or a separation of $G$ of order $O(n^{1/2})$. Whether there is an $O(n + m)$ time algorithm for this theorem was left as open problem. In this paper, we obtain a $O(n + m)$ time algorithm at the expense of the separator. Moreover, our algorithm exhibits a tradeoff between running time and the order of the separator. In particular, for any given $\epsilon \in [0, \frac{3}{2}]$, our algorithm either outputs a $K_{\lfloor \ell \rfloor}$-minor of $G$, or a separation of $G$ with order $O(n^{(2-\epsilon)/3})$ in $O(n^{1+\epsilon} + m)$ time.

Keywords: graph algorithm, separator, minor

1 Introduction

We consider graphs $G$ that are simple, finite, and undirected. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of $G$. Let $|G| := |V(G)|$ and $|E(G)| := |E(G)|$. A separation of $G$ is a pair $\{A, B\}$ of vertex sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, and there is no edge with one endpoint in $A \setminus B$ and the other endpoint in $B \setminus A$. The order of $\{A, B\}$ is $|A \cap B|$. The set $A \cap B$ is called a separator of $G$. A weighting of $G$ is a function $w : V(G) \to \mathbb{R}^+$. Let $w(S) := \sum_{v \in S} w(v)$ for all $S \subseteq V(G)$, and $w(G) := w(V(G))$. We say $(G, w)$ is a weighted graph. A separation $\{A, B\}$ of a weighted graph $(G, w)$ is an $\alpha$-separation if $w(A \setminus B) \leq \alpha \cdot w(G)$ and $w(B \setminus A) \leq \alpha \cdot w(G)$.

A ‘separator theorem’ is of the format: for some $0 < \alpha < 1$ and $0 < \epsilon < 1$, every graph $G$ from a certain family has an $\alpha$-separation of order $O(|G|^{1-\epsilon})$. Applications of separator theorems are numerous, and include VLSI circuit layout, approximation algorithms using the divide-and-conquer paradigm, solving sparse systems of linear equations, pebbling games, and graph drawing. See the recent monograph by Rosenberg and Heath [9] for more details.

A seminal theorem due to Lipton and Tarjan [5] states that every weighted planar graph $G$ has a $\frac{3}{2}$-separation of order $O(|G|^{1/2})$ that can be computed in $O(|G| + |E(G)|)$ time. This result was generalised

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for graphs with an excluded minor by Alon et al. [1] (see [2, 3, 7] for related results). A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges, in which case we say that $G$ has an $H$-minor. The Kuratowski-Wagner Theorem states that a graph is planar if and only if it has no $K_5$-minor and no $K_{3,3}$-minor. An $H$-model in $G$ is a set of disjoint connected subgraphs $\{X_v : v \in V(H)\}$ indexed by the vertices of $H$, such that for every edge $vw \in E(H)$, there is an edge $xy \in E(G)$ with $x \in X_v$ and $y \in X_w$. Clearly $G$ has an $H$-minor if and only if $G$ has an $H$-model. We choose to work with $H$-models rather than $H$-minors.

**Theorem 1 (Alon et al. [1])** There is an algorithm with running time $O((\ell \cdot |G|)^{1/2} \cdot (|G| + \|G\|))$ that, given $\ell \in \mathbb{Z}^+$ and a weighted graph $(G, w)$, either outputs:

(a) a $K_\ell$-model of $G$, or

(b) a $\frac{\ell}{3}$-separation of $(G, w)$ of order at most $\ell^{3/2} \cdot |G|^{1/2}$.

Suppose that $\ell$ is fixed. It follows from a result of Mader [6] (see Theorem 3) that Theorem 1 can be implemented in $O(|G|^{1/2} + \|G\|)$ time. Alon et al. [1] left as an open problem whether linear time is possible. The main result of this paper is the following partial answer to this question. We obtain a linear running time at the expense of a slightly larger separator (and a larger dependence on $\ell$). Moreover, our algorithm exhibits a tradeoff between running time (ranging from $O(n)$ to $O(n^{3/2})$) and the order of the separator (ranging from $O(n^{2/3})$ to $O(n^{1/2})$).

