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On the $L(p, 1)$-labelling of graphs

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In this paper we improve the best known bound for the $L(p, 1)$-labelling of graphs with given maximal degree.

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1 Introduction.

In all the paper we work on a graph G with maximal degree Δ. For a set of vertices $S \subset V(G)$, the graph $G\backslash S$ is the graph induced by $V(G)\backslash S$. The distance $d(u, v)$ between two vertices u and v is the number of edges in the shortest path from u to v. We say that v is a *d-neighbor* of u if $d(u, v) = d$. Let $N_d(v)$ be the set of d-neighbors of v. We will generally use the common term *neighbor* instead of 1-neighbor. A *L($\alpha_1, \alpha_2, ..., \alpha_k$)-labelling* of a graph G is a function $l : V(G) \to [0, \lambda]$ such that for any pair of vertices u and v if $d(u, v) = d \leq k$ then $|l(u) - l(v)| \geq \alpha_d$. The problem is to find an L($\alpha_1, \alpha_2, ..., \alpha_k$)-labelling of G that minimizes λ. We denote $\lambda_{\alpha_1,\alpha_2,...,\alpha_k}(G)$ such minimal λ. For a sequence of non-negative integers $S = (\alpha_1, \alpha_2, ..., \alpha_k)$, we will use the notation $\lambda_S(G)$ instead of $\lambda_{\alpha_1,\alpha_2,...,\alpha_k}(G)$.

This problem arises from the *channel assignement problem*. The channel assignement problem is to assign a channel to each radio transmitter so that close transmitters do not interfer and such that we use the minimum span of frequency. Roberts proposed to assign channels such that "close" transmitters receive different channels and "very close" transmitters receive channels that are at least two channels apart. This is a L(2,1)-labelling of a graph G where the vertices are the transmitters, the "very close" transmitters are adjacent vertices and the "close" transmitters are vertices at distance two in G. Since the constraints between transmitters disminish with the distance, the L($\alpha_1, \alpha_2, ..., \alpha_k$)-labelling of graph is interesting for this problem when the sequence $\alpha_1, \alpha_2, ..., \alpha_k$ is decreasing. Many work has been done on L(2,1)-labeling since the first paper of J.R.Griggs and R.K.Yeh [7]. Many papers deal with bounding $\lambda_{\alpha_1,\alpha_2}$ for some family of graphs or given some graphs invariants such as $\chi(G)$ and Δ (See for example [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14]). In their paper [7], Griggs and Yeh proved that $\lambda_{2,1}(G) \leq \Delta^2+2\Delta$ and made the following conjecture.

Conjecture 1 *For any graph G with maximal degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$.*

Actually they proved it for $\Delta = 2$ and for graphs of diameter at most two. They also proved that determining $\lambda_{2,1}(G)$ is NP-complete. In this paper we focus on bounding $\lambda_{p,1}$ according to Δ. In [3] the authors gave an algorithm for the L(2,1)-labeling and improved the upper bound of $\lambda_{2,1}$ to $\Delta^2 + \Delta$. In [4], with the same algorithm they obtained that $\lambda_{p,1}(G) \leq \Delta^2 + (p - 1)\Delta$. Let $\sigma(S, \Delta)$ be the function defined for any sequence $S = (\alpha_1, ..., \alpha_k)$ by $\sigma(S, \Delta) = \sum_{i=1}^{k} \alpha_i \Delta(\Delta - 1)^{i-1}$. With the algorithm used in [3, 4], we generalise their result as follow.

Proposition 1 *For any sequence of non-negative integers $S = (\alpha_1, \alpha_2, ..., \alpha_k)$, with $k \geq 1$, and any graph G with maximum degree Δ, we have that $\lambda_S(G) \leq \sigma(S, \Delta)$.*

But this is not the best known bound. In [9], Král and Škrekovski had a result on the list channel assignement problem. As a corollary of their result we have that :

Theorem 1 *For any sequence of non-negative integers $S = (\alpha_1, \alpha_2, ..., \alpha_k)$, such that $k \geq 2$ and $\alpha_1 > \alpha_2$, and any graph G with maximum degree $\Delta \geq 3$, we have that $\lambda_S(G) \leq \sigma(S, \Delta) - 1$.*

In this paper, we improve this last bound by two different ways for some specific sequences S.

