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Infinite families of accelerated series for some classical constants by the Markov-WZ Method

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In this article we show the Markov-WZ Method in action as it finds rapidly converging series representations for a given hypergeometric series. We demonstrate the method by finding new representations for $\log(2), \zeta(2)$ and $\zeta(3)$.

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A function $H(x,z)$, in the integer variables x and z, is called **hypergeometric** if $H(x+1,z)/H(x,z)$ and $H(x,z+1)/H(x,z)$ are rational functions of x and z. In this article we consider only those hypergeometric functions which are a ratio of products of factorials (we call such hypergeometric functions **pure-hypergeometric**). A **P-recursive** function is a function that satisfies a linear recurrence relation with polynomial coefficients. A pair (H,G) is called a Markov-WZ pair (MWZ-pair for short) if there exists a polynomial $P(x,z)$ in z of the form

$$P(x,z) = a_0(x) + a_1(x)z + \cdots + a_L(x)z^L, \tag{POLY}$$

for some non-negative integer L, and P-recursive functions $a_0(x), \dots, a_L(x)$ such that

$$H(x+1,z)P(x+1,z) - H(x,z)P(x,z) = G(x,z+1) - G(x,z) \quad . \tag{Markow-WZ}$$

We call $G(x,z)$ an MWZ mate of $H(x,z)$. We also require that the $a_i(x)'s$ satisfy the initial conditions

$$a_0(0) = 1, a_i(0) = 0, \text{ for } 1 \le i \le L \, .$$

First we will show that given a hypergeometric function $H(x,z)$, there always exists a polynomial with minimum degree that satisfies (Markow-WZ) .

1 Existence of MWZ-pair

In this section, $\deg(a)$ stands for the degree of a as a polynomial in z.

Theorem 1. *Given a hypergeometric term $H(x,z)$, there exist a non-negative integer L and a polynomial $P(x,z)$ of the form* (POLY) *associated with $H(x,z)$ such that $H(x,z)$ has an MWZ mate.*

Proof. We need to show that there exist $L \geq 0$, $a_i(x)$'s, $G(x,z)$, and $P(x,z)$ of the form (POLY) such that $H(x,z)P(x,z)$ and $G(x,z)$ satisfy (Markow-WZ). Moreover $G(x,z)$ has the form $G(x,z) = R(x,z)F(x,z)$, where $R(x,z)$ is a ratio of two P-recursive functions in (x,z).

Write

$$H(x+1,z)P(x+1,z) - H(x,z)P(x,z) = POL(z) \cdot \overline{H}(x,z) \ ,$$

where

$$POL(z) := A(z)\sum_{i=0}^{L} a_i(x+1)z^i - B(z)\sum_{i=0}^{L} a_i(x)z^i \ ,$$

$$\frac{H(x+1,z)}{H(x,z)} = \frac{A(z)}{B(z)} \text{ , and } \overline{H}(x,z) = \frac{H(x,z)}{B(z)} \ .$$

Since $\overline{H}(x,z)$ is a hypergeometric function divided by a polynomial, we can write the above expression as

$$H(x+1,z)P(x+1,z) - H(x,z)P(x,z) = \frac{a(z)}{b(z)} \cdot \frac{POL(z+1)}{POL(z)} \ ,$$

where

$$\frac{\overline{H}(x+1,z)}{\overline{H}(x,z)} = \frac{a(z)}{b(z)} \ .$$

Without loss of generality, we may assume that $gcd(a(z), b(z+h)) = 1$ for $h \geq 0$, otherwise we regroup and incorporate additional factors into the polynomial part, $POL(z)$. Then with $a(z), b(z)$ and $c(z) := POL(z)$ in parametric Gosper's algorithm [MZ] , look for a polynomial $X(z)$ that satisfies

$$a(z)X(z+1) - b(z-1)X(z) = c(z) \ . \qquad \text{(Gosper)}$$

We may consider only those X with

$$\deg(X) = \deg(c) - max\{\deg(a), \deg(b)\} \ ,$$

and the degree of $c(z)$ is easily seen to be

$$\deg(c) = L + max\{\deg(A), \deg(B)\}.$$

The unknowns are the $\deg(c) - max\{\deg(a), \deg(b)\} + 1$ coefficients of $X(z)$ and the $a_i's$ (there are a total of $2(L+1)$ unknowns). Comparing coefficients on both sides of (Gosper) gives $\deg(c)+1$ linear homogeneous equations. In order to guarantee a non-zero solution, we need

$$\text{\# of unknowns} - \# \ of \text{ equations } \geq 1 \ ,$$

and this holds if

$$2(L+1)-(\deg(c)+1)\geq 1 \, .$$

In particular, if we choose

$$L := max\{\deg(a),\deg(b)\} \, ,$$

we are guaranteed to get a non-trivial solution(!). This gives the $P(x,z)$ and the L. $G(x,z)$ is the anti-difference outputted by parametric Gosper [MZ]. □

