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Enumeration of walks reaching a line

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We enumerate walks in the plane $\mathbb{R}^2$, with steps East and North, that stop as soon as they reach a given line; these walks are counted according to the distance of the line to the origin, and we study the asymptotic behavior when the line has a fixed slope and moves away from the origin. When the line has a rational slope, we study a more general class of walks, and give exact as well as asymptotic enumerative results; for this, we define a nice bijection from our walks to words of a rational language. For a general slope, asymptotic results are obtained; in this case, the method employed leads us to find asymptotic results for a wider class of walks in $\mathbb{R}^m$.

Keywords: walk, generating function, rational language, singularity analysis

1 Introduction

In this work we consider primarily two classes of walks in the plane $\mathbb{R}^2$, noted $W_{a,\delta}^+$ and $W_{a,\delta}^-$, defined in the following manner:

**Definition 1** Let $a \in [0, 1] \text{ and } \delta \geq 0$ be real numbers. We denote by $D_{a,\delta}$ the line of $\mathbb{R}^2$ with slope $-a$, going through the point $(\delta, 0)$. An equation of $D_{a,\delta}$ is $y = -a(x - \delta)$. We denote by $W_{a,\delta}^+$ (resp. $W_{a,\delta}^-$) the set of walks in the plane $\mathbb{R}^2$ starting at the origin $O = (0, 0)$ with steps East or North, which end as soon as they reach the open (resp. closed) half plane above $D_{a,\delta}$. The cardinalities of the sets $W_{a,\delta}^+$ and $W_{a,\delta}^-$ are denoted respectively by $W_{a,\delta}^+$ and $W_{a,\delta}^-$. These definitions are illustrated on Figure 1.

These walks stop as soon as they cross the line $D_{a,\delta}$, those in $W_{a,\delta}^+$ having to go strictly beyond the line, whereas those in $W_{a,\delta}^-$ stop on it if they happen to touch it. We are interested in the enumeration of these walks according to the parameter $\delta$; that is, we fix the slope $-a$ of the line $D_{a,\delta}$, and study the numbers $W_{a,\delta}^+$ and $W_{a,\delta}^-$ in function of $\delta$. Note that, up to a constant factor $a$, $\delta$ represents the distance of the line $D_{a,\delta}$ to the origin.

We can now state our first theorem which gives all asymptotic results for $W_{a,\delta}^+$ and $W_{a,\delta}^-$ when $\delta$ goes to infinity.

**Theorem 1** Let $a \in [0, 1]$, and let $\lambda$ be the unique positive solution to the equation $\lambda^{-1} + \lambda^{-1/a} = 1$.

If $a = p/q > 0$ is a fixed rational number, where $p$ and $q$ are relatively prime positive integers, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{p(1 - \lambda^{-1/p})} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}\lambda^{[p\delta]/p}}$$

and

$$W_{a,\delta}^- \sim \frac{a}{p(\lambda^{1/p} - 1)} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}\lambda^{[p\delta]/p}}$$
hold when $\delta$ goes to infinity. If $a$ is irrational, then the asymptotic approximations

$$W^{+}_{a,\delta} \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta$$

and

$$W^{-}_{a,\delta} \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta$$

hold when $\delta$ goes to infinity.

As this theorem shows, the behavior of $W^{+}_{a,\delta}$ and $W^{-}_{a,\delta}$ depends on the rationality of the number $a$; if $a$ is rational, then we will find the generating function of the numbers $W^{+}_{a,n}$ and $W^{-}_{a,n}$. In this case, we will actually introduce another class of walks that includes $W^{+}_{a,n}$ and $W^{-}_{a,n}$ and find a bijection that sends walks to words of a rational language; various enumerative and asymptotic results derive from there. In the case of a general $a$, we will proceed differently, and start from an easily obtained functional equation to obtain asymptotic results. Our method is close to Erdős et al. (EHO+ 87), method that is also applicable to a wider class of walks defined in $\mathbb{R}^n$.

\section{Walks reaching a set of points}

As announced in the introduction, we now introduce a new class of walks that will include our original walks when the slope of $D_{a,\delta}$ is rational. The reader is advised to look at Figure 2 while reading the following definition.