**Theorem 2** There is an algorithm with running time $O(2^{3\ell^2 + 7\ell - 3}/2 \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|)$ that, given $\epsilon \in [0, \frac{1}{2}], \ell \in \mathbb{Z}^+$, and a weighted graph $(G, w)$, either outputs:

(a) a $K_\ell$-model of $G$, or

(b) a $\frac{\ell}{2}$-separation of $(G, w)$ of order at most $2^{(\ell^2 + 3\ell + 1)/2} \cdot |G|^{2 - \epsilon}/3$.

Note that for applications to divide-and-conquer algorithms a separation of order $O(|G|^{1-\epsilon})$, for some constant $\epsilon > 0$, is all that is needed.

The idea behind the proof of Theorem 2 is simple. We now outline the proof for fixed $\ell$ and with $\epsilon = 0$. Suppose that in $O(|G| + \|G\|)$ time, we can find a partition of $V(G)$ into $|G|^{2/3}$ connected subgraphs $\{S_1, S_2, \ldots, S_{|G|^{2/3}}\}$, each containing $O(|G|^{1/3})$ vertices. Let $H$ be the weighted graph obtained from $G$ by contracting each $S_i$ to a vertex $v_i$ with weight $w(v_i) = w(S_i)$. Then apply Theorem 1 to $H$ to either obtain a $K_\ell$-model in $H$, which defines a $K_\ell$-model in $G$, or a $\frac{\ell}{2}$-separation $\{A, B\}$ of $H$ with order $O(|H|^{1/2}) = O(|G|^{1/3})$, in which case $\bigcup\{S_i : v_i \in A\}, \bigcup\{S_i : v_i \in B\}$ is a $\frac{\ell}{2}$-separation of $G$ with order $O(|G|^{2/3})$. The running time is $O(|H|^{1/2} + \|H\|) \leq O(|G| + \|G\|)$. The proof of Theorem 2 is actually a little different from this outline. In particular, the subgraphs $S_i$ will not necessarily be connected, but we will still be able to convert the output from Theorem 1 applied to $H$ to the desired output for $G$. By relaxing the connectivity condition, we are able to prove that an appropriate partition exists.

We will use the following notation for a graph $G$. For $x \in V(G)$, let $N(x) := \{y \in V(G) : xy \in E(G)\}$. For a subgraph $X$ of $G$, let $N(X) := \bigcup\{N(x) \setminus X : x \in X\}$. Where there is no confusion, a set of vertices $S \subseteq V(G)$ will also refer to the subgraph of $G$ induced by $S$.

## 2 Mader's Theorem

This section contains a number of easily proved results—see the full version of the paper for details. We start with an algorithmic version of a theorem of Mader [6] (cf. [8]).
Lemma 1 Given a graph $G$ with $\|G\| \geq 2^k \cdot |G|$ (for some $k \in \mathbb{Z}^+$), a $K_{2k}$-model of $G$ can be computed in $O(\ell(\|G\| + \|G\|))$ time.

Note that if we ignore the running time, Theorem 3 is far from best possible. Kostochka [4] and Thomason [10] independently proved that if $\|G\| \in \Omega(\ell \sqrt{\log \ell} \cdot |G|)$ then $G$ has a $K_{2k}$-model. Theorem 3 implies the following slightly faster version of Theorem 1 (for fixed $\ell$)

Theorem 4 There is an algorithm with running time $O(2^{2\ell} \cdot |G|^{3/2} + |G|)$ that, given $\ell \in \mathbb{Z}^+$ and a weighted graph $(G, w)$, either outputs:

(a) a $K_{2\ell}$-model of $G$, or
(b) a $\frac{3}{2}$-separation of $(G, w)$ of order at most $\ell^{3/2} \cdot |G|^{1/2}$.