Theorem 2. *Let (H,G) be an MWZ-pair.*

(a) If $\lim_{j\to\infty} G(x,j)=0$ $\forall x\geq 0$, then

$$\sum_{z=0}^{\infty} H(0,z)=\sum_{x=0}^{\infty} G(x,0)-\lim_{i\to\infty}\sum_{z=0}^{\infty} H(i,z)P(i,z) \, ,$$

whenever both sides converge.

(b) If $\lim_{i\to\infty} H(i,z)P(i,z)=0$ $\forall z\geq 0$, then

$$\sum_{z=0}^{\infty} H(0,z)-\lim_{j\to\infty}\sum_{x=0}^{\infty} G(x,j)=\sum_{x=0}^{\infty} G(x,0) \, ,$$

whenever both sides converge.

Proof. **(a)** Let $P(x,z)$ be the polynomial that features in the MWZ-pair $(H(x,z),G(x,z))$ arising from $H(x,z)$.

Then apply theorem 7 [Z] to the 1-form

$$w=H(x,z)P(x,z)\delta z+G(x,z)\delta x \, , \tag{1}$$

and the region

$$\Omega=\{(x,z)\mid 0\leq z\leq\infty, 0\leq x\leq i\} \, ,$$

with the discrete boundary

$\{(0,z+1)\to(0,z)\mid z\geq 0\}\bigcup\{(x,0)\to(x+1,0)\mid 0\leq x\leq i\}\bigcup\{(i,z)\to(i,z+1)\mid z\geq\infty\}\bigcup\{(x+1,\infty)\to(x,\infty)\mid i-1\leq x\leq 0\}$,

and use the initial conditions $a_i(0)=\delta_{i0}$ for $0\leq i\leq L$.

(b) Replace the region in (a) by

$$\Omega=\{(x,z)\mid 0\leq x\leq\infty, 0\leq z\leq j\}$$

with the corresponding discrete boundary in the proof of (a), and apply to (1) together with the initial conditions $a_i(0)=\delta_{i0}$ for $0\leq i\leq L$.

□

Corollary 1. *If the limit in the conclusion of (a) or (b) is zero in addition to the given hypothesis, then*

$$\sum_{z=0}^{\infty} H(0,z) = \sum_{x=0}^{\infty} G(x,0) \ .$$

Theorem 3. *Let N_0 be a non-negative integer and (H,G) be an MWZ-pair. Then*

$$\sum_{z=0}^{\infty} H(0,z) = \sum_{x=0}^{\infty} (H(N_0+x,x)P(N_0+x,x)+G(N_0+x,x+1)) + \sum_{x=0}^{N_0-1} G(x,0) - \lim_{j\to\infty} \sum_{x=0}^{\infty} G(x,j) \ ,$$

whenever both sides converge.

Proof. Let $P(x,z)$ be the polynomial that features in the MWZ-pair $(H(x,z),G(x,z))$ arising from $H(x,z)$. Then the proof follows from theorem 7 [Z] by applying to the 1-form

$$w = H(x,z)P(x,z)\delta z + G(x,z)\delta x \ ,$$

and the region

$$\Omega = \{(x,z) \mid 0 \le z \le \infty, 0 \le x \le z+N_0\} \ ,$$

with the discrete boundary

$\partial\Omega_{N_0} := \{(0,z+1) \to (0,z) \mid z \ge 0\} \bigcup \{(x,0) \to (x+1,0) \mid 0 \le x \le N_0\} \bigcup \{(N_0+x,x) \to (N_0+x+1,x) \to (N_0+x+1,x+1) \mid x \ge 0\} \bigcup \{(x+1,\infty) \to (x,\infty) \mid x \ge 0\}$,

and using the initial conditions $a_i(0) = \delta_{i0}$ for $0 \le i \le L$. □

Corollary 2. *Let (H,G) be an MWZ-pair. If $\lim_{j\to\infty} \sum_{x=0}^{\infty} G(x,j) = 0$, then*

$$\sum_{z=0}^{\infty} H(0,z) = \sum_{x=0}^{\infty} (H(x,x)P(x,x)+G(x,x+1)) \ .$$

Proof. Set $N_0 = 0$ in theorem 3, and use the initial conditions $a_i(0) = \delta_{i0}$ for $0 \le i \le L$. □

