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A sufficient condition for bicolorable hypergraphs

David Défossez

1Laboratoire Leibniz-Imag, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France. E-mail: david.defossez@imag.fr

In this note we prove Sterboul’s conjecture, that provides a sufficient condition for the bicolorability of hypergraphs.

Keywords: hypergraphs, coloring, Sterboul’s conjecture

In [2], Fournier and Las Vergnas gave a sufficient condition for the bicolorability of hypergraphs. Their theorem was a weaker form of a conjecture due to Sterboul, that we prove here. These facts are reproduced in [1] and [3].

A hypergraph is a pair $H = (V, E)$, where the elements of $V$ are the vertices, and the elements of $E$ are subsets of $V$ and are called the edges. A function $c : V \to \{1, 2\}$ is called a bipartition of $V$, and for $x \in V$, we call $c(x)$ the color of $x$. If $c$ is such that for any $e \in E$ with $|e| \geq 2$ both colors occur, then $c$ is called a bicoloration of $H$. If only one color occurs, the edge is said to be monochromatic. If a hypergraph admits a bicoloration we say that it is bicolorable.

A sequence $(x_1, e_1, x_2, ..., e_k, x_1)$, where the $e_i$’s are distinct edges, the $x_i$’s are distinct vertices, and $k \geq 3$, is said to be a cycle if $x_i \in e_{i-1} \cap e_i$ for $i = 2, ..., k$ and $x_1 \in e_1 \cap e_k$. A cycle is said to be odd if it has an odd number of edges.

An odd cycle $(x_1, e_1, x_2, ..., e_k, x_1)$ such that two non-consecutive edges are disjoint and $|e_i \cap e_{i+1}| = 1$ for $i = 1, 2, ..., k - 1$, is called a Sterboul cycle. If a hypergraph $H$ has no Sterboul cycle, it is said to be a Sterboul hypergraph.

Then we can word Sterboul’s conjecture as follows:

Theorem 1 If $H$ is a Sterboul hypergraph, then $H$ is bicolorable.

Proof: The proof works by induction on the number of edges.

When the hypergraph has no edge, the theorem clearly holds.

The general step assumes that we have a hypergraph $H = (V, E)$ and $e_0 \in E$ such that $H \setminus e_0 = (V, E \setminus e_0)$ has a bicoloration $c : V \to \{1, 2\}$.

We can assume that $e_0$ has size at least 2, and that $c$ leaves $e_0$ monochromatic, or else we have nothing to do.
Now we use the following algorithm to transform the bipartition $c$ into a bicoloration of $H$. The algorithm switches successively the colors of some vertices of $H$ that are contained in a monochromatic edge in the current bipartition. It constructs an arborescence $G_0 = (V_0, E_0)$ and a mapping $g : V_0 \rightarrow E$ that keep track of the running of the algorithm: the vertices of $G_0$ are those whose colors were switched, and $g$ associates a vertex with the monochromatic edge that caused its color switch.

The vertices are chosen with a DFS (Depth-First Search) method, and to do so the algorithm uses a LIFO (Last In First Out) stack $P$ that contains the set of vertices whose colors have been switched, and so that might be in a monochromatic edge. $\text{top}(P)$ returns the last vertex entered in $P$; $\text{drop}(P)$ removes $\text{top}(P)$ out of $P$; and $\text{put}(x, P)$ enters a new vertex $x$ in $P$.

