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Infinite families of accelerated series for some classical constants by the Markov-WZ Method

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In this article we show the Markov-WZ Method in action as it finds rapidly converging series representations for a given hypergeometric series. We demonstrate the method by finding new representations for \( \log(2) \), \( \zeta(2) \) and \( \zeta(3) \).

Keywords: WZ theory, series convergence, hypergeometric series.

A function \( H(x,z) \), in the integer variables \( x \) and \( z \), is called hypergeometric if \( H(x+1,z)/H(x,z) \) and \( H(x,z+1)/H(x,z) \) are rational functions of \( x \) and \( z \). In this article we consider only those hypergeometric functions which are a ratio of products of factorials (we call such hypergeometric functions pure-hypergeometric). A P-recursive function is a function that satisfies a linear recurrence relation with polynomial coefficients. A pair \((H,G)\) is called a Markov-WZ pair (MWZ-pair for short) if there exists a polynomial \( P(x,z) \) in \( z \) of the form

\[
P(x,z) = a_0(x) + a_1(x)z + \cdots + a_L(x)z^L, \tag{POLY}
\]

for some non-negative integer \( L \), and P-recursive functions \( a_0(x), \ldots, a_L(x) \) such that

\[
H(x+1,z)P(x+1,z) - H(x,z)P(x,z) = G(x,z+1) - G(x,z). \tag{Markov-WZ}
\]

We call \( G(x,z) \) an MWZ mate of \( H(x,z) \). We also require that the \( a_i(x) \)'s satisfy the initial conditions

\[
a_0(0) = 1, a_i(0) = 0, \text{ for } 1 \leq i \leq L.
\]

First we will show that given a hypergeometric function \( H(x,z) \), there always exists a polynomial with minimum degree that satisfies \( (Markov-WZ) \).

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1 Existence of MWZ-pair

In this section, $\deg(a)$ stands for the degree of $a$ as a polynomial in $z$.

**Theorem 1.** Given a hypergeometric term $H(x, z)$, there exist a non-negative integer $L$ and a polynomial $P(x, z)$ of the form $[\text{POLY}^{\infty}]$ associated with $H(x, z)$ such that $H(x, z)$ has an MWZ mate.

**Proof.** We need to show that there exist $L \geq 0$, $a_i(x)$’s, $G(x, z)$, and $P(x, z)$ of the form $[\text{POLY}]$ such that $H(x, z)P(x, z)$ and $G(x, z)$ satisfy (Markow-WZ). Moreover $G(x, z)$ has the form $G(x, z) = R(x, z)F(x, z)$, where $R(x, z)$ is a ratio of two P-recursive functions in $(x, z)$.

Write

$$H(x+1, z)P(x+1, z) - H(x, z)P(x, z) = \text{POL}(z) \cdot \overline{P}(x, z),$$

where

$$\text{POL}(z) := A(z) \sum_{i=0}^{L} a_i(x + 1)z^i - B(z) \sum_{i=0}^{L} a_i(x)z^i,$$

$$\frac{H(x+1, z)}{H(x, z)} = \frac{A(z)}{B(z)}, \text{ and } \overline{P}(x, z) = \frac{H(x, z)}{B(z)}.$$  

Since $\overline{P}(x, z)$ is a hypergeometric function divided by a polynomial, we can write the above expression as

$$H(x+1, z)P(x+1, z) - H(x, z)P(x, z) = \frac{a(z)}{b(z)} \cdot \frac{\text{POL}(z+1)}{\text{POL}(z)},$$

where

$$\frac{\overline{P}(x+1, z)}{\overline{P}(x, z)} = \frac{a(z)}{b(z)}.$$  

Without loss of generality, we may assume that $\gcd(a(z), b(z + h)) = 1$ for $h \geq 0$, otherwise we regroup and incorporate additional factors into the polynomial part, $\text{POL}(z)$. Then with $a(z), b(z)$ and $c(z) := \text{POL}(z)$ in parametric Gosper’s algorithm [MZ], look for a polynomial $X(z)$ that satisfies