Remark. If $\lim_{j\to-\infty} G(x,j) = 0 \ \forall x$ and the hypothesis of theorem 1 (a) holds, then

$$\sum_{z=-\infty}^{\infty} H(x,z)P(x,z) \ ,$$

has a closed form evaluation (see example 10 below).

In the following examples, we use the Maple package MarkovWZ [MZ] which, for a given $H(x,z)$, outputs the polynomial $P(x,z)$ and the $G(x,z)$.

2 Examples of Accelerating Series

Let $H(a,b) := \dfrac{(ax+z)!}{(bx+z+1)!}$ in examples 1 through 9.

Example 1. Consider the hypergeometric term $(-1)^z H(0,1)$, and corresponding to this kernel determine a polynomial $P(x,z)$ in z with a minimum degree such that $((-1)^z H(0,1), G(x,z))$ is an MWZ-pair. Using the maple package MarkovWZ [MZ] , we see that the polynomial is

$$P(x,z) = \frac{x!}{2^x}\ ,$$

and the corresponding MWZ mate of $(-1)^z H(0,1)$ is

$$G(x,z) = \frac{(-1)^z x!}{2^{x+1}} H(0,1)\ .$$

It is not hard to check that $((-1)^z H(0,1), G(x,z))$ is indeed a MWZ-pair with the corresponding polynomial $P(x,z) = x!/2^x$.

Applying corollary 2 to the MWZ-pair we get,

$$\log(2) = \frac{3}{2}\sum_{x=0}^{\infty} \frac{(-1)^x x!(x+1)!}{(2x+2)!2^x} = 2\mathrm{arcsinh}\left(\frac{\sqrt{2}}{4}\right)\ .$$

Similarly, if we apply corollary 1 to the MWZ-pair, we find

$$\log(2) = \frac{1}{2}\sum_{x=0}^{\infty} \frac{1}{2^x(x+1)}\ .$$

In the remaining examples, we simply give the hypergeometric term $H(x,z)$, the polynomial $P(x,z)$ that features in the MWZ-pair, the corresponding $G(x,z)$, and then the identities that follow from the application of the corollaries above.

Example 2. Starting with the kernel $(-1)^z H(0,3)$, we find

$$P(x,z) = \frac{(3x)!}{8^x}\ ,$$

and

$$G(x,z) = \frac{32+63x^2+93x+22z+30xz+4z^2}{8(3x+z+2)(3x+z+3)} P(x,z)(-1)^z H(0,3)\ .$$

Application of corollary 1 gives

$$\log(2) = \frac{1}{8}\sum_{x=0}^{\infty} \frac{(-1)^x (x+1)!(3x)!(415x^2+487x+134)}{(4x+4)!8^x}\ ,$$

On the other hand if we apply corollary 2, we get

$$\log(2)=\sum_{x=0}^{\infty}\frac{(63x^2+93x+32)}{24(3x+2)(x+1)(3x+1)8^x}\ .$$

Example 3. By taking the kernel $(-1)^z H(0,6)$, we find

$$P(x,z)=\frac{(6x)!}{2^{6x}}\ ,$$

and

$$G(x,z)=\frac{Q(x,z)P(x,z)}{16(6x+z+2)(6x+z+3)(6x+z+4)(6x+z+5)(6x+z+6)}(-1)^z H(0,6)\ ,$$

where $Q(x,z)$ is a certain polynomial in x and z.