A $k$-clique of $G$ is a (not necessarily maximal) set of $k$ pairwise adjacent vertices of $G$. If every subgraph of $G$ has a vertex of degree at least $d$, then $G$ is $d$-degenerate. For example, Theorem 3 implies that a graph with no $K_{2k}$-minor is $2^{\ell-2}$-degenerate. It is easily proved that a $d$-degenerate graph $G$ with no $k$-clique has less than $d^{k-1} \cdot |G|$ cliques. Hence a graph $G$ with no $K_{2k}$-minor has less than $2^{(\ell-2)(\ell-1)} \cdot |G|$ cliques. For an algorithm, we have the following result.

Lemma 1 Given a graph $G$ with no $k$-clique and at least $2^{(\ell-2)(k-1)} \cdot |G|$ cliques (for some $\ell \in \mathbb{Z}^+$), a $K_{2k}$-minor of $G$ can be computed in $O(\ell(\|G\| + \|G\|))$ time.

3 Proof of Theorem 2

Let $G$ be a graph. Let $A$ be a partition of $V(G)$. Let $H$ be the graph obtained from $G$ by collapsing each part $S \in A$ to a single vertex $v$, and replacing parallel edges by a single edge. Denote $H_v := S$. We say 

$$\{H_v : v \in V(H)\}$$

is an $H$-partition of $G$. Furthermore, $\{H_v : v \in V(H)\}$ is a connected $H$-partition of $G$ if for every $e \in E(H)$ if and only if there is an edge of $G$ between every component of $H_v$ and every component of $H_w$. We prove the following lemma.

Lemma 2 There is an algorithm with running time $O(2^{2\ell} \cdot |G| + \|G\|)$ that, given $\ell, k \in \mathbb{Z}^+$ and a graph $G$, outputs a connected $H$-partition of $G$ such that either:

(a) $H$ has a $K_{2\ell}$-model (which is also output), or
(b) $|H| \leq 2^{\ell^2+\ell-1} \cdot |G| \cdot k^{-1}$, and $|H_v| < 2k$ for all $v \in V(H)$.

Proof of Theorem 2 assuming Lemma 2 Apply Lemma 2 with $k = \lfloor |G|^{(1-2\ell)/3} \rfloor$. First suppose that $G$ has a $K_{2\ell}$-model $\{S_1, S_2, \ldots, S_{2\ell}\}$ of $H$. Thus each $S_i$ is a connected subgraph of $H$. Choose a connected component $Z_v$ of $H_v$ for each $v \in V(H)$. Let $T_v := \bigcup\{Z_v : v \in S_i\}$. Then $\{T_1, T_2, \ldots, T_{2\ell}\}$ is a $K_{2\ell}$-model of $G$.

Otherwise $|H| \leq 2^{\ell^2+\ell-1} \cdot |G|^{2(1+\ell)/3}$, and $|H_v| < 2|G|^{(1-2\ell)/3}$ for all $v \in V(H)$. Let $w(v) := w(H_v)$ for all $v \in V(H)$. Apply Theorem 4 to $(H, w)$. The running time is $O(2^{2\ell} |H|^{3/2+\ell} \|H\|) \leq O(2^{2\ell} \cdot 2^{2\ell^2+\ell-1} \cdot |G|^{2(1+\ell)/3} \cdot |G|^{1+\epsilon+\ell} \|G\|) \leq O(2^{2\ell^2+7\ell-3}/2 \cdot |G|^{1+\epsilon+\ell} \|G\|)$.

We either obtain a $K_{2\ell}$-model of $H$, or a $\frac{3}{2}$-separation of $H$ with order at most $\ell^{3/2} \cdot |H|^{1/2}$. In the first case, $G$ has a $K_{2\ell}$-model as proved above.

Now suppose that we obtain a $\frac{3}{2}$-separation $\{A, B\}$ of $(H, w)$ with order $|A \cap B| \leq \ell^{3/2} \cdot |H|^{1/2} \leq \ell^{3/2} \cdot (2^{\ell^2+\ell-1} |G|^{2(1+\ell)/3})^{1/2} \leq 2^{(\ell^2+2\ell-1)/2} \cdot |G|^{(1+\epsilon)/3}$.
Let $X := \bigcup \{ H_v : v \in A \}$ and $Y := \bigcup \{ H_v : v \in B \}$. Then $\{ X, Y \}$ is a separation of $G$ with order $|X \cap Y| = \big| \bigcup \{ H_v : v \in A \cap B \} \big| \leq 2^{(\ell^2 + 3\ell - 1)/2} \cdot |G|^{(1+\varepsilon)/3} \cdot 2|G|^{(1-2\varepsilon)/3} \leq 2^{(\ell^2 + 3\ell + 1)/2} \cdot |G|^{(2-\varepsilon)/3}$.

