Fast separation in a graph with an excluded minor

Bruce Reed\(^1\) and David R. Wood\(^2\)

\(^1\)School of Computer Science, McGill University, Montréal, Canada (breed@cs.mcgill.ca)
\(^2\)Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain
(david.wood@upc.edu)

Let \(G\) be an \(n\)-vertex \(m\)-edge graph with weighted vertices. A pair of vertex sets \(A, B \subseteq V(G)\) is a \(\frac{2}{3}\)-separation of order \(|A \cap B|\) if \(A \cup B = V(G)\), there is no edge between \(A \setminus B\) and \(B \setminus A\), and both \(A \setminus B\) and \(B \setminus A\) have weight at most \(\frac{2}{3}\) the total weight of \(G\). Let \(\ell \in \mathbb{Z}^+\) be fixed. Alon, Seymour and Thomas [\(J. \text{Amer. Math. Soc.} \ 1990\)] presented an algorithm that in \(O(n^{1/2})\) time, either outputs a \(K_\ell\)-minor of \(G\), or a separation of \(G\) of order \(O(n^{1/2})\). Whether there is a \(O(n + m)\) time algorithm for this theorem was left as open problem. In this paper, we obtain a \(O(n + m)\) time algorithm at the expense of \(O(n^{2/3})\) separator. Moreover, our algorithm exhibits a tradeoff between running time and the order of the separator. In particular, for any given \(\epsilon \in [0, \frac{1}{2}]\), our algorithm either outputs a \(K_\ell\)-minor of \(G\), or a separation of \(G\) with order \(O(n^{(2-\epsilon)/3})\) in \(O(n^{1+\epsilon} + m)\) time.

**Keywords:** graph algorithm, separator, minor

1 Introduction

We consider graphs \(G\) that are simple, finite, and undirected. Let \(V(G)\) and \(E(G)\) denote the vertex and edge sets of \(G\). Let \(|G| := |V(G)|\) and \(||G|| := |E(G)|\). A separation of \(G\) is a pair \(\{A, B\}\) of vertex sets \(A, B \subseteq V(G)\) such that \(A \cup B = V(G)\), and there is no edge with one endpoint in \(A \setminus B\) and the other endpoint in \(B \setminus A\). The order of \(\{A, B\}\) is \(|A \cap B|\). The set \(A \cap B\) is called a separator of \(G\). A weighting of \(G\) is a function \(w : V(G) \rightarrow \mathbb{R}^+\). Let \(w(S) := \sum_{v \in S} w(v)\) for all \(S \subseteq V(G)\), and \(w(G) := w(V(G))\). We say \((G, w)\) is a weighted graph. A separation \(\{A, B\}\) of a weighted graph \((G, w)\) is an \(\alpha\)-separation if \(w(A \setminus B) \leq \alpha \cdot w(G)\) and \(w(B \setminus A) \leq \alpha \cdot w(G)\).

A ‘separator theorem’ is of the format: for some \(0 < \alpha < 1\) and \(0 < \epsilon \leq 1\), every graph \(G\) from a certain family has an \(\alpha\)-separation of order \(O(||G||^{1-\epsilon})\). Applications of separator theorems are numerous, and include VLSI circuit layout, approximation algorithms using the divide-and-conquer paradigm, solving sparse systems of linear equations, pebbling games, and graph drawing. See the recent monograph by Rosenberg and Heath [9] for more details.

A seminal theorem due to Lipton and Tarjan [5] states that every weighted planar graph \(G\) has a \(\frac{2}{3}\)-separation of order \(O(||G||^{1/2})\) that can be computed in \(O(||G|| + ||G||)\) time. This result was generalised...
for graphs with an excluded minor by Alon et al. [1] (see [2, 3, 7] for related results). A graph $H$ is a
minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges, in which case we
say that $G$ has an $H$-minor. The Kuratowski-Wagner Theorem states that a graph is planar if and only
if it has no $K_5$-minor and no $K_{3,3}$-minor. An $H$-model in $G$ is a set of disjoint connected subgraphs
$\{X_v : v \in V(H)\}$ indexed by the vertices of $H$, such that for every edge $vw \in E(H)$, there is an edge
$xy \in E(G)$ with $x \in X_v$ and $y \in X_w$. Clearly $G$ has an $H$-minor if and only if $G$ has an $H$-model. We
choose to work with $H$-models rather than $H$-minors.