Corollary 2 gives

$$\log(2)=\sum_{x=0}^{\infty}\frac{(-1)^x(6x)!(x+1)!P(x)}{(7x+7)!64^x}\ ,$$

where

$$P(x):=1648544x^5+4584284x^4+4905938x^3+2511703x^2+610829x+55914\ ,$$

and corollary 1 gives

$$\log(2)=\sum_{x=0}^{\infty}\frac{40824x^5+129924x^4+158814x^3+92655x^2+25605x+2654}{384(6x+1)(3x+1)(2x+1)(3x+2)(5+6x)(x+1)64^x}\ .$$

Example 4. Starting with $H(0,2)^2$, we find that

$$P(x,z)=\frac{\sqrt{\pi}((2x)!)^3}{16^x\Gamma(2x+1/2)}\ ,$$

and

$$G(x,z)=\frac{Q(x,z)}{2((1+4x)(3+4x)(2x+z+2)^2)}P(x,z)H(0,2)^2\ ,$$

where

$$Q(x,z):=120x^4+372x^3+136x^3z+56x^2z^2+426x^2+316x^2z \\ +242xz+86xz^2+8xz^3+213x+39+33z^2+6z^3+61z\ .$$

Application of corollary 2 gives

$$\zeta(2)=\frac{\sqrt{\pi}}{8}\sum_{x=0}^{\infty}\frac{(2912x^4+7100x^3+6381x^2+2494x+355)((x+1)!)^2((2x)!)^3}{\Gamma(2x+5/2)((3x+3)!)^2 16^x}\ .$$

On the other hand, corollary 1 yields

$$\zeta(2)=\frac{3\sqrt{\pi}}{32}\sum_{x=0}^{\infty}\frac{(20x^2+32x+13)(2x)!}{(2x+1)(x+1)\Gamma(2x+5/2)16^x}\ .$$

Example 5. By taking the kernel $H(1,2)^2$, we get

$$P(x,z) = \frac{(x!)^3\sqrt{\pi}}{4^x\Gamma(x+1/2)} ,$$

$$G(x,z) = \frac{21x^3+55x^2+47x+13+28x^2z+48xz+20z+13xz^2+11z^2+2z^3}{2(2x+1)(2x+z+2)^2}F(x,z) ,$$

where $F(x,z) := P(x,z)H(1,2)^2$.

If we apply corollary 2 we get

$$\zeta(2) = \frac{1}{9\sqrt{\pi}}\sum_{x=0}^{\infty}\frac{(145x^2+186x+59)(x!)^5\Gamma(x+1/2)4^x}{((3x+2)!)^2} .$$

On the other hand, corollary 1 yields

$$\zeta(2) = \frac{\pi^{3/2}}{64}\sum_{x=0}^{\infty}\frac{(21x+13)x!^3}{(64)^x(\Gamma(x+3/2))^3} .$$

Example 6. Corresponding to $H(1,3)^2$, we find that

$$P(x,z) = \frac{\sqrt{\pi}(2x)!^3}{16^x\Gamma(2x+1/2)} ,$$

and

$$G(x,z) = \frac{Q(x,z)}{2(3+4x)(1+4x)(3x+z+2)^2(3x+z+3)^2}P(x,z)(-1)^zH(1,3)^2 ,$$

where $Q(x,z)$ is a polynomial in x and z. Application of corollary 2 gives

$$\zeta(2) = \frac{\pi^{3/2}}{2048}\sum_{x=0}^{\infty}\frac{((2x)!)^3(10920x^4+27908x^3+25962x^2+10275x+1421)}{(\Gamma(2x+5/2))^3(4096)^x} ,$$

and corollary 1 gives

$$\zeta(2) = \frac{\sqrt{\pi}}{72}\sum_{x=0}^{\infty}\frac{P(x)(x!)^2((2x)!)^2}{16^x\Gamma(2x+5/2)((3x+2)!)^2} ,$$

where

$$P(x) := 2912x^4+7100x^3+6381x^2+2494x+355 .$$

Example 7. Corresponding to $H(1,5)^2$, we find that

$$P(x,z) = \frac{\sqrt{2\pi}(4x)!^3}{4(256)^x\sin(1/8\pi)\sin(3/8\pi)\Gamma(4x+1/2)} ,$$

and a corresponding MWZ mate $G(x,z)$. If we apply corollary 2, we find

$$\zeta(2)=\frac{\sqrt{2\pi}}{3200\sin(3/8\pi)\sin(1/8\pi)}\sum_{x=0}^{\infty}\frac{P(x)((4x)!)^3(x!)^2}{(256)^x\Gamma(4x+9/2)((5x+4)!)^2}\ ,$$

where

$$\begin{aligned}P(x) &:= 3333245952x^{10}+18842142336x^9+47204597136x^8+68964524342x^7+65011852179x^6\\ &+41280848445x^5+17862102186x^4+5194331883x^3+970166319x^2+104901994x+4974228\ .\end{aligned}$$

The terms of this series are $O((\frac{256}{9765625})^j)\approx O(10^{-5j})$.