Theorem 2 *For any sequence $S = (\alpha_1, ..., \alpha_k)$ with $k \geq 2$ and such that $\alpha_1 > \alpha_2 \geq \alpha_3 \geq ... \geq \alpha_k = 1$, and any connected graph G with maximum degree $\Delta \geq 3$, there is an ordering of the vertices, v_0, v_1, ..., v_n and a $L(\alpha_1, ..., \alpha_k)$-labelling l of G such that :*

- $l(v_0) \leq \sigma(S, \Delta) - 1$
- $l(v_j) \leq \sigma(S, \Delta) - j$ *for* $1 \leq j < k$
- $l(v_j) \leq \sigma(S, \Delta) - k$ *for* $k \leq j$

This implies that just a constant number of vertices, k, are labelled more than $\sigma(S, \Delta) - k$.

Theorem 3 *For any sequence $S = (p, 1)$ with $p \geq 2$ and any graph G with maximum degree $\Delta \geq 3$, we have that $\lambda_{p,1}(G) \leq \sigma(S, \Delta) - 2 = \Delta^2 + (p - 1)\Delta - 2$.*

So, for the L(2,1)-labelling we obtain that $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ and we get a little closer to Conjecture 1. To prove Theorem 3 we need the following structural lemma.

Lemma 1 *Every graph G with maximal degree $\Delta \geq 3$ has either :*

(i) *a vertex v with degree less than Δ.*

(ii) *a cycle of length three.*

(iii) *two cycle of length four passing through the same vertex v.*

(iv) *a vertex v with three neighbors u, x and y, such that there is a cycle of length four passing through the edge uv and such that the graph $G \backslash \{x, y\}$ is connected.*

(v) *a vertex u with two adjacent vertices v and w such that the graph $G \backslash X$ is connected, where X is the set $(N(v) \bigcup N(u)) \backslash \{w\}$.*

For proving Theorem 2, the following corollary of Lemma 1 is sufficient.

Corollary 1 *Every graph G with maximal degree $\Delta \geq 3$ has either :*

(i) *a vertex v with degree less than Δ.*

(ii) *a cycle of length ≤ 4.*

(iii) *a vertex v with two neighbors x and y such that the graph $G \backslash \{x, y\}$ is connected.*

In this abstract we do not prove Lemma 1 and Theorem 3, but most of the arguments used in the proof of Theorem 3 are in the proof of Theorem 2. In section 2 we generalise the labeling algorithm presented in [3] and we obtain Proposition 1. In section 3 we modify it to prove Theorem 2.