Theorem 1 (Alon et al. [1]) There is an algorithm with running time $O((\ell \cdot |G|)^{1/2} \cdot (|G| + \|G\|))$ that,
given $\ell \in \mathbb{Z}^+$ and a weighted graph $(G, w)$, either outputs:
(a) a $K_\ell$-model of $G$, or
(b) a $2^\ell$-separation of $(G, w)$ of order at most $\ell^{3/2} \cdot |G|^{1/2}$.

Suppose that $\ell$ is fixed. It follows from a result of Mader [6] (see Theorem 3) that Theorem 1 can be
implemented in $O(|G|^{1/2} + \|G\|)$ time. Alon et al. [1] left as an open problem whether linear time is
possible. The main result of this paper is the following partial answer to this question. We obtain a linear
running time at the expense of a slightly larger separator (and a larger dependence on $\ell$). Moreover, our
algorithm exhibits a tradeoff between running time (ranging from $O(n)$ to $O(n^{3/2})$) and the order of the
separator (ranging from $O(n^{2/3})$ to $O(n^{1/2})$).

Theorem 2 There is an algorithm with running time $O(2^{(3\ell^2 + 7\ell - 3)/2} \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|)$ that, given
$\epsilon \in [0, \frac{1}{2}]$, $\ell \in \mathbb{Z}^+$, and a weighted graph $(G, w)$, either outputs:
(a) a $K_\ell$-model of $G$, or
(b) a $2^\ell$-separation of $(G, w)$ of order at most $2^{(\ell^2 + 3\ell + 1)/2} \cdot |G|^{(2-\epsilon)/3}$.

Note that for applications to divide-and-conquer algorithms a separation of order $O(|G|^{1-\epsilon})$, for some
constant $\epsilon > 0$, is all that is needed.

The idea behind the proof of Theorem 2 is simple. We now outline the proof for fixed $\ell$ and with $\epsilon = 0$.
Suppose that in $O(|G| + \|G\|)$ time, we can find a partition of $V(G)$ into $|G|^{2/3}$ connected subgraphs
$\{S_1, S_2, \ldots, S_{|G|^{2/3}}\}$, each containing $O(|G|^{1/3})$ vertices. Let $H$ be the weighted graph obtained from
$G$ by contracting each $S_i$ to a vertex $v_i$ with weight $w(v_i) = w(S_i)$. Then apply Theorem 1 to $H$ to
either obtain a $K_\ell$-model in $H$ which defines a $K_\ell$-model in $G$, or a $2^\ell$-separation $\{A, B\}$ of $H$ with order
$O(|H|^{1/2}) = O(|G|^{1/3})$, in which case $\{\bigcup\{S_i : v_i \in A\}, \bigcup\{S_i : v_i \in B\}\}$ is a $2^\ell$-separation of $G$
with order $O(|G|^{2/3})$. The running time is $O(|H|^{1/2} + \|H\|) \leq O(|G| + \|G\|)$. The proof of Theorem 2
is actually a little different from this outline. In particular, the subgraphs $S_i$ will not necessarily be
connected, but we will still be able to convert the output from Theorem 1 applied to $H$ to the desired
output for $G$. By relaxing the connectivity condition, we are able to prove that an appropriate partition
exists.

We will use the following notation for a graph $G$. For $x \in V(G)$, let $N(x) := \{y \in V(G) : xy \in E(G)\}$. For a subgraph $X$ of $G$, let $N(X) := \bigcup\{N(x) \setminus X : x \in X\}$. Where there is no confusion, a set
of vertices $S \subseteq V(G)$ will also refer to the subgraph of $G$ induced by $S$.

2 Mader’s Theorem

This section contains a number of easily proved results—see the full version of the paper for details. We
start with an algorithmic version of a theorem of Mader [6] (cf. [8]).
Given a graph $G$ with $|G| \geq 2^{2k-3} \cdot |G|$ (for some $k \in \mathbb{Z}^+$), a $K_\ell$-model of $G$ can be computed in $O((|G|+|G|^2))$ time.

Note that if we ignore the running time, Theorem 3 is far from best possible. Kostochka [4] and Thomason [10] independently proved that if $|G| \in \Omega(\sqrt{\ell \log \ell} \cdot |G|)$ then $G$ has a $K_\ell$-model. Theorem 3 implies the following slightly faster version of Theorem 1 (for fixed $\ell$).