Example 8. Similarly for the kernel $H(0,2)^3$, we get

$P(x,z)=\frac{(-1)^x(x!(2x)!)^3}{(3x)!}$ and

$G(x,z)=\frac{Q(x,z)}{6(3x+1)(3x+2)(2x+z+2)^3}H(0,2)^3P(x,z)$,

where $Q(x,z)$ is a certain polynomial in x and z.

By using corollary 2, we get

$$\zeta(3)=\sum_{x=0}^{\infty}\frac{(-1)^x(2x)!^3(x+1)!^6P(x)}{2(x+1)^2((3x+3)!)^4}\ ,$$

where

$$P(x):=40885x^5+124346x^4+150160x^3+89888x^2+26629x+3116,$$

and application of corollary 1 gives

$$\zeta(3)=\sum_{x=0}^{\infty}\frac{(-1)^x(56x^2+80x+29)(x!)^3}{4(2x+1)^2(3x+3)!}\ .$$

Example 9. Starting with the kernel $H(1,3)^3$, we get

$P(x,z)=\frac{(-1)^x(x!(2x)!)^3}{(3x)!}$ and

$G(x,z)=\frac{Q(x,z)}{6(3x+2)(3x+1)(3x+z+2)^3(3x+z+3)^3}P(x,z)H(1,3)^3$,

where

$$\begin{aligned}Q(x,z) =:\ &448x^5+624zx^4+1760x^4+1932zx^3+2728x^3+348z^2x^3+2214x^2z+2084x^2+792z^2x^2\\ &+90z^3x^2+594xz^2+1113xz+9z^4x+132z^3x+784x+6z^4+207z+48z^3+147z^2+116\ .\end{aligned}$$

In this example, we show all the steps to demonstrate the application of theorem 2

Let

$$F(x,z) := H(x,z)P(x,z) \; .$$

Define $M(n)$, $for\ n = 0,1,2,3,4,\ldots$, by

$$M(n) := \sum_{x=0}^{n-1} G(x,0) + \sum_{x=0}^{\infty} (F(x+n,x) + G(x+n,x+1)) \; .$$

Then theorem 2 says that $\zeta(3) = M(n),\ \forall n = 0,1,2,3,4,\ldots$.

In particular

$$\zeta(3) = M(0) = \frac{1}{24} \sum_{x=0}^{\infty} \frac{(x!)^3 (2x)!^6 (-1)^x P(x)}{(3x+2)!((4x+3)!)^3} \; , \tag{2}$$

where

$$P(x) := 126392x^5 + 412708x^4 + 531578x^3 + 336367x^2 + 104000x + 12463 \; .$$

On the other hand, application of corollary 1 gives

$$\zeta(3) = \frac{1}{162} \sum_{x=0}^{\infty} \frac{P(x)(x!)^6((2x)!)^3(-1)^x}{((3x+2)!)^4} \; ,$$

where

$$P(x) := 40885x^5 + 124346x^4 + 150160x^3 + 89888x^2 + 26629x + 3116 \; .$$

The series (2) was first derived in [AZ] and used by S. Wedeniwski (1999) to obtain up to 128 million correct decimal places. The terms of the series in (2) are $O((110592)^{-j}) \approx O(10^{-5j})$, while the terms of the second series are $O((\frac{64}{531441})^j) \approx O(10^{-4j})$.

Instead, if we take $H(1,5)^3$, we get

$$P(x,z) = \frac{2\sqrt{3}}{3\sqrt{\pi}} \frac{(2x-1/2)^3 (2x)!^5 (4096)^x}{(729)^x \Gamma(2x+2/3)\Gamma(2x+1/3)} \; ,$$

and a corresponding $G(x,z)$. Let

$$F(x,z) = H(1,5)^3 P(x,z) \; ,$$

and let $M(n)$ be as above.

Then theorem 2 gives $\zeta(3) = M(n)$,$\forall n = 0,1,2,3,4,\ldots$ and in particular

$$\zeta(3) = M(0) = \frac{16}{81} \sum_{x=0}^{\infty} \frac{P(x)(4096)^x((4x)!)^3((2x)!)^2((2x+1)!)^4(-1)^x}{((6x+5)!)^4} \; , \tag{3}$$

where

$$P(x) := 5561689253120x^{13} + 41827852352256x^{12} + 143295193251200x^{11} + 295842983236608x^{10} + 410324548816928x^{9} + 403368918753744x^{8} + 288879369092920x^{7} + 152460289970616x^{6} + 59240414929957x^{5} + 16722886152858x^{4} + 3330604771504x^{3} + 442815051024x^{2} + 35195802021x + 1261871244\,.$$

The terms of this series are $O((\frac{4096}{282429536481})^j) \approx O(10^{-8j})$.