$$a(z)X(z + 1) - b(z - 1)X(z) = c(z). \quad \text{(Gosper)}$$

We may consider only those $X$ with

$$\deg(X) = \deg(c) - \max\{\deg(a), \deg(b)\},$$

and the degree of $c(z)$ is easily seen to be

$$\deg(c) = L + \max\{\deg(A), \deg(B)\}.$$  

The unknowns are the $\deg(c) - \max\{\deg(a), \deg(b)\} + 1$ coefficients of $X(z)$ and the $a_i$’s (there are a total of $2(L + 1)$ unknowns). Comparing coefficients on both sides of (Gosper) gives $\deg(c) + 1$ linear homogeneous equations. In order to guarantee a non-zero solution, we need

$$\# \text{ of unknowns} - \# \text{ of equations} \geq 1,$$
and this holds if
\[ 2(L + 1) - (\deg(c) + 1) \geq 1. \]
In particular, if we choose
\[ L := \max\{\deg(a), \deg(b)\}, \]
we are guaranteed to get a non-trivial solution(). This gives the \( P(x, z) \) and the \( L \). \( G(x, z) \) is the anti-difference outputted by parametric Gosper [MZ].

**Theorem 2.** Let \((H, G)\) be an MWZ-pair.

(a) If \( \lim_{j \to \infty} G(x, j) = 0 \ \forall x \geq 0 \), then
\[
\sum_{z=0}^{\infty} H(0, z) = \sum_{x=0}^{\infty} G(x, 0) - \lim_{i \to \infty} \sum_{z=0}^{\infty} H(i, z)P(i, z),
\]
whenever both sides converge.

(b) If \( \lim_{i \to \infty} H(i, z)P(i, z) = 0 \ \forall z \geq 0 \), then
\[
\sum_{z=0}^{\infty} H(0, z) - \lim_{j \to \infty} \sum_{x=0}^{\infty} G(x, j) = \sum_{x=0}^{\infty} G(x, 0),
\]
whenever both sides converge.

**Proof.** (a) Let \( P(x, z) \) be the polynomial that features in the MWZ-pair \((H(x, z), G(x, z))\) arising from \( H(x, z) \).

Then apply theorem 7 [Z] to the 1-form
\[
w = H(x, z)P(x, z)\delta z + G(x, z)\delta x, \tag{1}
\]
and the region
\[
\Omega = \{(x, z) \mid 0 \leq z \leq \infty, 0 \leq x \leq i\},
\]
with the discrete boundary
\[
\{(0, z+1) \to (0, z) \mid z \geq 0\} \cup \{(x, 0) \to (x+1, 0) \mid 0 \leq x \leq i\} \cup \{(i, z) \to (i, z+1) \mid z \geq \infty\} \cup \{(x+1, \infty) \to (x, \infty) \mid i - 1 \leq x \leq 0\},
\]
and use the initial conditions \( a_i(0) = \delta_{i0} \) for \( 0 \leq i \leq L \).

(b) Replace the region in (a) by
\[
\Omega = \{(x, z) \mid 0 \leq x \leq \infty, 0 \leq z \leq j\}
\]
with the corresponding discrete boundary in the proof of (a), and apply to (1) together with the initial conditions \( a_i(0) = \delta_{i0} \) for \( 0 \leq i \leq L \).
Corollary 1. If the limit in the conclusion of (a) or (b) is zero in addition to the given hypothesis, then
\[ \sum_{z=0}^{\infty} H(0,z) = \sum_{x=0}^{\infty} G(x,0) . \]

Theorem 3. Let \( N_0 \) be a non-negative integer and \((H, G)\) be an MWZ-pair. Then
\[ \sum_{z=0}^{\infty} H(0,z) = \sum_{x=0}^{\infty} (H(N_0 + x,x)P(N_0 + x,x) + G(N_0 + x,x + 1)) + \sum_{x=0}^{N_0-1} G(x,0) - \lim_{j \to \infty} \sum_{x=0}^{\infty} G(x,j) , \]
whenever both sides converge.

