Enumeration of walks reaching a line

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We enumerate walks in the plane $\mathbb{R}^2$, with steps East and North, that stop as soon as they reach a given line; these walks are counted according to the distance of the line to the origin, and we study the asymptotic behavior when the line has a fixed slope and moves away from the origin. When the line has a rational slope, we study a more general class of walks, and give exact as well as asymptotic enumerative results ; for this, we define a nice bijection from our walks to words of a rational language. For a general slope, asymptotic results are obtained; in this case, the method employed leads us to find asymptotic results for a wider class of walks in $\mathbb{R}^m$.

Keywords: walk, generating function, rational language, singularity analysis

1 Introduction

In this work we consider primarily two classes of walks in the plane $\mathbb{R}^2$, noted $\mathcal{W}^+_{a,\delta}$ and $\mathcal{W}^-_{a,\delta}$, defined in the following manner :

Definition 1 *Let $a \in [0,1[$ and $\delta \geqslant 0$ be real numbers. We denote by $\mathcal{D}_{a,\delta}$ the line of $\mathbb{R}^2$ with slope $-a$, going through the point $(\delta,0)$. An equation of $\mathcal{D}_{a,\delta}$ is $y=-a(x-\delta)$. We denote by $\mathcal{W}^+_{a,\delta}$ (*resp. *$\mathcal{W}^-_{a,\delta}$) the set of walks in the plane $\mathbb{R}^2$ starting at the origin $O=(0,0)$ with steps East or North, which end as soon as they reach the open (*resp. *closed) half plane above $\mathcal{D}_{a,\delta}$. The cardinalities of the sets $\mathcal{W}^+_{a,\delta}$ and $\mathcal{W}^-_{a,\delta}$ are denoted respectively by $W^+_{a,\delta}$ and $W^-_{a,\delta}$.*

These definitions are illustrated on Figure 1.

These walks stop as soon as they cross the line $\mathcal{D}_{a,\delta}$, those in $\mathcal{W}^+_{a,\delta}$ having to go strictly beyond the line, whereas those in $\mathcal{W}^-_{a,\delta}$ stop on it if they happen to touch it. We are interested in the enumeration of these walks according to the parameter δ; that is, we fix the slope $-a$ of the line $\mathcal{D}_{a,\delta}$, and study the numbers $W^+_{a,\delta}$ and $W^-_{a,\delta}$ in function of δ. Note that, up to a constant factor a, δ represents the distance of the line $\mathcal{D}_{a,\delta}$ to the origin.

We can now state our first theorem which gives all asymptotic results for $W^+_{a,\delta}$ and $W^-_{a,\delta}$ when δ goes to infinity.

Theorem 1 *Let $a \in]0,1]$, and let λ be the unique positive solution to the equation $\lambda^{-1}+\lambda^{-1/a}=1$.*

If $a=p/q>0$ is a fixed rational number, where p and q are relatively prime positive integers, then the asymptotic approximations

$$W^+_{a,\delta} \underset{\infty}{\sim} \frac{a}{p(1-\lambda^{-1/p})}\cdot\frac{1}{1-(1-a)\lambda^{-1}}\lambda^{\lfloor p\delta\rfloor/p} \quad \text{and} \quad W^-_{a,\delta} \underset{\infty}{\sim} \frac{a}{p(\lambda^{1/p}-1)}\cdot\frac{1}{1-(1-a)\lambda^{-1}}\lambda^{\lfloor p\delta\rfloor/p}$$

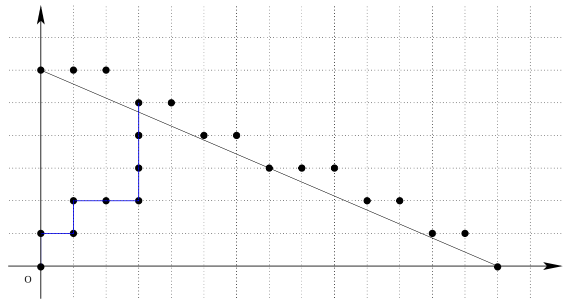

Fig. 1: An example of walk in $\mathcal{W}^+_{a,\delta}$ with $a = \frac{3}{7}$ and $n = 14$.

hold when δ goes to infinity. If a is irrational, then the asymptotic approximations

$$W^+_{a,\delta} \underset{\infty}{\sim} \frac{a}{\ln \lambda} \cdot \frac{1}{1-(1-a)\lambda^{-1}} \lambda^{\delta} \quad \textit{and} \quad W^-_{a,\delta} \underset{\infty}{\sim} \frac{a}{\ln \lambda} \cdot \frac{1}{1-(1-a)\lambda^{-1}} \lambda^{\delta}$$

hold when δ goes to infinity.