**INPUT:** A hypergraph $H = (V, E)$ and a bipartition $c$ such that $e_0 \in E$ is the only monochromatic edge.

**OUTPUT:** A bicoloration of $H$ or an Error message only if $H$ is not a Sterboul hypergraph.

let $x_0 \in e_0$
$G_0 := (\{x_0\}, \emptyset)$
$g(x_0) := e_0$
switch $c(x_0)$
put $(x_0, P)$
While $P \neq \emptyset$ do
  let $v = \text{top}(P)$
  If there exists $e \in E$, $|e| \geq 2$, monochromatic such that $v \in e$ then
    If $e \setminus V_0 = \emptyset$ then
      return Error
    else
      let $w \in e \setminus V_0$
      $V_0 := V_0 \cup w$ ; $E_0 := E_0 \cup (vw)$
      $g(w) := e$
      switch $c(w)$
      put $(w, P)$
    end If
  else
    drop $(P)$
  end If
end While

First we remark that $G_0$ is indeed an arborescence since the end point of each new arc of $G_0$ is a new vertex. Then for a given $x \in V_0$ there is a unique path in $G_0$ from $x_0$ to $x$. Moreover when $x$ is at the top of $P$, then $P$ contains exactly the vertices of that path (because $P$ is a LIFO stack).

We can also remark that if the algorithm does not return Error, then at each iteration either a new vertex is put into $P$, or a vertex is dropped out of $P$. Since a vertex appears at most once in $G_0$ and thus can be put at most once in $P$, we have at most $2|V|$ iterations, and the algorithm ends.
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We note $\mathcal{P}^{(i)}$, $G_0^{(i)} = (V_0^{(i)}, E_0^{(i)})$, $c^{(i)}$, $g^{(i)}$ the values of $\mathcal{P}$, $G_0 = (V_0, E_0)$, $c$, $g$ (respectively) at the beginning of the $i$-th iteration. We also note $c^{(0)}$ the original bipartition (which is different from $c^{(1)}$ because of the switch of $c(x_0)$).

To prove the validity of the algorithm, we have to prove that:

- Error cannot be returned if $H$ is a Sterboul hypergraph.
- The output of the algorithm if no Error occurs is a bicoloration.

Before proving those points, we claim the following:

Claim 1 Suppose that $H$ is a Sterboul hypergraph. Consider the beginning of the $i$-th iteration. Let $\mathcal{P}^{(i)} = (x_k \ldots x_0)$, and $e_j = g^{(i)}(x_j)$ for $j = 0, \ldots, k$. Then we have:

(a) For each $j = 0, \ldots, k$, $x_j$ is the only vertex of its color in $e_j$.
(b) For each $j = 0, \ldots, k-1$ we have $e_j \cap e_{j+1} = \{x_j\}$.
(c) Two non-consecutive edges are disjoint.

Proof: The proof works by induction on $i$.

For $i = 1$ the claim clearly holds since $\mathcal{P}^{(1)} = (x_0)$.

We now consider $i \geq 1$ and we suppose the claim holds at iteration $i$. We are going to prove that it also holds at iteration $i + 1$. Let $\mathcal{P}^{(i)} = (x_k \ldots x_0)$, and $e_j = g^{(i)}(x_j)$ for $j = 0, \ldots, k$.

If during the $i$-th iteration the algorithm dropped $x_k$ out of $\mathcal{P}$ (that is $\mathcal{P}^{(i+1)} = (x_{k-1} \ldots x_0)$), the claim clearly holds at iteration $i + 1$. Thus we assume that the algorithm found $e_{k+1} \in \mathcal{E}$ with $x_k \in e_{k+1}$ that is monochromatic for $c^{(i)}$, and $x_{k+1} \in e_{k+1} \setminus V_0^{(i)}$ (because we assume that there is a $(i + 1)$-th iteration or else the claim is true) so that $\mathcal{P}^{(i+1)} = (x_{k+1}x_k \ldots x_0)$.

Since (a) holds at iteration $i$, we know that if $w \in e_k \setminus x_k$ then $c^{(i)}(w) \neq c^{(i)}(x_k)$. As $x_k \in e_{k+1}$ and $e_{k+1} + 1$ is monochromatic for $c^{(i)}$, then $e_k \cap e_{k+1} = \{x_k\}$ and (b) holds at iteration $i + 1$.