Proof. Let \( P(x,z) \) be the polynomial that features in the MWZ-pair \((H(x,z), G(x,z))\) arising from \( H(x,z) \). Then the proof follows from theorem 7 \([Z]\) by applying to the 1-form
\[ w = H(x,z)P(x,z)\delta z + G(x,z)\delta x , \]
and the region
\[ \Omega = \{(x,z) \mid 0 \leq z \leq \infty, 0 \leq x \leq z + N_0 \} , \]
with the discrete boundary
\[ \partial \Omega_{N_0} := \{(0,0) \to (0, z+1) \mid z \geq 0\} \cup \{(x,0) \to (x+1,0) \mid 0 \leq x \leq N_0\} \cup \{(N_0 + x,x) \to (N_0 + x + 1,x+1) \mid x \geq 0\} \cup \{(x+1,\infty) \to (x,\infty) \mid x \geq 0\} , \]
and using the initial conditions \( a_i(0) = \delta_{i0} \) for \( 0 \leq i \leq L \).

Corollary 2. Let \((H, G)\) be an MWZ-pair. If \( \lim_{j \to -\infty} \sum_{x=0}^{\infty} G(x,j) = 0 \), then
\[ \sum_{z=0}^{\infty} H(0,z) = \sum_{x=0}^{\infty} (H(x,x)P(x,x) + G(x,x + 1)) . \]

Proof. Set \( N_0 = 0 \) in theorem 3 and use the initial conditions \( a_i(0) = \delta_{i0} \) for \( 0 \leq i \leq L \).

Remark. If \( \lim_{j \to -\infty} G(x,j) = 0 \forall x \) and the hypothesis of theorem 1 (a) holds, then
\[ \sum_{z=-\infty}^{\infty} H(x,z)P(x,z) , \]
has a closed form evaluation (see example 10 below).

In the following examples, we use the Maple package MarkovWZ \([MZ]\) which, for a given \( H(x,z) \), outputs the polynomial \( P(x,z) \) and the \( G(x,z) \).
2 Examples of Accelerating Series

Let $H(a, b) := \frac{(ax + z)!}{(bx + z + 1)!}$ in examples 1 through 9.

**Example 1.** Consider the hypergeometric term $(-1)^z H(0, 1)$, and corresponding to this kernel determine a polynomial $P(x, z)$ in $z$ with a minimum degree such that $((-1)^z H(0, 1), G(x, z))$ is an MWZ-pair. Using the maple package MarkovWZ, we see that the polynomial is

$$P(x, z) = \frac{x!}{2x},$$

and the corresponding MWZ mate of $(-1)^z H(0, 1)$ is

$$G(x, z) = \frac{(-1)^x x!}{2x+1} H(0, 1).$$

It is not hard to check that $((-1)^z H(0, 1), G(x, z))$ is indeed a MWZ-pair with the corresponding polynomial $P(x, z) = x!/2x$.

Applying corollary 2 to the MWZ-pair we get,

$$\log(2) = \frac{3}{2} \sum_{x=0}^{\infty} \frac{(-1)^x x!(x+1)!}{(2x+2)!2^x} = 2\arcsinh\left(\frac{\sqrt{2}}{4}\right).$$

Similarly, if we apply corollary 1 to the MWZ-pair, we find

$$\log(2) = \frac{1}{2} \sum_{x=0}^{\infty} \frac{1}{2^x(x+1)}.$$

In the remaining examples, we simply give the hypergeometric term $H(x, z)$, the polynomial $P(x, z)$ that features in the MWZ-pair, the corresponding $G(x, z)$, and then the identities that follow from the application of the corollaries above.

