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Pairwise Intersections and Forbidden Configurations

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Let \(f_m(a, b, c, d)\) denote the maximum size of a family \(\mathcal{F}\) of subsets of an \(m\)-element set for which there is no pair of subsets \(A, B \in \mathcal{F}\) with

\[
|A \cap B| \geq a, \quad |\bar{A} \cap B| \geq b, \quad |A \cap \bar{B}| \geq c, \quad \text{and} \quad |\bar{A} \cap \bar{B}| \geq d.
\]

By symmetry we can assume \(a \geq d\) and \(b \geq c\). We show that \(f_m(a, b, c, d)\) is \(\Theta(m^{a+b-1})\) if either \(b > c\) or \(a, b \geq 1\).

We also show that \(f_m(0, b, b, 0)\) is \(\Theta(m^b)\) and \(f_m(a, 0, 0, d)\) is \(\Theta(m^a)\). This can be viewed as a result concerning forbidden configurations and is further evidence for a conjecture of Anstee and Sali. Our key tool is a strong stability version of the Complete Intersection Theorem of Ahlswede and Khachatrian, which is of independent interest.

**Keywords:** forbidden configurations, extremal set theory, intersecting set systems, uniform set systems, \((0,1)\)-matrices

Let \(f_m(a, b, c, d)\) denote the maximum size of a family \(\mathcal{F}\) of subsets of an \(m\)-element set for which there is no pair of subsets \(A, B \in \mathcal{F}\) with

\[
|A \cap B| \geq a, \quad |\bar{A} \cap B| \geq b, \quad |A \cap \bar{B}| \geq c, \quad \text{and} \quad |\bar{A} \cap \bar{B}| \geq d.
\]

By symmetry we can assume \(a \geq d\) and \(b \geq c\).

**Theorem 1** Suppose \(a \geq d\) and \(b \geq c\). Then \(f_m(a, b, c, d)\) is \(\Theta(m^{a+b-1})\) if either \(b > c\) or \(a, b \geq 1\). Also \(f_m(a, 0, 0, d)\) is \(\Theta(m^a)\) and \(f_m(0, b, b, 0)\) is \(\Theta(m^b)\).

Some motivation for studying this function comes from the forbidden configuration problem for matrices popularised by the first author. We can identify a family \(\mathcal{A} = \{A_1, \ldots, A_n\}\) of subsets of \([m]\) with an \(m \times n\) \((0,1)\)-matrix \(A\) determined by incidence, i.e. \(A_{ij} = 1\) if \(i \in A_j\), otherwise 0. Such a matrix is simple, by which we mean it has no repeated columns. Let \(F\) be a \((0,1)\)-matrix (not necessarily simple). We define \(\text{forb}(m, F)\) to be the largest \(n\) for which there is a simple \(m \times n\) \((0,1)\)-matrix \(A\) that does not contain an \(F\) configuration, i.e. a submatrix which is a row and column permutation of \(F\). If we interpret...
A, F as incidence matrices of systems A, F (the latter possibly having sets with multiplicity) then A has an F configuration exactly when A has F as a trace, i.e. F ⊂ \{A ∩ X : A ∈ A\} for some X ⊂ [m].

The first forbidden configuration result was obtained independently by Sauer [6], Perles, Shelah [7], Vapnik and Chervonenkis [8]. When F is the k × 2^k (0, 1)-matrix with all possible distinct columns they showed that \text{forb}(m, F) = \sum_{i=0}^{k-1} \binom{m}{i}. For a general k-row matrix F, Füredi obtained an O(m^k) upper bound on \text{forb}(m, F), but it seems hard to determine the order of magnitude of \text{forb}(m, F) for each F. This was achieved when F has 2 rows by Anstee, Griggs and Sali [2] and for 3 rows by Anstee and Sali [3], but is open in general.

It is not hard to see that if F consists of a single column with s 0’s and t 1’s then \text{forb}(m, F) is \Theta(m^\max(s-1,t-1)). In this paper we solve the problem when F has two columns. Let F_{abcd} be the (a + b + c + d) × 2 (0, 1)-matrix which has a rows of [11], b rows of [10], c rows of [01], d rows of [00]. Then \text{forb}(m, F_{abcd}) = f_m(a, b, c, d) as defined above.

In [3] a conjecture was made for the asymptotic behaviour of \text{forb}(m, F) as a function of m and F. In particular, a restricted set of constructions of simple matrices were described in [3] that were conjectured to predict the asymptotics of \text{forb}(m, F). These were used in this paper to predict the asymptotics in Theorem 1 as well as to provide construction. This is further evidence for the conjecture in [3].

Our key tool is a strong stability version of the Complete Intersection Theorem of Ahlswede and Khachatrian [1], which is of independent interest. Strong stability results have been employed with success by the second author, for example in [4],[5]. First we recall some notation. Let numbers k, r_1, r_2 be given and suppose G and H are disjoint sets with |G| = k - r_1 + r_2. We define Tk_{r_1,r_2} on the pair (H,G) to be the family consisting of all sets of size k in G ∪ H that intersect G in at least k - r_1 = |G| - r_2 points. Note that any two sets in Tk_{r_1,r_2} have at least |G| - 2r_2 = k - r_1 - r_2 points in common, i.e. Tk_{r_1,r_2} is (k - r)-intersecting, where r = r_1 + r_2.

We also define F_{r_1,r_2} on the pair (H,G) to be the family consisting of all sets of size k in G ∪ H that intersect G in exactly k - r_1 = |G| - r_2 points. Clearly this is a subsystem of Tk_{r_1,r_2} and |Tk_{r_1,r_2} \setminus F_{r_1,r_2}| is of a lower order of magnitude than |Tk_{r_1,r_2}| and |F_{r_1,r_2}|. In particular, if the systems are defined on the ground set [m] with k = \Theta(m) then |Tk_{r_1,r_2}| and |F_{r_1,r_2}| are \Theta(m^r), whereas |Tk_{r_1,r_2} \setminus F_{r_1,r_2}| < m^{r-2}.

The Complete Intersection Theorem, conjectured by Frankl, and proved by Ahlswede and Khachatrian [1], is that any k-uniform, (k - r)-intersecting family of maximum size on a given ground set is isomorphic to Tk_{r-p,p}, for some 0 ≤ p ≤ r, which depends on the size of the ground set. We prove the following result.

**Theorem 2.** Suppose A is a k-uniform (k - r)-intersecting set system on [m] of size at least (5r)^{r}m^{r-1}. Then A ⊂ Tk_{r-p,p} for some 0 ≤ p ≤ r.

We use this theorem in our proofs of the upper bounds in Theorem 1 in cases where A is a k-uniform (k - r)-intersecting set system satisfying some additional properties. If |A| is small, we can ignore it for the purposes of upper bounds. If |A| is large enough to matter for the upper bounds, we can use the fact that A ⊂ Tk_{r-p,p} to deduce structure in A (e.g. the partition G, H above) which we can exploit in our proofs.

**References**