As this theorem shows, the behavior of $W^+_{a,\delta}$ and $W^-_{a,\delta}$ depends on the rationality of the number a; if a is rational, then we will find the generating function of the numbers $W^+_{a,n}$ and $W^-_{a,n}$. In this case, we will actually introduce another class of walks that includes $\mathcal{W}^+_{a,n}$ and $\mathcal{W}^-_{a,n}$ and find a bijection that sends walks to words of a rational language; various enumerative and asymptotic results derive from there. In the case of a general a, we will proceed differently, and start from an easily obtained functional equation to obtain asymptotic results. Our method is close to Erdös et al. (EHO+87), method that is also applicable to a wider class of walks defined in $\mathbb{R}^n$.

2 Walks reaching a set of points

As announced in the introduction, we now introduce a new class of walks that will include our original walks when the slope of $\mathcal{D}_{a,\delta}$ is rational. The reader is advised to look at Figure 2 while reading the following definition.

Definition 2 **($V_{d,n}$ and $\mathcal{W}_{d,n}$)** *Let $d = (d_i)_{i\geqslant 1}$ be an infinite sequence of positive integers, and let $e = (e_i)_{i\in\mathbb{N}}$ be the corresponding sequence of partial sums, defined by $e_0 = 0$ and $e_k = d_1 + d_2 + \cdots + d_k$, for $k \geqslant 1$. We associate to d a set of points V_d in the plane, with integer coordinates: the set $V_d \subset \mathbb{Z}\times\mathbb{N}$ consists in the origin O together with, for every $k \geqslant 1$, the d_k points with y-coordinate equal to k and x-coordinate in $[\![-e_k, -e_{k-1}-1]\!]$.*

For any integer n, $V_{d,n}$ is defined as the translated of V_d by the vector $(n, 0)$. That is, $V_{d,n} = V_d + (n, 0)$. The generalized set of walks *$\mathcal{W}_{d,n}$ consists of the walks that start at the origin O, make steps East or North, and have their last points, and no other one, in $V_{d,n}$.*

These walks are a generalization of our walks $\mathcal{W}^+_{a,n}$ and $\mathcal{W}^-_{a,n}$. Indeed, let d^+_a and d^-_a be the sequences whose kth terms are given respectively by $\lceil \frac{k}{a} \rceil - \lceil \frac{k-1}{a} \rceil$ and $\lfloor \frac{k}{a} \rfloor - \lfloor \frac{k-1}{a} \rfloor$. Then we have the following proposition :

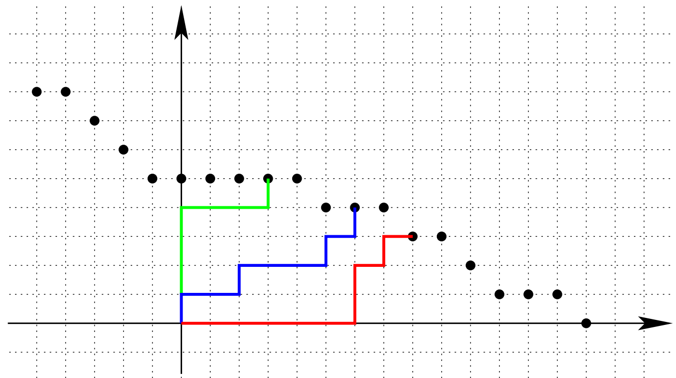

Fig. 2: A set $V_{d,n}$ with examples of walks in $W_{d,n}$. Here $n = 14$ and $d = (3, 1, 2, 3, 6, 1, 1, \ldots)$

Proposition 1 *For every $n \in \mathbb{N}$ and $a \in]0, 1]$, we have the equalities*

$$\mathcal{W}^+_{a,n} = \mathcal{W}_{d^+_a,n+1} \text{ and } \mathcal{W}^-_{a,n} = \mathcal{W}_{d^-_a,n}.$$

An interesting case happens when the sequence d is periodic. It is easy to see that d^+_a and d^-_a are periodic exactly when a is a rational number. If $d = (d_1, \ldots, d_p)$ is a finite sequence, we will note $V_{d,n} = V_{\bar{d},n}$ and $\mathcal{W}_{d,n} = \mathcal{W}_{\bar{d},n}$, where $\bar{d}$ is the periodic infinite sequence $(d_1, \ldots, d_p, d_1, \ldots, d_p, \ldots)$.

From now on d will stand for a finite sequence $d = (d_1, \ldots, d_p)$ of positive integers. We define $q = d_1 + \cdots d_p$, and $a = p/q$. To such a sequence we attach the following language on a finite alphabet (recall that a *run* in a finite word is a maximal factor composed of identical letters)

Definition 3 (Language $\mathcal{L}_d$) *The language $\mathcal{L}_d$ is the set of words w on the alphabet $\Sigma = \{a_0, a_1, \ldots, a_{p-1}\}$ that satisfy the following conditions (where we set by convention $d_0 = d_p$ and $a_p = a_0$):*

C1. w is the empty word, or its initial letter belongs to $\{a_0, a_1\}$

C2. for all i, a run of a_i in w is terminal or is followed by a run of a_{i+1};

C3. for all i, the runs of a_i in w are of length at least d_i; this constraint does not apply to the last run, and, if w begins with a_0, it does not apply to the first run either.