**Example 2.** Starting with the kernel $(-1)^z H(0, 3)$, we find

$$P(x, z) = \frac{(3x)!}{8x},$$

and

$$G(x, z) = \frac{32 + 63x^2 + 93x + 22z + 30xz + 4z^2}{8(3x + z + 2)(3x + z + 3)} P(x, z) (-1)^z H(0, 3).$$

Application of corollary 1 gives

$$\log(2) = \frac{1}{8} \sum_{x=0}^{\infty} \frac{(-1)^x (x+1)! (3x)! (415x^2 + 487x + 134)}{(4x + 4)! 8^x},$$

In examples 1 through 9.
On the other hand if we apply corollary 2, we get
\[
\log(2) = \sum_{x=0}^{\infty} \frac{(63x^2 + 93x + 32)}{24(3x + 2)(x + 1)(3x + 1)8^x}.
\]

Example 3. By taking the kernel \((-1)^2H(0, 6)\), we find
\[
P(x, z) = \frac{(6x)!}{26x^4},
\]
and
\[
G(x, z) = \frac{Q(x, z)P(x, z)}{16(6x + z + 2)(6x + z + 3)(6x + z + 4)(6x + z + 5)(6x + z + 6)} (-1)^2H(0, 6),
\]
where \(Q(x, z)\) is a certain polynomial in \(x\) and \(z\).

Corollary 2 gives
\[
\log(2) = \sum_{x=0}^{\infty} \frac{(-1)^x(6x)!}{(7x + 6)!4^x},
\]
where
\[
P(x) := 1648544x^5 + 4584284x^4 + 4905938x^3 + 2511703x^2 + 610829x + 55914,
\]
and corollary 1 gives
\[
\log(2) = \sum_{x=0}^{\infty} \frac{40824x^5 + 129924x^4 + 158814x^3 + 92655x^2 + 25605x + 2654}{384(6x + 1)(3x + 1)(5x + 2)(x + 1)64^x}.
\]

Example 4. Starting with \(H(0, 2)^2\), we find that
\[
P(x, z) = \frac{\sqrt{\pi}(2x)!^3}{16\Gamma(2x + 1/2)}
\]
and
\[
G(x, z) = \frac{Q(x, z)}{2((1 + 4x)(3 + 4x)(2x + z + 2)^2)} P(x, z) H(0, 2)^2,
\]
where
\[
Q(x, z) := 120x^4 + 372x^3 + 136x^2 z + 56x^2 z^2 + 426x^2 + 316x z
\]
\[+ 242xz + 86xz^2 + 8x^2 z^2 + 213x + 39 + 33z^2 + 6z^2 + 61z.
\]

Application of corollary 2 gives
\[
\zeta(2) = \frac{\sqrt{\pi}}{8} \sum_{x=0}^{\infty} \frac{(2912x^4 + 7100x^3 + 6381x^2 + 2494x + 355)((x + 1)!2((2x)!)^3}{\Gamma(2x + 5/2)((3x + 3)!)^216^x}.
\]

On the other hand, corollary 1 yields
\[
\zeta(2) = \frac{3\sqrt{\pi}}{32} \sum_{x=0}^{\infty} \frac{(20x^6 + 32x^4 + 13)(2x)!}{(2x + 1)(x + 1)!16^x}.
\]
Example 5. By taking the kernel $H(1, 2)^2$, we get

$$P(x, z) = \frac{(x!)^3 \sqrt{\pi}}{4^x \Gamma(x + 1/2)},$$

$$G(x, z) = \frac{21x^3 + 55x^2 + 47x + 13 + 28x^2z + 48xz + 20z + 13xz^2 + 11z^2 + 2x^3}{2(2x + 1)(2x + z + 2)^2} F(x, z),$$

where $F(x, z) := P(x, z)H(1, 2)^2$.