\textbf{Definition 2 (V}_{d,n} \text{ and W}_{d,n}) Let $d = (d_i)_{i \geq 1}$ be an infinite sequence of positive integers, and let $e = (e_i)_{i \in \mathbb{N}}$ be the corresponding sequence of partial sums, defined by $e_0 = 0$ and $e_k = d_1 + d_2 + \cdots + d_k$, for $k \geq 1$. We associate to $d$ a set of points $V_d$ in the plane, with integer coordinates: the set $V_d \subseteq \mathbb{Z} \times \mathbb{N}$ consists in the origin $O$ together with, for every $k \geq 1$, the $d_k$ points with $y$-coordinate equal to $k$ and $x$-coordinate in $[-e_k, -e_{k-1} - 1]$.

For any integer $n$, $V_{d,n}$ is defined as the translated of $V_d$ by the vector $(n, 0)$. That is, $V_{d,n} = V_d + (n, 0)$. The generalized set of walks $W_{d,n}$ consists of the walks that start at the origin $O$, make steps East or North, and have their last points, and no other one, in $V_{d,n}$.

These walks are a generalization of our walks $W_{a,n}^+$ and $W_{a,n}^-$. Indeed, let $d_a^+$ and $d_a^-$ be the sequences whose $k$th terms are given respectively by $\lfloor k/a \rfloor - \lfloor k-1/a \rfloor$ and $\lfloor k/a \rfloor - \lfloor k-1/a \rfloor$. Then we have the following proposition:
Proposition 1 For every \( n \in \mathbb{N} \) and \( a \in [0, 1] \), we have the equalities
\[
\mathcal{W}^+_{a,n} = \mathcal{W}^+_{d^+_{a,n+1}} \text{ and } \mathcal{W}^-_{a,n} = \mathcal{W}^-_{d^-_{a,n}}.
\]

An interesting case happens when the sequence \( d \) is periodic. It is easy to see that \( d^+_i \) and \( d^-_i \) are periodic exactly when \( a \) is a rational number. If \( d = (d_1, \ldots, d_p) \) is a finite sequence, we will note \( V_{d,n} = V_{\bar{d},n} \) and \( W_{d,n} = W_{\bar{d},n} \), where \( \bar{d} \) is the periodic infinite sequence \((d_1, \ldots, d_p, d_1, \ldots, d_p, \ldots)\).

From now on \( d \) will stand for a finite sequence \( d = (d_1, \ldots, d_p) \) of positive integers. We define \( q = d_1 + \cdots + d_p \), and \( a = p/q \). To such a sequence we attach the following language on a finite alphabet (recall that a run in a finite word is a maximal factor composed of identical letters)

Definition 3 (Language \( \mathcal{L}_d \)) The language \( \mathcal{L}_d \) is the set of words \( w \) on the alphabet \( \Sigma = \{a_0, a_1, \ldots, a_{p-1}\} \) that satisfy the following conditions (where we set by convention \( d_0 = d_p \) and \( a_p = a_0 \)):

C1. \( w \) is the empty word, or its initial letter belongs to \( \{a_0, a_1\} \)

C2. for all \( i \), a run of \( a_i \) in \( w \) is terminal or is followed by a run of \( a_{i+1} \);

C3. for all \( i \), the runs of \( a_i \) in \( w \) are of length at least \( d_i \); this constraint does not apply to the last run, and, if \( w \) begins with \( a_0 \) it does not apply to the first run either.

We can finally state the theorem announced in the introduction:

Theorem 2 Let \( n \geq 0 \) be an integer. There exists an explicit bijection between walks in \( W_{d,n} \) and words of \( \mathcal{L}_d \) of length \( n \).

The language \( \mathcal{L}_d \) is rational, and we give an unambiguous rational expression that represents it. Then the existence of a bijection as stated in Theorem 2 allows us to explicit the generating function \( W_d(x) = \sum_{k=0}^\infty W_{d,k} x^k \) of the sequence \( (W_{d,n})_{n \in \mathbb{N}} \):

Theorem 3 The generating function \( W_d(x) \) has the following expression:

\[
W_d(x) = \frac{N(x)}{(1 - x)^p - x^q}, \text{ with } N(x) = (1 - x)^{p-2} + \sum_{i=1}^{p-2} x^{e_i + 1}(1 - x)^{p-2-i} + \sum_{k=e_{p-1}+1}^{e_{p-1}-1} x^k.
\]
Given a rational function, we can easily have access to asymptotic approximations of the coefficients of its series expansion, and we show that the first part of Theorem 1 can thus be obtained as a consequence of Theorem 3.