This improves the previous record (2).

Example 10. If we start with

$$H(x,z) = \binom{x+a}{a+z}\binom{x+b}{b+z}\,,$$

we get

$$P(x,z) = \frac{(a+b)!(x+a)!(x+b)!}{(a+2x+b)!a!b!}\,,$$

and

$$G(x,z) = \frac{(3x^2+2xa+2xb+6x-2xz+2b+2a-2z+3-za+ab-zb)(a+z)(b+z)}{(a+2x+1+b)(2x+b+2+a)(x+1-z)^2)}H(x,z)P(x,z)\,.$$

One can easily check that $G(x,\pm\infty) = 0$.

Hence, we get

$$\sum_{z=-\infty}^{\infty}\binom{x+a}{a+z}\binom{x+b}{b+z} = \frac{(a+2x+b)!a!b!}{(a+b)!(x+a)!(x+b)!}\,.$$

This is a derivation of the classical Chu-Vandermonde summation formula, in the framework of the MWZ-method. The Markov-WZ method can sometimes lead to a discovery of new identities with appropriate $H(x,z)$.

Example 11. Let

$$H_s(x,z) := \left(\frac{(-1)^z(m)_z}{(m+\delta)_{x+z}}\right)^s\,.$$

In this example we will show how to use implementations of some numerical methods together with the Markov-WZ Method to give new WZ-pairs. The steps are:

(a) Take the output from Markov in MarkovWZ (see [MZ]), which is a system of first order linear recurrence relation(s) for the unknown coefficient functions $a_i(x)'s$.

(b) Crank out some terms for the unknown coefficients, i.e. use the recurrence equation outputted by the program and find the first few terms.

(c) Use the Salvy-Zimmermann `gfun` program in the Algolib library available from `algo.inria.fr`, or **findrec** in **EKHAD** †, to find a recurrence equation satisfied by the coefficient functions.

(d) Finally, solve the recurrence relations to find a closed form for the coefficients (if there exists one) (for example, in Maple, use `rsolve`).

11.1 Starting with $H_2(x,z)$, we find that $L=0$ and

$$P(x,z) := \frac{\Gamma(\delta+x)^3\Gamma(\delta-1/2)}{4^x\Gamma(\delta+x-1/2)\Gamma(\delta)^3} .$$

Therefore we get a WZ-pair $(F,G)(notMWZ!)$, where $F(x,z) := H_2(x,z)P(x,z)$, and

$$G(x,z) := F(x,z)\frac{(3x+2z+2m-2+3\delta)}{2(2x+2\delta-1)} ,$$

and by applying corollary 1, we get the identity

$$\sum_{z=0}^{\infty}\frac{\Gamma(z+m)^2\Gamma(m+\delta)^2}{\Gamma(m)^2\Gamma(m+\delta+z)^2} = \frac{1}{2}\sum_{x=0}^{\infty}\frac{(3x+3\delta+2m-2)\Gamma(\delta+x)^3\Gamma(\delta-1/2)\Gamma(m+\delta)^2}{\Gamma(\delta+x-1/2)\Gamma(\delta)^3\Gamma(m+x+\delta)^2(2x+2\delta-1)}\left(\frac{1}{4}\right)^x ,$$

for $\delta = 0,1,2,3,\ldots, m = 0,1,2,3,\ldots$. If we specialize to $m=1$ and $\delta=1$, we get the formula for $\zeta(2)$, which is

$$\zeta(2) = \frac{3\sqrt{\pi}}{4}\sum_{x=0}^{\infty}\frac{\Gamma(x+1)}{(x+1)\Gamma(3/2+x)}\left(\frac{1}{4}\right)^x = \frac{3}{2}\,{}_3F_2\left(\begin{matrix}1, ,1, 1\\ 2, \frac{3}{2}\end{matrix};\frac{1}{4}\right)$$

11.2 Starting with $H_3(x,z)$ we find that $L=1$ and there is a vector first order recurrence relations for the polynomials $a_0(x), a_1(x),$. That means if we set