We have $w(X \setminus Y) = w(A \setminus B) \leq \frac{1}{2} w(H) = \frac{1}{3} w(G)$. Similarly $w(B \setminus A) \leq \frac{1}{2} w(G)$.

**Proof of Lemma 2:**

**Step 1:** Using a breadth-first search algorithm, compute a maximal set $A$ of connected subgraphs of $G$ such that $|S| = k$ for all $S \in A$. Let $B$ be the set of connected components of $G \setminus \bigcup \{ S \in A \}$. Then $A \cup B$ is a partition of $V(G)$, and there is no edge of $G$ between distinct $T_1, T_2 \in B$. Note that $|T| < k$ for all $T \in B$, as otherwise $T$ would contain a connected subgraph $X$ with $|X| = k$, which could be added to $A$.

**Step 2:** Let $H$ be the graph obtained from $G$ by contracting each set $S \in A \cup B$ into a single vertex $v$ with $H_v := S$, and replacing parallel edges by a single edge. Since each $S \in A \cup B$ is connected in $G$, \{ $H_v : v \in V(H)$ \} is a connected $H$-partition of $G$. Let $A := \{ v \in V(H) : H_v \in A \}$ and $B := \{ v \in V(H) : H_v \in B \}$. A vertex $v$ of $H$ is big if $|H_v| \geq k$. A vertex $v$ of $H$ is small if $|H_v| < k$. Observe that every vertex in $A$ is big. $B$ is an independent set of $H$, and every vertex in $B$ is small. Partition $B = C \cup D \cup E$, where $C := \{ v \in B : \deg_H(v) \geq 2^{\ell-2} \}$, $D := \{ v \in B : \ell - 1 \leq \deg_H(v) < 2^{\ell-2} \}$, and $E := \{ v \in B : \deg_H(v) \leq \ell - 2 \}$.

**Step 3:** Suppose that $|C| \geq |A|$. Then the subgraph $C \cup A$ of $H$ has at least $2^{\ell-2} \cdot |C|$ edges and at most $2|C|$ vertices. By Theorem 3, a $K_\ell$-model of $C \cup A$ can be computed in $O(\ell \cdot |G|)$ time. We now assume that $|C| < |A|$.

**Step 4:** For each vertex $v \in D \cup E$, if there is a pair $x, y \in A$ of distinct neighbours of $v$, such that $\{ x, y \}$ has not been assigned any vertex in $D \cup E$, then assign $v$ to $\{ x, y \}$. This step can be implemented in $O(2^{2\ell} \cdot |G|)$ time, since each vertex in $D \cup E$ has degree at most $2^{\ell-2}$.

Suppose that there is a vertex $v \in D$ that is not assigned. Let the neighbourhood of $v$ be $\{ x_1, x_2, \ldots, x_d \}$. Then $d \geq \ell - 1$. Thus for all $1 \leq i < j \leq d$, there is a distinct vertex $v_{i,j}$ that is assigned to the pair $\{ x_i, x_j \}$, and $v_{i,j}$ is adjacent to both $x_i$ and $x_j$. In the graph obtained from $H$ by contracting each edge $x_i v_{i,j}$, the subgraph $\{ x_1, x_2, \ldots, x_d, v \}$ is a clique on at least $\ell$ vertices. Thus $H$ has a $K_\ell$-model. We now assume that every vertex in $D$ is assigned.