If we apply corollary 2 we get

$$\zeta(2) = \frac{1}{9\sqrt{\pi}} \sum_{x=0}^{\infty} \frac{(145x^2 + 186x + 59)(x!)^5 \Gamma(x + 1/2)4^x}{((3x + 2)!)^2}.$$  

On the other hand, corollary 1 yields

$$\zeta(2) = \frac{\pi}{64} \sum_{x=0}^{\infty} \frac{(21x + 13)x!^3}{(64x^3(\Gamma(x + 3/2))^3}.$$  

Example 6. Corresponding to $H(1, 3)^2$, we find that

$$P(x, z) = \frac{\sqrt{\pi}(2x)!^3}{16^x \Gamma(2x + 1/2)},$$

and

$$G(x, z) = \frac{Q(x, z)}{2(3 + 4x)(1 + 4x)(3x + z + 2)^2(3x + z + 3)^2} P(x, z)(-1)^2 H(1, 3)^2,$$

where $Q(x, z)$ is a polynomial in $x$ and $z$. Application of corollary 2 gives

$$\zeta(2) = \frac{\pi^{3/2}}{2048} \sum_{x=0}^{\infty} \frac{((2x)!^3(10920x^4 + 27908x^3 + 25962x^2 + 10275x + 1421)}{(\Gamma(2x + 5/2)))^3(4096)^x},$$

and corollary 1 gives

$$\zeta(2) = \frac{\sqrt{\pi}}{72} \sum_{x=0}^{\infty} \frac{P(x)(x!)^2((2x!)^2}{16^x \Gamma(2x + 5/2)((3x + 2)!^2},$$

where

$$P(x) := 2912x^4 + 7100x^3 + 6381x^2 + 2494x + 355.$$  

Example 7. Corresponding to $H(1, 5)^2$, we find that

$$P(x, z) = \frac{\sqrt{2\pi}(4x)!^3}{4(256)^x \sin(1/8\pi) \sin(3/8\pi) \Gamma(4x + 1/2)},$$
and a corresponding MWZ mate \( G(x, z) \). If we apply corollary 2 we find

\[
\zeta(2) = \frac{\sqrt{2\pi}}{3200 \sin(3/8\pi) \sin(1/8\pi)} \sum_{x=0}^{\infty} \frac{P(x)((4x)!)^3(x!)^2}{(256)^x \Gamma((4x + 9/2)(5x + 4)!)^2},
\]

where

\[
P(x) := 3333245952x^{10} + 18842142336x^9 + 47204597136x^8 + 68964524342x^7 + 65011852179x^6 + 41280848445x^5 + 17862102186x^4 + 519433183x^3 + 970166319x^2 + 104901994x + 4974228.
\]

The terms of this series are \( O((\frac{256}{9765625})^j) \approx O(10^{-5j}) \).

**Example 8.** Similarly for the kernel \( H(0, 2)^3 \), we get

\[
P(x, z) = \frac{(-1)^x(x!(2x)!)^3}{(3x)!}
\]

and

\[
G(x, z) = \frac{Q(x, z)}{6(3x + 1)(3x + 2)(2x + z + 2)^3} H(0, 2)^3 P(x, z),
\]

where \( Q(x, z) \) is a certain polynomial in \( x \) and \( z \).

By using corollary 2, we get

\[
\zeta(3) = \sum_{x=0}^{\infty} \frac{(-1)^x(2x)!(2x+1)!6P(x)}{2(x+1)^2((3x+3)!)^4},
\]

where

\[
P(x) := 40885x^5 + 124346x^4 + 150160x^3 + 89888x^2 + 26629x + 3116,
\]

and application of corollary 1 gives

\[
\zeta(3) = \sum_{x=0}^{\infty} \frac{(-1)^x(56x^2 + 80x + 29)(x!)^3}{4(2x+1)^2((3x+3)!)}.
\]

**Example 9.** Starting with the kernel \( H(1, 3)^3 \), we get

\[
P(x, z) = \frac{(-1)^x(x!(2x)!)^3}{(3x)!}
\]

and

\[
G(x, z) = \frac{Q(x, z)}{6(3x + 2)(3x + 1)(3x + z + 2)^3(3x + z + 3)^3} P(x, z) H(1, 3)^3,
\]

where

\[
Q(x, z) := 448x^5 + 624zx^4 + 1760x^4 + 1932zx^3 + 2728x^3 + 348z^2x^3 + 2214x^2z^2 + 2084x^2 + 792z^2x^2 + 90z^3x^2 + 594zx^2 + 1113x + 9z^4x + 132z^3x + 784x + 6z^4 + 207z + 48z^3 + 147z^2 + 116.
\]
In this example, we show all the steps to demonstrate the application of theorem 2. Let
\[ F(x, z) := H(x, z)P(x, z). \]