**Theorem 3** There is an algorithm with running time $O(2^{2\ell} \cdot |G|^3/2 + \ell \cdot |G|)$ that, given $\ell \in \mathbb{Z}^+$ and a weighted graph $(G, w)$, either outputs:

(a) a $K_\ell$-model of $G$, or
(b) a $\frac{3}{2}$-separation of $(G, w)$ of order at most $\ell^{3/2} \cdot |G|^{1/2}$.

A $k$-clique of $G$ is a (not necessarily maximal) set of $k$ pairwise adjacent vertices of $G$. If every subgraph of $G$ has a vertex of degree at most $d$, then $G$ is $d$-degenerate. For example, Theorem 3 implies that a graph with no $K_\ell$-minor is $2^{2\ell-2}$-degenerate. It is easily proved that a $d$-degenerate graph $G$ with no $k$-clique has less than $d^{k-1} \cdot |G|$ cliques. Hence a graph $G$ with no $K_\ell$-minor has less than $2^{(\ell-2)(k-1)} \cdot |G|$ cliques.

For an algorithm, we have the following result.

**Lemma 1** Given a graph $G$ with no $k$-clique and at least $2^{(\ell-2)(k-1)} \cdot |G|$ cliques (for some $\ell \in \mathbb{Z}^+$), a $K_\ell$-minor of $G$ can be computed in $O(\ell(|G|+|G|^2))$ time.

## 3 Proof of Theorem 2

Let $G$ be a graph. Let $A$ be a partition of $V(G)$. Let $H$ be the graph obtained from $G$ by collapsing each part $S \in A$ to a single vertex $v$, and replacing parallel edges by a single edge. Denote $H_v := S$. We say $\{H_v : v \in V(H)\}$ is an $H$-partition of $G$. Furthermore, $\{H_v : v \in V(H)\}$ is a connected $H$-partition of $G$ if $uv \in E(H)$ if and only if there is an edge of $G$ between every component of $H_v$ and every component of $H_w$. We prove the following lemma.

**Lemma 2** There is an algorithm with running time $O(2^{2\ell} \cdot |G| + |G|^2)$ that, given $\ell, k \in \mathbb{Z}^+$ and a graph $G$, outputs a connected $H$-partition of $G$ such that either:

(a) $H$ has a $K_\ell$-model (which is also output), or
(b) $|H| \leq 2^{\ell^2 + \ell - 1} \cdot |G| \cdot k^{-1}$, and $|H_x| < 2k$ for all $x \in V(H)$.

**Proof of Theorem 2 assuming Lemma 2:** Apply Lemma 2 with $k = \lfloor (G^{(1+\frac{1}{3})}/3 \rfloor$. First suppose that Lemma 2 outputs a $K_\ell$-model $\{S_1, S_2, \ldots, S_t\}$ of $H$. Thus each $S_i$ is a connected subgraph of $H$. Choose a connected component $Z_v$ of $H_v$ for each $v \in V(H)$. Let $T_i := \bigcup\{Z_v : v \in S_i\}$. Then $\{T_1, T_2, \ldots, T_t\}$ is a $K_\ell$-model of $G$.

Otherwise $|H| \leq 2^{\ell^2 + \ell - 1} \cdot |G|^{2(1+\ell)/3}$, and $|H_x| < 2|G|^{(1-2\ell)/3}$ for all $x \in V(H)$. Let $w(v) := w(H_v)$ for all $v \in V(H)$. Apply Theorem 4 to $(H, w)$. The running time is

$O(2^{2\ell} \cdot |H|^{3/2} + \ell \cdot |H|) \leq O(2^{2\ell} \cdot |G|^{2(1+\ell)/3} \cdot |G|^{3/2} + \ell \cdot |G|) \leq O(2^{(3\ell^2 + 7\ell - 3)/2} \cdot |G|^{1+\ell + \ell \cdot |G|})$.

We either obtain a $K_\ell$-model of $H$, or a $\frac{3}{2}$-separation of $H$ with order at most $\ell^{3/2} \cdot |H|^{1/2}$. In the first case, $G$ has a $K_\ell$-model as proved above.

