Local chromatic number and topological properties of graphs

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The local chromatic number of a graph, introduced by Erdős et al. in [3], is the minimum number of colors that must appear in the closed neighborhood of some vertex in any proper coloring of the graph. This talk, based on the papers [13, 14, 15], would like to survey some of our recent results on this parameter. We give a lower bound for the local chromatic number in terms of the lower bound of the chromatic number provided by the topological method introduced by Lovász. We show that this bound is tight in many cases. In particular, we determine the local chromatic number of certain odd chromatic Schrijver graphs and generalized Mycielski graphs. We further elaborate on the case of 4-chromatic graphs and, in particular, on surface quadrangulations.

Keywords: graph coloring, topological method, Schrijver graphs, Mycielski graphs, surface quadrangulation

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1 Introduction

In 1978, proving the conjecture of Kneser, Lovász [6] introduced a topological technique to bound the chromatic number $\chi(G)$ of a graph G from below. In the same year, Bárány [2] found another short proof of Kneser's conjecture, also using topology. Still in 1978 this latter proof was generalized by Schrijver [12] showing that the same lower bound is true for the chromatic number of a family of induced subgraphs of Kneser graphs, that not only have their chromatic number equal to the so obtained lower bound, but are also vertex color-critical. Recall that Kneser graphs $KG(n,k)$ are defined on the k-element sets of an n-set as vertices and two vertices form an edge if the corresponding k-subsets are disjoint. The family of vertex color-critical induced subgraphs discovered by Schrijver is the following.

Definition 1 ([12]) *The* Schrijver graph $SG(n,k)$ *is defined as follows. Its vertices are those k-element subsets of the set $[n]=\{1,\dots,n\}$ that do not contain cyclically consecutive elements $i, i+1$ or $n, 1$. Two such vertices are adjacent if they represent disjoint k-subsets.*

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The above mentioned results state that $\chi(SG(n,k)) = \chi(KG(n,k)) = n - 2k + 2$.

Later the chromatic number of other families of graphs were also determined by this topological method. Two examples are generalized Mycielski graphs in [16], [4] and some finite subgraphs of Borsuk graphs in [7]. (For the definition of Borsuk graphs we also refer to [7], while the definition of generalized Mycielski graphs can be found in [4], [8], or [17].)

By now the topological method for bounding the chromatic number from below has several variations. Many of them can be described via introducing a so-called box complex assigned to the graph and express the lower bound on the chromatic number in terms of topological invariants of this complex. The paper [9] analyzes several of the possible box complexes and establishes a hierarchy of the bounds one can obtain by their use. All these bounds give the same sharp estimation of the chromatic number for the graphs mentioned above. We single out two of these possible topological bounds that we now simply denote by $b_0(G)$ and $b_s(G)$ for graph G. They satisfy $b_s(G) \leq b_0(G) \leq \chi(G)$. We call a graph topologically t-chromatic if $b_0(G) \geq t$ and strongly topologically t-chromatic if $b_s(G) \geq t$. (By $b_s(G) \leq b_0(G)$ strong topological t-chromaticity implies topological t-chromaticity.)

Our results show that if a graph is topologically t-chromatic then this implies a lower bound on another graph coloring parameter, its local chromatic number, and this bound is also sharp in many cases. In fact, the results of [13] also have implications on yet another coloring parameter, the circular chromatic number (cf. [19]) that we do not discuss here. The talk is based on the upcoming papers [13, 14, 15].

2 Local chromatic number

In short, the local chromatic number is the fewest number of colors that can appear in the most colorful closed neighborhood of a vertex in a proper coloring of the graph. Introduced by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [3], the formal definition is as follows.

Definition 2 ([3]) *The* local chromatic number $\psi(G)$ *of a graph G is*

$$\psi(G) := \min_{c} \max_{v \in V(G)} |\{c(u) : u \in N(v)\}| + 1,$$

where $N(v) = \{u : uv \in E(G)\}$ and the minimum is taken over all proper colorings c of G.

It is obvious that the chromatic number $\chi(G)$ is an upper bound on $\psi(G)$. It is less obvious, that $\psi(G) < \chi(G)$ is possible, moreover, there exist graphs G with $\psi(G) = 3$ and $\chi(G)$ arbitrarily large, cf. [3].

It was observed in [5] that the fractional chromatic number $\chi_f(G)$ (see [11] for definitions) bounds the local chromatic number from below, that is, $\chi_f(G) \leq \psi(G)$ is always true. This motivates the study of the local chromatic number of graphs that have a large gap between their fractional and ordinary chromatic numbers. Standard examples of such graphs are Kneser graphs and Mycielski graphs (see [11]), and one easily sees that their variants, Schrijver graphs and generalized Mycielski graphs also have this property. These are all graphs the chromatic number of which can be determined by the topological method discussed above.

This is how we were led to investigate the relevance of topological lower bounds of the chromatic number for the local chromatic number. It turned out that if G is a topologically $\chi(G)$-chromatic graph, that is one for which the topological method gives the chromatic number exactly, then its local chromatic number should be at least about the half of its chromatic number, and this lower bound is tight in many cases. In particular, we have the following result for Schrijver graphs.

Theorem 3 ([13]) *If $t = n - 2k + 2 > 2$ is odd and $n \geq 4t^2 - 7t$ then*

$$\psi(SG(n,k)) = \left\lceil \frac{t}{2} \right\rceil + 1.$$

This theorem easily implies that for even $t = n - 2k + 2 > 2$ and large enough n the value of $\psi(SG(n,k))$ is one of $t/2 + 1$ and $t/2 + 2$.

The following proposition shows that some lower bound on n is really needed in Theorem 3.

Proposition 4 ([13]) $\psi(SG(n,2)) = n - 2 = \chi(SG(n,2))$ *for every* $n \geq 4$.

While the lower bound part of Theorem 3 is topological, the matching upper bound is obtained via combinatorial methods. Both the upper and the lower estimation work in a more general setting resulting in similar results for generalized Mycielski graphs and Borsuk graphs of certain parameters. We refer to [13] for further details, as well as, for some topological consequences.

3 4-chromatic graphs and surface quadrangulations

Theorem 3 leaves open the question whether (large enough) 4-chromatic Schrijver graphs have local chromatic number 3 or 4. In other words, Theorem 3 does not decide whether the smallest chromatic number t for which a t-chromatic Schrijver graph with smaller local than ordinary chromatic number exists is 4 or 5. In [14] we have shown that this smallest number is 5, thus the following holds.

Theorem 5 ([14])

$$\psi(SG(2k+2,k)) = 4.$$

This theorem is again true in a more general setting. In fact, we show that all strongly topologically 4-chromatic graphs have local chromatic number 4. The same implication does not hold if G is only topologically 4-chromatic. For further details we refer to [14].

It is known that generalized Mycielski graphs of chromatic number 4 quadrangulate the projective plane. It turns out that 4-chromatic Schrijver graphs are closely related to quadrangulations of the Klein bottle. The chromatic number of surface quadrangulations is a widely investigated topic, see [1, 10, 18], and the above mentioned connections suggest that analogs of Theorem 5 may be true for certain quadrangulations of non-orientable surfaces. Indeed, one can show that non-bipartite quadrangulations of the projective plane have local chromatic number 4, generalizing a celebrated result of Youngs [18] stating that such graphs are never 3-chromatic. In [15] we also prove that certain quadrangulations of the Klein bottle that are shown to be 4-chromatic in [1] and [10] have local chromatic number 4. Surprisingly, however, quadrangulations of other non-orientable surfaces exist that are 4-chromatic by the same results but their local chromatic number is 3.

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