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Hamiltonian cycles in torical lattices

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We establish sufficient conditions for a toric lattice $T_{m,n}$ to be Hamiltonian. Also, we give some asymptotics for the number of Hamiltonian cycles in $T_{m,n}$.

Keywords: Hamiltonian cycle, toric lattice, Hardy–Littlewood method.

Let $T_{m,n} = J_m \times J_n$ be a toric lattice, i.e., the Cartesian product of two directed cycles lengths $m$ and $n$ respectively.

Erdős problem [1]. When $T_{m,n}$ contains Hamiltonian cycles?

The next theorem was proved by A.A.Evdokimov [2].

Theorem 1 $T_{m,n}$ is Hamiltonian iff there are solutions of the following Diophantine system

\[
\begin{align*}
x + y &= \gcd(m, n), \\
\gcd(x, m) &= 1, \quad \gcd(y, n) = 1
\end{align*}
\]

(\(\gcd\) means the greatest common divisor).

Let $J_{m,n}$ be the number of solutions of the system (1). We obtain estimates for $J_{m,n}$ in two special cases. Let

\[
m = \prod_{i=1}^{r} p_i^{a_i}, \quad n = \prod_{j=1}^{s} q_j^{b_j}
\]

are prime decompositions for $m, n$. We use the following notations

\[
P = \prod_{i=1}^{r} p_i, \quad Q = \prod_{j=1}^{s} q_j, \quad \lambda(P, Q) = \prod_{r|\text{lcm}(P, Q)} \left(1 - \frac{1}{r}\right)
\]

(lcm means the least common multiple).

Theorem 2 $J_{m,n} \geq 1$ if $\gcd(m, n) > \left[ \prod_{i=1}^{r} (1 + p_i) + \prod_{j=1}^{s} (1 + q_j) \right] \left(4\lambda(P, Q)\right)^{-1}$.

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The proofs of the theorems 1, 2 are based on the following analytic and combinatorial results.

Let

\[ J_N(u) = \sum_{(a, N) = 1} u^a, \quad N = p_1^{\alpha_1} \cdots p_k^{\alpha_k}. \]

**Lemma 1** \( J_N(u) = \frac{1}{1-u} - \sum_{i=1}^{k} \frac{1}{1-u^{p_i}} + \sum_{1 \leq i < j \leq k} \frac{1}{1-u^{p_ip_j}} - \cdots \)

This formula can be easily proved by inclusion - exclusion principle.

Let \( S_r(m, n) \) be the number of solutions of the system

\[ x + y = r, \quad \gcd(x, m) = 1, \quad \gcd(y, n) = 1. \]  \( \text{(2)} \)

The generating function for \( S_r(m, n) \) is related with \( J_n(u) \) by the following formula.

**Lemma 2**

\[ \sum_{r=1}^{\infty} S_r(m, n) u^r = J_m(u) J_n(u). \]  \( \text{(3)} \)

Formula (3) implies an expression for the number of solutions of the system (1).

**Lemma 3** Let \( N = \gcd(m, n) + 1 \). Then the following equation holds

\[
J_{m,n} = \gcd(m, n) \sum_{u \mid P, v \mid Q} \frac{\mu(u)\mu(v)}{\gcd(u, v)} + \sum_{u \mid P, v \mid Q} \frac{\mu(u)\mu(v)(u + v)}{2\gcd(u, v)} + \sum_{u \mid P, v \mid Q} \frac{\mu(v)}{\gcd(u, v)} \sum_{\alpha^u = 1} \frac{1}{\alpha^{N-1}(\alpha - 1)}.
\]  \( \text{(4)} \)

In sums of type

\[ \sum_{\alpha^u = 1} \frac{1}{\alpha^{N-1}(\alpha^v - 1)} \]  \( \text{(5)} \)

the summation is over those roots of equation \( \alpha^u = 1 \) that are not the roots of equation \( \alpha^v = 1 \).

Sums (5) are called Dedekind sums. They are well-known in combinatorial analysis (e.g., see [3]).

To simplify (4) we use identities about Möbius function. They are 2-dimensional analogues of the classical formula

\[
\sum_{d \mid n} \frac{\mu(d)}{d} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right).
\]

An example of these identities is given by the following Lemma.

**Lemma 4** ([4]) \[ \sum_{u \mid m, v \mid n} \frac{\mu(u)\mu(v)}{\gcd(u, v)} = \prod_{r \mid \gcd(P, Q)} \left(1 - \frac{1}{r}\right). \]
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Dealing with Dedekind sums (5) we use the following useful statement. Let

\[ S_n(a) = \sum_{\alpha^k = 1} \frac{1}{\alpha^n(\alpha^a - 1)}, \]  

(6)

where summation is over those roots of equation \( x^b = 1 \) that are not the roots of equation \( x^a = 1 \). By \( m_0 \) we denote the smallest positive solution of equation

\[ ax \equiv -(n + a) \pmod{b}. \]

Let \( w(a, b) = m_0 - 1 \).

**Lemma 5**

\[ S_n(a) = \frac{b}{2} - \frac{\gcd(a, b)}{2 \lcm(a, b)} - w(a, b). \]  

(7)

**References**


