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Hamiltonian cycles in torical lattices

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We establish sufficient conditions for a toric lattice $T_{m,n}$ to be Hamiltonian. Also, we give some asymptotics for the number of Hamiltonian cycles in $T_{m,n}$.

**Keywords:** Hamiltonian cycle, toric lattice, Hardy–Littlewood method.

Let $T_{m,n} = J_m \times J_n$ be a toric lattice, i.e., the Cartesian product of two directed cycles lengths $m$ and $n$ respectively.

**Erdős problem** [1]. When $T_{m,n}$ contains Hamiltonian cycles?

The next theorem was proved by A.A.Evdokimov [2].

**Theorem 1** $T_{m,n}$ is Hamiltonian iff there are solutions of the following Diophantine system

\[
\begin{align*}
x + y &= \gcd(m, n), \\
\gcd(x, m) &= 1, \ \gcd(y, n) &= 1
\end{align*}
\]  

(\gcd means the greatest common divisor).

Let $J_{m,n}$ be the number of solutions of the system (1). We obtain estimates for $J_{m,n}$ in two special cases. Let

\[m = \prod_{i=1}^{r} p_i^{a_i}, \quad n = \prod_{j=1}^{s} q_j^{b_j}\]

are prime decompositions for $m, n$. We use the following notations

\[P = \prod_{i=1}^{r} p_i, \quad Q = \prod_{j=1}^{s} q_j, \quad \lambda(P, Q) = \prod_{r | \text{lcm}(P,Q)} \left(1 - \frac{1}{r}\right)\]

(lcm means the least common multiple).

**Theorem 2** $J_{m,n} \geq 1$ if $\gcd(m, n) > \left[ \prod_{i=1}^{r} (1 + p_i) + \prod_{j=1}^{s} (1 + q_j) \right] \left(4\lambda(P, Q)\right)^{-1}$.

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The proofs of the theorems 1, 2 are based on the following analytic and combinatorial results.

Let

\[ J_N(u) = \sum_{(a,N)=1} u^a, \quad N = p_1^{\alpha_1} \cdots p_k^{\alpha_k}. \]

**Lemma 1**

\[ J_N(u) = \frac{1}{1-u} - \sum_{i=1}^k \frac{1}{1-u^{p_i}} + \sum_{1 \leq i < j \leq k} \frac{1}{1-u^{p_ip_j}} - \cdots \]

This formula can be easily proved by inclusion - exclusion principle.

Let \( S_r(m, n) \) be the number of solutions of the system

\[ x + y = r, \quad \gcd(x, m) = 1, \quad \gcd(y, n) = 1. \]

(2)

The generating function for \( S_r(m, n) \) is related with \( J_n(u) \) by the following formula.

**Lemma 2**

\[ \sum_{r=1}^{\infty} S_r(m, n)u^r = J_m(u)J_n(u). \]

(3)

Formula (3) implies an expression for the number of solutions of the system (1).

**Lemma 3**

Let \( N = \gcd(m, n) + 1 \). Then the following equation holds

\[ J_{m,n} = \gcd(m, n) \sum_{u \mid P, v \mid Q} \frac{\mu(u)\mu(v)}{\text{lcm}(u, v)} + \sum_{u \mid P, v \mid Q} \frac{\mu(u)\mu(v)(u+v)}{2\text{lcm}(u,v)} + \sum_{u \mid P, v \mid Q} \frac{\mu(u)}{u} \sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^u-1)} + \sum_{u \mid P, v \mid Q} \frac{\mu(v)}{v} \sum_{\alpha^v=1} \frac{1}{\alpha^{N-1}(\alpha^v-1)}. \]

(4)

In sums of type

\[ \sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^u-1)} \]

the summation is over those roots of equation \( \alpha^u = 1 \) that are not the roots of equation \( \alpha^v = 1 \).

Sums (5) are called Dedekind sums. They are well-known in combinatorial analysis (e.g., see [3]).

To simplify (4) we use identities about Möbius function. They are 2-dimensional analogues of the classical formula

\[ \sum_{d \mid n} \frac{\mu(d)}{d} = \prod_{p \mid n} \left( 1 - \frac{1}{p} \right). \]

An example of these identities is given by the following Lemma.

**Lemma 4 ([4])**

\[ \sum_{u \mid m, v \mid n} \frac{\mu(u)\mu(v)}{\text{lcm}(u, v)} = \prod_{r \mid \text{lcm}(P,Q)} \left( 1 - \frac{1}{r} \right). \]
Dealing with Dedekind sums (5) we use the following useful statement. Let

\[ S_n(a) = \sum_{\alpha^b = 1, \alpha \neq \alpha^a} \frac{1}{\alpha^n(\alpha^a - 1)}, \quad (6) \]

where summation is over those roots of equation \( x^b = 1 \) that are not the roots of equation \( x^a = 1 \). By \( m_0 \) we denote the smallest positive solution of equation

\[ ax \equiv -(n + a) \quad (\text{mod } b). \]

Let \( w(a, b) = m_0 - 1 \).

**Lemma 5**

\[ S_n(a) = \frac{b}{2} - \frac{\gcd(a, b)}{2 \lcm(a, b)} - w(a, b). \quad (7) \]

**References**