Define \( M(n) \), for \( n = 0, 1, 2, 3, 4, \ldots \), by
\[ M(n) := \sum_{x=0}^{n-1} G(x, 0) + \sum_{x=0}^{\infty} (F(x+n, x) + G(x+n, x+1)). \]

Then theorem 2 says that \( \zeta(3) = M(n) \), \( \forall n = 0, 1, 2, 3, 4, \ldots \).

In particular
\[ \zeta(3) = M(0) = \frac{1}{24} \sum_{x=0}^{\infty} \frac{(x!)^3 (2x)!^6 (-1)^x P(x)}{(3x+2)!((4x+3)!)^3}, \tag{2} \]

where
\[ P(x) := 126392x^5 + 412708x^4 + 531578x^3 + 336367x^2 + 104000x + 12463. \]

On the other hand, application of corollary 1 gives
\[ \zeta(3) = \frac{1}{162} \sum_{x=0}^{\infty} \frac{P(x)(x!)^6 ((2x)!)^3 (-1)^x}{((3x+2)!)^4}, \]

where
\[ P(x) := 40885x^5 + 124346x^4 + 150160x^3 + 89888x^2 + 26629x + 3116. \]

The series (2) was first derived in [AZ] and used by S. Wedeniwski (1999) to obtain up to 128 million correct decimal places. The terms of the series in (2) are \( O\left((\frac{64}{531441})^j\right) \approx O(10^{-5j}) \), while the terms of the second series are \( O\left((\frac{110592}{162})^j\right) \approx O(10^{-4j}) \).

Instead, if we take \( H(1,5)^3 \), we get
\[ P(x, z) = \frac{2\sqrt{3}}{3\sqrt{\pi}} \frac{(2x-1/2)^3 (2x)^5 (4096)^x}{(729)^x \Gamma(2x+2/3) \Gamma(2x+1/3)}, \]

and a corresponding \( G(x, z) \). Let
\[ F(x, z) = H(1,5)^3 P(x, z), \]

and let \( M(n) \) be as above.

Then theorem 2 gives \( \zeta(3) = M(n) \), \( \forall n = 0, 1, 2, 3, 4, \ldots \) and in particular
\[ \zeta(3) = M(0) = \frac{16}{81} \sum_{x=0}^{\infty} \frac{P(x)(4096)^x ((4x)!)^3 ((2x)!)^2 ((2x+1)!)^4 (-1)^x}{((6x+5)!)^4}, \tag{3} \]
where
\[ P(x) := 5561689253120x^{13} + 41827852352256x^{12} + 143295193251200x^{11} + 29584298326608x^{10} 
+ 4103245488192x^9 + 403368918753744x^8 + 288879369092920x^7 + 152460289970616x^6 
+ 59240414929957x^5 + 16722886152858x^4 + 3330604771504x^3 
+ 442815051024x^2 + 35195802021x + 1261871244. \]

The terms of this series are \( O\left(\frac{4096}{282429536481}\right)^j \approx O\left(10^{-8j}\right) \).

This improves the previous record \( \text{(2)} \).

**Example 10.** If we start with
\[ H(x, z) = \frac{(x+a)(x+b)}{(a+z)(b+z)}, \]
we get
\[ P(x, z) = \frac{(a+b)(x+a)(x+b)!}{(a+2x+b)!a!b!}, \]
and
\[ G(x, z) = \frac{(3x^2 + 2ax + 2xb + 6x - 2xz + 2b + 2a - 2z + 3 - za + ab - zb)(a+z)(b+z)}{(a+2x+1+b)(2x+b+2+a)(x+1-z)^2} H(x, z)P(x, z). \]

One can easily check that \( G(x, \pm \infty) = 0. \)