$a(x) := [a_0(x), a_1(x)]^T$,

then there is a 2 by 2 matrix $\mathbf{A}(x)$ such that $a(x+1) = \mathbf{A}(x)a(x)$, and by using findrec in EKHAD we get

$$a_0(x) := \frac{(-1)^x\Gamma(x+\delta)^3(x+\delta-1)\Gamma(\delta-1/2)}{\Gamma(\delta)^3}, \text{ and } a_1(x) := \frac{2(-1)^x\Gamma(\delta+x)^3\Gamma(\delta-1/2)}{\Gamma(\delta)^3} .$$

Hence our polynomial is $P(x,z) = a_0(x)+a_1(x)(z+m),$, and the corresponding WZ-pair is $(H_3(x,z)P(x,z), G(x,z))$, where

$$G(x,z) := \frac{2x+2\delta+z+m-1}{2z+2m+\delta+x-1}P(x,z)H_3(x,z) ,$$

as outputted by zeil in EKHAD. Applying corollary 1, we get the identity

$$\sum_{z=0}^{\infty}\frac{(-1)^z(2z+2m+\delta-1)\Gamma(m+z)^3}{\Gamma(m)^3\Gamma(m+\delta+z)^3} = \sum_{x=0}^{\infty}\frac{(-1)^x(2x+2\delta+m-1)\Gamma(x+\delta)^3}{\Gamma(\delta)^3\Gamma(m+\delta+x)^3} ,$$

for $\delta = 0,1,2,3,\ldots$, and $m = 0,1,2,3,\ldots$.

† download-able free from: `http://www.math.rutgers.edu/~zeilberg/`

11.3 Starting with $H_4(x,z)$ we find that $L=1$ and there is a first order vector recurrence relations for the polynomials $a_0(x), a_1(x)$. Using findrec in EKHAD we get

$$a_0(x) := \frac{(-1)^x \Gamma(\delta+x)^5(\delta+x-1)\Gamma(\delta-1/2)}{\Gamma(\delta+x-1/2)\Gamma(\delta)^5 4^x} ,$$

and

$$a_1(x) := \frac{2(-1)^x \Gamma(\delta+x)^5 \Gamma(\delta-1/2)}{4^x \Gamma(\delta+x-1/2)\Gamma(\delta)^5} .$$

This leads to the WZ-pair $(F(x,z)(a_0(x)+a_1(x)(m+z)), G)$, where G is

$$G := \frac{5x^2+6mx+10\delta x+6m\delta+5\delta^2+2m^2+6xz-6x+6\delta z-6\delta+4mz-4m+2z^2-4z+2}{2(2x+2\delta-1)(2m+2z+x+\delta-1)} .$$

Application of corollary 1 yields the identity

$$\sum_{z=0}^{\infty} \frac{\Gamma(m+z)^4(2m+2z+\delta-1)}{\Gamma(m+\delta+z)^4} = \frac{1}{4}\sum_{x=0}^{\infty} \frac{\Gamma(m)^4\Gamma(x+\delta)^5\Gamma(\delta-1/2)P(x)}{\Gamma(x+1/2+\delta)\Gamma(m+\delta+x)^4\Gamma(\delta)^5}\left(\frac{-1}{4}\right)^x ,$$

that holds for $\delta = 0,1,2,3,\ldots$, and $m = 0,1,2,3,4,\ldots$, where

$$P(x) := 5x^2+10x\delta+6xm+2m^2+5\delta^2+6\delta m+2-6x-4m-6\delta .$$

If we specialize to $m=1$ and $\delta=1$, we find the motivation for Andrei Markov's beautiful work, namely

$$\zeta(3) = \frac{5\sqrt{\pi}}{4}\sum_{x=0}^{\infty} \frac{\Gamma(x+1)}{(x+1)^2\Gamma(x+3/2)}\left(\frac{-1}{4}\right)^x = \frac{5}{4}\, {}_4F_3\left(\begin{matrix}1,,1,1,1\\ 2,2,\frac{3}{2}\end{matrix};\frac{-1}{4}\right) .$$

11.4 Starting with $H_5(x,z)$ we found that $L=3$. The corresponding polynomial satisfies a recurrence relation of order ≥ 2, for which we couldn't find an explicit closed form solution for the polynomial. Nonetheless, as described in [MZ] , we have an accelerating formula for $\zeta(5)$ (see [MZ] for $5 \leq n \leq 9$).

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