2 The basic algorithm.

In [3] the authors present an algorithm that $L(2,1)$-label graphs and establish that for a graph G of maximal degree Δ we have $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$. Here we present an extended version of this algorithm that $L(\alpha_1, ..., \alpha_k)$-label graphs and establishes Proposition 1.

```
i = 0 ;
while there are unlabelled vertices do
    for v_j = v_n to v_0 do
        if v_j is unlabelled and v_j can be labelled i then
            let v_j be labelled i;
        end
    end
    i = i + 1;
end
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In this algorithm a vertex v_j *can be labelled* i if it has no d-neighbor already labelled x with $i - \alpha_d < x < i + \alpha_d$.

Let us denote $l(v)$ the value the algorithm assigns to the vertex v. Observe that if the vertex v is not labelled i it cannot be because its d-neighbor u is labelled $l(u)$, with $i < l(u) < i + \alpha_d$. Indeed, when the algorithm "proposed" v to be labelled i, the vertex u was still unlabelled. So, a vertex u which has been labelled $l(u)$ could only "forbid" its d-neighbor v to be labelled $l(u)$, $l(u)+1$,..., and $l(u)+\alpha_d - 1$. Let us denote $F(u,v)$, the set of values which have been forbiden by u to v during the execution of the algorithm, we have that $F(u,v) = \{l(u), l(u)+1, ..., l(u)+\alpha_d - 1\}$, if $d(u,v) = d$. The set $F(v)$ of all the values that have been forbiden to v is the union on all the vertices u of $F(u,v)$, $F(v) = \bigcup_{u \in V(G)} F(u,v)$. Note that the algorithm labels v with the smallest value which is not in $F(v)$. So $l(v) \leq |F(v)|$, since there are $|F(v)| + 1$ values in the interval $[0, |F(v)|]$. The set $F(v)$ being a union of possibly disjoint sets we have $|F(v)| \leq \sum_{u \in V(G)} |F(u,v)|$. In a graph of maximal degree Δ, one can easily see by induction on i that there are at most $\Delta(\Delta-1)^{i-1}$ vertices in $N_i(v)$. Since if u is a i-neighbor of v we have $|F(u,v)| = \alpha_i$, we obtain that $l(v) \leq \sum_{i=1}^{k} \alpha_i \Delta(\Delta-1)^{i-1}$.

3 The improved algorithm and proof of Theorem 2.

To improve the bound we have in Proposition 1, we have to be more carefull on the order the algorithm considers the vertices. If we have two vertices v_p and v_q, with $p < q$ and $d(v_p, v_q) = d \leq k$, the vertex v_p only forbids $\alpha_d - 1$ values to v_q. Indeed, the vertex v_p does not forbid to v_q the value $l(v_p)$, when the algorithm considered the possibility to label the vertex v_q with the value $l(v_p)$ the vertex v_p, being considered after v_q by the algorithm, was still unlabelled. This observation reduces the size of $F(v_p, v_q)$ by one and so the bound on $l(v_q)$. So, if for a vertex v_q there are x vertices v_p, with $p < q$ and $d(v_p, v_q) = d \leq k$, then $l(v_q) \leq |F(v_q)| = \sigma(S, \Delta) - x$. It would be interesting to have an order such that many vertices have some d-neighbors, with $d \leq k$, posterior to them. It is not possible for all the vertices, the vertex v_0 being the last vertex considered by the algorithm, we cannot reduce the size of $F(v_0)$ with this observation.

Given a spanning tree T of G rooted in r, numbering the vertices of G according to a preorder traversal of T we obtain that the vertices of G numbered v_0, ..., v_n are such that :

- $v_0 = r$
- If $i < j < k$ then $d(v_i, v_j) \leq k$.
- If $k \leq j$, there are at least k vertices v_i such that $d(v_i, v_j) \leq k$.

With such numbering of the vertices, by the previous observation, we clearly prove the two last points of Theorem 2. Now we are going to show how to choose T and r in order to obtain the first point. To do that, consider the case of Corollary 1 we are in.

Case (i) If there is a vertex of degree less than Δ, let r be this vertex and consider any spanning tree T of G. In this case, since there are at most $\Delta - 1$ vertices in $N_1(v_0)$, we easily bound $|F(v_0)|$ by $\sigma(S, \Delta) - \alpha_1$.

Case (ii) If there is a cycle of length ≤ 4, let r be a vertex of this cycle and consider any spanning tree T of G. In this case, since there are at most $\Delta(\Delta - 1) - 1$ vertices in $N_2(v_0)$, we easily bound $|F(v_0)|$ by $\sigma(S, \Delta) - \alpha_2$.

Case (iii) If there is a vertex with two neigbors x and y such that the graph $G \backslash \{x, y\}$ is connected, let r be this vertex. We construct T from any spanning tree of $G \backslash \{x, y\}$ by adding the edges rx and ry. We then number the vertices by a preorder traversal of T such that x and y are the two last numbered vertices. It is possible since x and y are leafs in T. So we have that $v_0 = r$, $v_{n-1} = x$ and $v_n = y$. Since v_n is the first vertex considered by the algorithm, it clearly labels it 0. Since $d(v_n, v_{n-1}) = 2$ (else, see the previous case), the algorithm cannot label v_{n-1} less than α_2. We consider two cases according to the label of v_{n-1}.

- If v_{n-1} is labelled α_2, since $\alpha_1 > \alpha_2$, the value α_2 is in both $F(v_{n-1}, v_0)$ and $F(v_n, v_0)$. This implies that $|F(v_0)| \leq \sigma(S, \Delta) - 1$.
- If v_{n-1} is not labelled α_2, since there was no vertex labelled α_2 when the algorithm considered this value for v_{n-1}, there is a vertex v_k labelled l such that $d(v_k, v_{n-1}) = d$ and $l + \alpha_d > \alpha_2$. Since $l < \alpha_2$ and $\alpha_k = 1$ we have that $d < k$. This implies that $d(v_k, v_0) \leq k$ and that the value l is in both $F(v_k, v_0)$ and $F(v_n, v_0)$. This implies that $|F(v_0)| \leq \sigma(S, \Delta) - 1$.

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