Suppose $j_0 = \max \{0 \leq j \leq k-1 | e_j \cap e_{k+1} \neq \emptyset \}$ exists, and let $y \in e_{j_0} \cap e_{k+1}$. If $k - j_0$ is odd, then $(y, e_{j_0}, x_{j_0}, \ldots, e_{k+1}, y)$ is a Sterboul cycle (because (c) holds at iteration $i$ and (b) holds at iteration $i + 1$), and $k - j_0$ is even. But then since (a) holds at iteration $i$, we have $c^{(i)}(y) \neq c^{(i)}(x_{j_0})$ (by definition of $j_0$), $c^{(i)}(x_{j_0}) \neq c^{(i)}(x_{j_0}+1)$, ..., $c^{(i)}(x_{k-1}) \neq c^{(i)}(x_k)$ and then $c^{(i)}(y) \neq c^{(i)}(x_k)$, which is impossible because $e_{k+1}$ is monochromatic for $c^{(i)}$. Hence $j_0$ does not exist, and (c) holds at iteration $i + 1$.

Thus $x_{k+1} \notin e_j$ for all $j = 0, \ldots, k$. Since the only color switch done during the $i$-th iteration concerns $x_{k+1}$, then (a) holds at iteration $i + 1$.

This achieves to prove the claim.

Now we are able to prove the validity of the algorithm.

- Suppose that $H$ is a Sterboul hypergraph. Consider an iteration $i$, and let $\mathcal{P}^{(i)} = (x_k \ldots x_0)$.

If $k = 1$ then from (b) of the claim we have $x_1 \notin e_0$. If $k \geq 2$ then from (c) of the claim we also have $x_k \notin e_0$. This proves that we always have $e_0 \cap V_0 = \{x_0\}$.

If Error is returned, it means that at a given iteration $i_0$, the algorithm found an edge $e$ monochromatic for $c^{(i_0)}$ such that $e \setminus V_0^{(i_0)} = \emptyset$. Then $e$ was also monochromatic for $c^{(0)}$, but $e_0$ was the only such edge,
so we have a contradiction because we have just seen that we must have $e_0 \cap V_{i_0} = \{x_0\}$. Thus if $H$ is a Sterboul hypergraph, Error cannot be returned.

- Finally, if the bipartition obtained by the algorithm is not a bicoloration then we have some $e \in \mathcal{E}$ that is monochromatic. Consider $i_0$ the last iteration during which the algorithm dropped a vertex of $e$ out of $\mathcal{P}$ ($i_0$ exists or else $e$ was monochromatic with $e^{(0)}$, but $e_0$ was the only such edge and $e_0$ is not monochromatic at the end of the algorithm), and let $y_0$ be that vertex. Then $e$ was not monochromatic with $e^{(i_0)}$ or else the algorithm would have considered $e$ during the iteration $i_0$ instead of dropping $y_0$. But since no color switch concerning a vertex of $e$ occurs afterwards (by choice of $i_0$), we have a contradiction. Thus the final bipartition is a bicoloration.

So the algorithm is correct and the theorem is proved. 

We can slightly modify the algorithm so that it gives a Sterboul cycle instead of just returning Error when the hypergraph is not Sterboul (in order to have a certificate that the hypergraph is not Sterboul).

To do so we just have to check at each iteration that the properties of Claim 1 still hold. If not it means that the monochromatic edge considered intersects the path induced by the stack, and a Sterboul cycle can be easily found.

Our algorithm finds a bicoloration for Sterboul hypergraphs in polynomial time. However it cannot be used to recognize bicolorable hypergraphs (since a Sterboul hypergraph may be bicolorable) and neither to recognize Sterboul hypergraphs (since it may happen that it gives a bicoloration for a hypergraph that is not Sterboul).

The problem of recognizing bicolorable hypergraphs is well-known to be NP-complete [4]. But we leave the following question open: what is the complexity of recognizing Sterboul hypergraphs?

References