Now suppose that we obtain a $\frac{3}{2}$-separation $\{A, B\}$ of $(H, w)$ with order $|A \cap B| \leq \ell^{3/2} \cdot |H|^{1/2} \leq \ell^{3/2} \cdot (2^{2\ell + \ell - 1} \cdot |G|^{2(1+\ell)/3})^{1/2} \leq 2^{(2\ell^2 + 3\ell - 1)/2} \cdot |G|^{(1+\ell)/3}$. 

Let $X := \bigcup \{H_v : v \in A\}$ and $Y := \bigcup \{H_v : v \in B\}$. Then $\{X, Y\}$ is a separation of $G$ with order $|X \cap Y| = |\bigcup \{H_v : v \in A \cap B\}| \leq 2^{(\ell^2 + 3\ell - 1)/2} \cdot |G|^{(1+\epsilon)/3} \cdot 2 \cdot |G|^{(1-2\epsilon)/3} \leq 2^{(\ell^2 + 3\ell + 1)/2} \cdot |G|^{(2-\epsilon)/3}.$

We have $w(X \setminus Y) = w(A \setminus B) \leq \frac{3}{2} w(H) = \frac{3}{2} w(G)$. Similarly $w(B \setminus A) \leq \frac{3}{2} w(G)$.

**Proof of Lemma 2:**

**Step 1:** Using a breadth-first search algorithm, compute a maximal set $A$ of connected subgraphs of $G$ such that $|S| = k$ for all $S \subseteq A$. Let $B$ be the set of connected components of $G \setminus \bigcup \{S \in A\}$. Then $A \cup B$ is a partition of $V(G)$, and there is no edge of $G$ between distinct $T_1, T_2 \in B$. Note that $|T| < k$ for all $T \in B$, as otherwise $T$ would contain a connected subgraph $X$ with $|X| = k$, which could be added to $A$.

**Step 2:** Let $H$ be the graph obtained from $G$ by contracting each set $S \in A \cup B$ into a single vertex $V(S)$ with $H_v := V$, and replacing parallel edges by a single edge. Since each $S \in A \cup B$ is connected in $G$, $\{H_v : v \in V(H)\}$ is a connected $H$-partition of $G$. Let $A := \{v \in V(H) : H_v \in A\}$ and $B := \{v \in V(H) : H_v \in B\}$. A vertex $v$ of $H$ is big if $|H_v| \geq k$. A vertex $v$ of $H$ is small if $|H_v| < k$. Observe that every vertex of $A$ is big, $B$ is an independent set of $H$, and every vertex in $B$ is small. Partition $B = C \cup D \cup E$, where $C := \{v \in B : \deg_H(v) \geq 2^{\ell - 2}\}$, $D := \{v \in B : \ell - 1 \leq \deg_H(v) < 2^{\ell - 2}\}$, and $E := \{v \in B : \deg_H(v) \leq \ell - 2\}$.

**Step 3:** Suppose that $|C| \geq |A|$. Then the subgraph $C \cup A$ of $H$ has at least $2^{\ell - 2} \cdot |C|$ vertices and at most $2|C|$ vertices. By Theorem 3, a $K_{\ell}$-model of $C \cup A$ can be computed in $O(\ell \cdot |G|)$ time. We now assume that $|C| < |A|$.

**Step 4:** For each vertex $v \in D \cup E$, if there is a pair $x, y \in A$ of distinct neighbours of $v$, such that $\{x, y\}$ has not been assigned any vertex in $D \cup E$, then assign $v$ to $\{x, y\}$. This step can be implemented in $O(2^{2\ell} \cdot |G|)$ time, since each vertex in $D \cup E$ has degree at most $2^{\ell}$.

Suppose that there is a vertex $v \in D$ that is not assigned. Let the neighbourhood of $v$ be $\{x_1, x_2, \ldots, x_d\}$. Then $d \geq \ell - 1$. Thus for all $1 \leq i < j \leq d$, there is a distinct vertex $v_{i,j}$ assigned to the pair $\{x_i, x_j\}$, and $v_{i,j}$ is adjacent to both $x_i$ and $x_j$. In the graph obtained from $H$ by contracting each edge $x_iv_{i,j}$, the subgraph $\{x_1, x_2, \ldots, x_d, v\}$ is a clique on at least $\ell$ vertices. Thus $H$ has a $K_{\ell}$-model. We now assume that every vertex in $D$ is assigned.