Hence, we get
\[ \sum_{z=-\infty}^{\infty} \frac{(x+a)}{(a+z)} \frac{(x+b)}{b+z} = \frac{(a+2x+b)!a!b!}{(a+b)!(x+a)!(x+b)!}. \]

This is a derivation of the classical Chu-Vandermonde summation formula, in the framework of the MWZ-method. The Markov-WZ method can sometimes lead to a discovery of new identities with appropriate \( H(x, z). \)

**Example 11.** Let
\[ H_s(x, z) := \left(\frac{(-1)^s(m)_z}{(m+8)_{x+z}}\right)^s. \]

In this example we will show how to use implementations of some numerical methods together with the Markov-WZ Method to give new WZ-pairs. The steps are:

(a) Take the output from Markov in MarkovWZ (see \( \text{[MZ]} \) ), which is a system of first order linear recurrence relation(s) for the unknown coefficient functions \( a_i(x)'s. \)

(b) Crank out some terms for the unknown coefficients, i.e. use the recurrence equation outputted by the program and find the first few terms.
(c) Use the Salvy-Zimmermann gfun program in the Algolib library available from algo.inria.fr, or findrec in EKHAD\footnote{download-able free from: http://www.math.rutgers.edu/~zeilberg/} to find a recurrence equation satisfied by the coefficient functions.

(d) Finally, solve the recurrence relations to find a closed form for the coefficients (if there exists one) (for example, in Maple, use rsolve).

11.1 Starting with $H_2(x,z)$, we find that $L = 0$ and
\[ P(x,z) := \frac{\Gamma(\delta + x)^3 \Gamma(\delta - 1/2)}{4^x \Gamma(\delta + x - 1/2) \Gamma(\delta)^3} \, . \]
Therefore we get a WZ-pair $(F,G)(notMWZ!)$, where $F(x,z) := H_2(x,z)P(x,z)$, and
\[ G(x,z) := F(x,z) \frac{(3x + 2z + 2m - 2 + 3\delta)}{2(2x + 2\delta - 1)} \, , \]
and by applying corollary\footnote{download-able free from: http://www.math.rutgers.edu/~zeilberg/} we get the identity
\[ \sum_{z=0}^{\infty} \frac{\Gamma(z + m)^2 \Gamma(m + \delta)^2}{\Gamma(m)^2 \Gamma(m + \delta + z)^2} = \frac{1}{2} \sum_{x=0}^{\infty} \frac{(3x + 2z + 2m - 2 + 3\delta)^3 \Gamma(\delta - 1/2) \Gamma(m + \delta)^2}{\Gamma(\delta + x - 1/2) \Gamma(\delta)^3 \Gamma(m + x + \delta)^2 (2x + 2\delta - 1)} \left( \frac{1}{4} \right)^x \, , \]
for $\delta = 0, 1, 2, 3, \ldots, m = 0, 1, 2, 3, \ldots$. If we specialize to $m = 1$ and $\delta = 1$, we get the formula for $\zeta(2)$, which is
\[ \zeta(2) = \frac{3\sqrt{\pi}}{4} \sum_{x=0}^{\infty} \frac{\Gamma(x + 1)}{(x + 1) \Gamma(3/2 + x)} \left( \frac{1}{4} \right)^x = \frac{3}{2} \, {}_3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{1}{4} \right) \, . \]