Let $E^* := \{ \{ x, y \} \in D \cup E \mid \{ x, y \}$ have not been assigned any vertex in $D \cup E \}$ assigned to the pair $\{ x, y \}$. This graph has $|E^*|$ vertices and at least $|D| + |E^*|$ edges. Thus if $|D| + |E^*| \geq 2^{\ell-3} \cdot |A|$, then by Theorem 3, $H$ has a $K_\ell$-model that can be computed in $O(\ell \cdot |G|)$ time. We now assume that $|D| + |E^*| < 2^{\ell-3} \cdot |A|$.

**Step 5:** Partition $E \setminus E^* = \bigcup \{ P_1, P_2, \ldots, P_s \}$ such that for all $u, v \in E \setminus E^*$, we have $N(u) = N(v)$ if and only if both $u, v \in P_i$ for some $1 \leq i \leq s$. For all $1 \leq i \leq s$, partition $P_i = \bigcup \{ P_{i,1}, P_{i,2}, \ldots, P_{i,t_i} \}$ such that for all $1 \leq j \leq t_i - 1$, $k \leq |\bigcup \{ H_v : v \in P_{i,j} \}| < 2k$, and $|\bigcup \{ H_v : v \in P_{i,t_i} \}| < k$. This is possible since $|H_v| < k$ for all $v \in P_i$. Collapse each set $P_{i,j}$ into a single vertex $p_{i,j}$ in $H$, whose associated subgraph in $G$ is $H_{p_{i,j}} := \bigcup \{ H_v : v \in P_{i,j} \}$. Since the vertices in $P_{i,j}$ have the same neighbourhood, $\{ H_v : v \in V(H) \}$ remains a connected partition of $G$. Let $E_{big} = \{ p_{i,j} : 1 \leq i \leq s, 1 \leq j \leq t_i - 1 \}$ and $E_{small} = \{ p_{i,t_i} : 1 \leq i \leq s \}$. Then every vertex in $E_{big}$ is big and every vertex in $E_{small}$ is small.
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Suppose that $|E_{\text{small}}| \geq 2^\ell^2 \cdot |A|$. Let $X$ be the graph obtained from $A$ by adding a clique on $N(v)$ for each vertex $v \in E_{\text{small}}$. Since distinct vertices in $E_{\text{small}}$ have distinct neighbourhoods, this process adds at least $|E_{\text{small}}| \geq 2^\ell^2 \cdot |A|$ cliques. Thus by Lemma 1, a $K_\ell$-model of $X$ can be computed in $O(|G|)$ time. For every edge $x_i, x_j$ in this $K_\ell$-model that is in $X$ but not in $A$, we have $x_i, x_j \in N(v)$ for some $v \in E_{\text{small}}$. Since $v$ is not assigned, there is a vertex $u \in D \cup E^*$ assigned to $\{x_i, x_j\}$, and $u$ is adjacent to both $x_i$ and $x_j$. Since $u$ is not in the $K_\ell$-model, we can include $u$ in the connected subgraph of the $K_\ell$-model that contains $x_i$ or $x_j$, and we obtain a $K_\ell$-model in $A \cup D \cup E^*$ (in particular, without the edge $x_i x_j$). Now assume that $|E_{\text{small}}| < 2^\ell^2 \cdot |A|$.  

**Step 6:** We have now partitioned $V(H)$ into sets $A \cup E_{\text{big}}$ of big vertices, and sets $C \cup D \cup E^* \cup E_{\text{small}}$ of small vertices. We have proved that $|C| < |A|$, $|D| + |E^*| < 2^{\ell-3} \cdot |A|$, and $|E_{\text{small}}| < 2^\ell^2 \cdot |A|$. Thus the number of small vertices is less than $(1 + 2^{\ell-3} + 2^\ell + 1) \cdot |A| \leq 2^{\ell^2 + \ell - 2} \cdot |A|$. By definition, the number of big vertices in $H$ is at most $|G| \cdot k^{-1}$. In particular, $|A| \leq |G| \cdot k^{-1}$. Thus $|H| \leq 2^\ell^2 + \ell - 1 \cdot |G| \cdot k^{-1}$. Moreover, every $|H_u| < 2k^2$ for every vertex $v \in V(H)$. ∎

**References**