Let $E^*$ be the set of assigned vertices in $E$. Consider the graph obtained from $A \cup D \cup E^*$ by contracting the edge $v_w$ for each $v \in D \cup E^*$ assigned to the pair $\{x, y\}$. This graph has $|A|$ vertices and at least $|D| + |E^*|$ edges. Thus if $|D| + |E^*| \geq 2^{\ell - 3} \cdot |A|$, then by Theorem 3, $H$ has a $K_{\ell}$-model that can be computed in $O(\ell \cdot |G|)$ time. We now assume that $|D| + |E^*| < 2^{\ell - 3} \cdot |A|$.

**Step 5:** Partition $E \setminus E^* = \bigcup \{P_1, P_2, \ldots, P_s\}$ such that for all $u, v \in E \setminus E^*$, we have $N(u) = N(v)$ if and only if both $u, v \in P_i$ for some $1 \leq i \leq s$. For all $1 \leq i \leq s$, partition $P_i = \{P_{i,1}, P_{i,2}, \ldots, P_{i,k_i}\}$ such that for all $1 \leq j \leq t_i - 1, k \leq |\{H_u : v \in P_{i,j}\}| < 2k$, and $|\{H_u : v \in P_{i,t_i}\}| < k$. This is possible since $|H_u| < k$ for all $v \in P_i$. Collapse each set $P_{i,j}$ into a single vertex $p_{i,j}$ in $H$, whose associated subgraph in $G$ is $H_{p_{i,j}} := \bigcup \{H_u : v \in P_{i,j}\}$. Since the vertices in $P_{i,j}$ have the same neighbourhood, $\{H_u : v \in V(H)\}$ remains a connected partition of $G$. Let $E_{\text{big}} = \{p_{i,j} : 1 \leq i \leq s, 1 \leq j \leq t_i - 1\}$ and $E_{\text{small}} = \{p_{i,t_i} : 1 \leq i \leq s\}$. Then every vertex in $E_{\text{big}}$ is big and every vertex in $E_{\text{small}}$ is small.
Suppose that $|E_{\text{small}}| \geq 2^{|E|} \cdot |A|$. Let $X$ be the graph obtained from $A$ by adding a clique on $N(v)$ for each vertex $v \in E_{\text{small}}$. Since distinct vertices in $E_{\text{small}}$ have distinct neighbourhoods, this process adds at least $|E_{\text{small}}| \geq 2^{|E|} \cdot |A|$ cliques. Thus by Lemma 1, a $K_{\ell}$-model of $X$ can be computed in $O(|(G)|)$ time. For every edge $x_i, x_j$ in this $K_{\ell}$-model that is in $X$ but not in $A$, we have $x_i, x_j \in N(v)$ for some $v \in E_{\text{small}}$. Since $v$ is not assigned, there is a vertex $u \in D \cup E^*$ assigned to $\{x_i, x_j\}$, and $u$ is adjacent to both $x_i$ and $x_j$. Since $u$ is not in the $K_{\ell}$-model, we can include $u$ in the connected subgraph of the $K_{\ell}$-model that contains $x_i$ or $x_j$, and we obtain a $K_{\ell}$-model in $A \cup D \cup E^*$ (in particular, without the edge $x_i, x_j$). Now assume that $|E_{\text{small}}| < 2^{|E|} \cdot |A|$. 

Step 6: We have now partitioned $V(H)$ into sets $A \cup E_{\text{big}}$ of big vertices, and sets $C \cup D \cup E^* \cup E_{\text{small}}$ of small vertices. We have proved that $|C| < |A|$, $|D| + |E^*| < 2^{|E|} \cdot |A|$, and $|E_{\text{small}}| < 2^{|E|} \cdot |A|$. Thus the number of small vertices is less than $(1 + 2^{\ell - 3} + 2^{\ell} + 1) \cdot |A| \leq 2^{|E| + \ell - 2} \cdot |A|$. By definition, the number of big vertices in $H$ is at most $|G| \cdot k^{-1}$. In particular, $|A| \leq |G| \cdot k^{-1}$. Thus $|H| \leq 2^{|E| + \ell - 1} \cdot |G| \cdot k^{-1}$. Moreover, every $|H_u| < 2k$ for every vertex $v \in V(H)$. 

References