11.2 Starting with $H_3(x,z)$ we find that $L = 1$ and there is a vector first order recurrence relations for the polynomials $a_0(x), a_1(x)$. That means if we set
\[ a(x) := [a_0(x), a_1(x)]^T \, , \]
then there is a 2 by 2 matrix $A(x)$ such that $a(x + 1) = A(x)a(x)$, and by using findrec in EKHAD we get
\[ a_0(x) := \frac{(-1)^3 \Gamma(x + \delta)^3 (x + \delta - 1) \Gamma(\delta - 1/2)}{\Gamma(\delta)^3} \, , \quad \text{and} \quad \frac{a_1(x)}{2} := \frac{2(-1)^3 \Gamma(\delta + x)^3 \Gamma(\delta - 1/2)}{\Gamma(\delta)^3} \, . \]
Hence our polynomial is $P(x,z) = a_0(x) + a_1(x)(z + m)$, and the corresponding WZ-pair is $(H_3(x,z)P(x,z), G(x,z))$, where
\[ G(x,z) := \frac{2x + 2\delta + z + m - 1}{2z + 2m + \delta + x - 1} P(x,z) H_3(x,z) \, , \]
as outputted by zeil in EKHAD. Applying corollary 1, we get the identity
\[ \sum_{z=0}^{\infty} \frac{(-1)^3 (2z + 2m + \delta - 1) \Gamma(m + z)^3}{\Gamma(m)^3 \Gamma(m + \delta + z)^3} = \sum_{x=0}^{\infty} \frac{(-1)^3 (2x + 2\delta + m - 1) \Gamma(x + \delta)^3}{\Gamma(\delta)^3 \Gamma(m + \delta + x)^3} \, , \]
for $\delta = 0, 1, 2, 3, \ldots$, and $m = 0, 1, 2, 3, \ldots$.
11.3 Starting with $H_4(x,z)$ we find that $L = 1$ and there is a first order vector recurrence relations for the polynomials $a_0(x), a_1(x).$ Using findrec in EKHAD we get

$$a_0(x) := \frac{(-1)^x \Gamma(\delta + x)^3(\delta + x - 1)\Gamma(\delta - 1/2)}{\Gamma(\delta + x - 1/2)\Gamma(\delta)^3 4^x},$$

and

$$a_1(x) := \frac{2(-1)^x \Gamma(\delta + x)^3 \Gamma(\delta - 1/2)}{4^x \Gamma(\delta + x - 1/2)\Gamma(\delta)^3}.$$

This leads to the WZ-pair $(F(x,z)(a_0(x) + a_1(x)(m + z)), G),$ where $G$ is

$$G := \frac{5x^2 + 6mx + 10\delta x + 6m\delta + 5\delta^2 + 2m^2 + 6xz - 6x + 6\delta z - 6\delta + 4mz - 4m + 2z^2 - 4z + 2}{2(2x + 2\delta - 1)(2m + 2z + x + \delta - 1)}.$$

Application of corollary yields the identity

$$\sum_{z=0}^{\infty} \frac{\Gamma(m + z)^4(2m + 2z + \delta - 1)}{\Gamma(m + \delta + z)^4} = \frac{1}{4} \sum_{x=0}^{\infty} \frac{\Gamma(m)^4 \Gamma(x + \delta)^3 \Gamma(\delta - 1/2) P(x)}{\Gamma(x + 1/2 + \delta)\Gamma(m + \delta + x)^3 \Gamma(\delta)^3} \left(\frac{-1}{4}\right)^x,$$

that holds for $\delta = 0, 1, 2, 3, \ldots,$ and $m = 0, 1, 2, 3, 4, \ldots,$ where

$$P(x) := 5x^2 + 10x\delta + 6xm + 2m^2 + 5\delta^2 + 6\delta m + 2 - 6x - 4m - 6\delta.$$

If we specialize to $m = 1$ and $\delta = 1,$ we find the motivation for Andrei Markov’s beautiful work, namely

$$\zeta(3) = \frac{5\sqrt{\pi}}{4} \sum_{x=0}^{\infty} \frac{\Gamma(x + 1)}{(x + 1)^2 \Gamma(x + 3/2)} \left(\frac{-1}{4}\right)^x = \frac{5}{4} 4F3 \left(\frac{1, 1, 1, 1, -1}{2, 2, \frac{3}{2}}; -1\right).$$

11.4 Starting with $H_5(x,z)$ we found that $L = 3.$ The corresponding polynomial satisfies a recurrence relation of order $\geq 2,$ for which we couldn’t find an explicit closed form solution for the polynomial. Nonetheless, as described in [MZ], we have an accelerating formula for $\zeta(5)$ (see [MZ] for $5 \leq n \leq 9$).

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References