We can finally state the theorem announced in the introduction:

Theorem 2 *Let $n \geqslant 0$ be an integer. There exists an explicit bijection between walks in $\mathcal{W}_{d,n}$ and words of $\mathcal{L}_d$ of length n.*

The language $\mathcal{L}_d$ is rational, and we give an unambiguous rational expression that represents it. Then the existence of a bijection as stated in Theorem 2 allows us to explicit the generating function $W_d(x) = \sum_{k=0}^{\infty} W_{d,k} x^k$ of the sequence $(W_{d,n})_{n \in \mathbb{N}}$:

Theorem 3 *The generating function $W_d(x)$ has the following expression:*

$$W_d(x) = \frac{N(x)}{(1-x)^p - x^q}, \text{ with } N(x) = (1-x)^{p-2} + \sum_{i=1}^{p-2} x^{e_i+1}(1-x)^{p-2-i} + \sum_{k=e_{p-1}+1}^{e_p-1} x^k.$$

Given a rational function, we can easily have access to asymptotic approximations of the coefficients of its series expansion, and we show that the first part of Theorem 1 can thus be obtained as a consequence of Theorem 3.

In fact, thanks to the bijection of Theorem 2, we can even find the bivariate generating function of the numbers $(W_{d,n,k})_{n,k}$ which enumerate walks in $\mathcal{W}_{d,n}$ of length k. By the techniques of singularity analysis exposed in chapter 8 of (FS), we can then prove that the average length of a walk in $\mathcal{W}_{d,n}$ is asymptotically $C_a \cdot n$ when n goes to infinity, where C_a is positive constant depending only on a.

3 Asymptotic results in the general case

Let W_a^+ be the function defined on $\mathbb{R}$ by $W_a^+(\delta) = 1$ if $\delta < 0$, and by $W_a^+(\delta) = W_{a,\delta}^+$ if $\delta \geqslant 0$. Then, by decomposing walks according to their first step, one shows that W_a^+ satisfies the following functional equation :

$$\forall \delta \geqslant 0, \quad W_a^+(\delta) = W_a^+(\delta - 1/a) + W_a^+(\delta - 1). \tag{1}$$

This equation and related ones have appeared in various contexts, and have been studied in numerous works, including (CG01; FK74; Pip93). Here we use a method inspired by the paper (EHO$^+$87). This consists in interpreting Equation 1 as a "renewal equation", so that its asymptotic behavior is given by the celebrated *Renewal Limit Theorem* (RLT) of probability theory; see Feller (Fel71) for all necessary background. Application of the RLT immediately leads to a proof of Theorem 1 as far as $W_{a,\delta}^+$ is concerned. It is then extended to $W_{a,\delta}^-$ by finding simple relations between the two numbers.

Our walks have a natural generalization in any dimension. Let $\bar{a} = (a_1, \ldots, a_m)$ be a vector in $\mathbb{R}^m$, with all coordinates being positive, and $\mathcal{H}_\delta$ be the hyperplane of equation $\mathcal{H}_\delta : \quad a_1x_1 + \cdots + a_{m-1}x_{m-1} + a_m(x_m - \delta) = 0$. Then define $W_{\bar{a},\delta}^+$ (*resp.* $W_{\bar{a},\delta}^-$) to be the numbers of walks in $\mathbb{R}^m$ from the origin with steps in $\{e_i\}_{1 \leqslant i \leqslant m}$ defined by the fact that their last points, and no other one, are "above $\mathcal{H}_\delta$" (*resp.* "above or on $\mathcal{H}_\delta$").

Assume $1 = a_m \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{m-1}$, and let λ designate the unique positive solution to $\sum_{i=1}^m \lambda^{-a_i} = 1$. If all a_i are rational numbers and we write $a_i = p_i/q_i$ in reduced form for each i, we define $q = \mathrm{lcm}(q_i)$. Then the proof of the following theorem is proved along the same lines as described above :

Theorem 4 *Let λ and q be defined as above. Then we have the following asymptotics when δ tends to ∞:*

(i) if at least one a_i is irrational, then

$$W_{\bar{a},\delta}^+ \underset{\infty}{\sim} \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta \quad \textit{and} \quad W_{\bar{a},\delta}^- \underset{\infty}{\sim} \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta,$$

(ii) and if all a_i are rational, then

$$W_{\bar{a},\delta}^+ \underset{\infty}{\sim} \frac{m-1}{q(1-\lambda^{-1/q}) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q} \quad \textit{and} \quad W_{\bar{a},\delta}^- \underset{\infty}{\sim} \frac{m-1}{q(\lambda^{1/q}-1) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q}.$$

In fact, the same reasoning shows that similar approximations hold in the irrational case when the steps are allowed to be any finite number of non zero vectors with nonnegative coordinates.

Acknowledgements

I would like to thank Yves Verhoeven who is at the origin of this work and helped me on many occasions during my researches.

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