In fact, thanks to the bijection of Theorem 2, we can even find the bivariate generating function of the numbers \((W_{d,n,k})_{n,k}\) which enumerate walks in \(W_{d,n}\) of length \(k\). By the techniques of singularity analysis exposed in chapter 8 of (FS), we can then prove that the average length of a walk in \(W_{d,n}\) is asymptotically \(C_a \cdot n\) when \(n\) goes to infinity, where \(C_a\) is positive constant depending only on \(a\).

### 3 Asymptotic results in the general case

Let \(W^+_a\) be the function defined on \(\mathbb{R}\) by \(W^+_a(\delta) = 1\) if \(\delta < 0\), and by \(W^+_a(\delta) = W^+_{a,\delta}\) if \(\delta \geq 0\). Then, by decomposing walks according to their first step, one shows that \(W^+_a\) satisfies the following functional equation:

\[
\forall \delta \geq 0, \quad W^+_a(\delta) = W^+_a(\delta - 1/a) + W^+_a(\delta - 1).
\]

This equation and related ones have appeared in various contexts, and have been studied in numerous works, including (CG01; FK74; Pip93). Here we use a method inspired by the paper (EHO+87). This consists in interpreting Equation 1 as a “renewal equation”, so that its asymptotic behavior is given by the celebrated Renewal Limit Theorem (RLT) of probability theory; see Feller (Fel71) for all necessary background. Application of the RLT immediately leads to a proof of Theorem 1 as far as \(W^+_{a,\delta}\) is concerned.

It is then extended to \(W^-_{a,\delta}\) by finding simple relations between the two numbers.

Our walks have a natural generalization in any dimension. Let \(\vec{a} = (a_1, \ldots, a_m)\) be a vector in \(\mathbb{R}^m\), with all coordinates being positive, and \(H_\delta\) be the hyperplane of equation \(H_\delta : \quad a_1x_1 + \cdots + a_{m-1}x_{m-1} + a_m(x_m - \delta) = 0\). Then define \(W^+_{\vec{a},\delta}\) (resp. \(W^-_{\vec{a},\delta}\)) to be the numbers of walks in \(\mathbb{R}^m\) from the origin with steps in \(\{e_i\}_{1 \leq i \leq m}\) defined by the fact that their last points, and no other one, are “above \(H_\delta\)” (resp. “above or on \(H_\delta\)”).

Assume \(1 = a_m \leq a_1 \leq a_2 \leq \ldots \leq a_{m-1}\), and let \(\lambda\) designate the unique positive solution to \(\sum_{i=1}^m \lambda^{-a_i} = 1\). If all \(a_i\) are rational numbers and we write \(a_i = p_i/q_i\) in reduced form for each \(i\), we define \(q = \text{lcm}(q_i)\). Then the proof of the following theorem is proved along the same lines as described above:

**Theorem 4** Let \(\lambda\) and \(q\) be defined as above. Then we have the following asymptotics when \(\delta\) tends to \(\infty\):

(i) if at least one \(a_i\) is irrational, then

\[
W^+_{\vec{a},\delta} \sim \frac{m - 1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i} \cdot \lambda^\delta} \quad \text{and} \quad W^-_{\vec{a},\delta} \sim \frac{m - 1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i} \cdot \lambda^\delta},
\]

(ii) and if all \(a_i\) are rational, then

\[
W^+_{\vec{a},\delta} \sim \frac{m - 1}{q(1 - \lambda^{-1/q}) \cdot \sum_{i=1}^m a_i \lambda^{-a_i} \lambda^{[q\delta]}/q} \quad \text{and} \quad W^-_{\vec{a},\delta} \sim \frac{m - 1}{q(\lambda^{1/q} - 1) \cdot \sum_{i=1}^m a_i \lambda^{-a_i} \lambda^{[q\delta]}/q}.
\]

In fact, the same reasoning shows that similar approximations hold in the irrational case when the steps are allowed to be any finite number of non zero vectors with nonnegative coordinates.
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