Survival probability of a critical multi-type branching process in random environment

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Introduction

Branching processes in random environment constitute an important part of the theory of branching processes (see, for example, (1, 2), (4-7), (9-14)). A branching process in random environment was first considered by Smith and Wilkinson (10). The subsequent papers (3, 7), (11) investigated single- and multi-type Galton-Watson processes in random environment. The asymptotics of the survival probability of the critical branching processes in a random environment generated by a sequence of independent identically distributed random variables under the condition $\mathbb{E}X^2 < \infty$ for the increment $X$ of the associated random walk was found in (6), (9) for single-type processes, and in (4) for multi-type processes. Recent papers (1), (5), (12-14) study the survival probability for an extended class of the critical single-type branching processes in random environment where the case $\mathbb{E}X^2 = \infty$ is not excluded and, moreover, $\mathbb{E}X$ may not exist. The present paper investigates an extended class of multi-type critical branching processes in random environment whose associated random walks satisfy the Doney-Spitzer condition. In particular, we generalize some results established in (1) and (4) concerning the asymptotic behavior of survival probability.

Let $Z(n) = (Z_1(n), ..., Z_p(n))$, $n = 0, 1, ...$, be a $p$-type Galton-Watson branching process in a random environment. This process can be described as follows.

Let $N_0 = \{0, 1, 2, \ldots\}$ and $N_0^p$ be the set of all vectors $t = (t_1, ..., t_p)$ with non-negative integer coordinates. Denote by $(\Delta, B(\Delta))$ a set of probability measures on $N_0^p$ with $\sigma$-algebra $B(\Delta)$ of Borel sets endowed with the metric of total variation, and by $(\Delta, B(\Delta))$ the $p$-times product of the space $(\Delta, B(\Delta))$ on itself. Let $F = (F^{(1)}, ..., F^{(p)})$ be a random variable (random measure) taking values in $(\Delta, B(\Delta))$. An infinite sequence $\Pi = (F_0, F_1, F_2, \ldots)$ of independent identically distributed copies of $F$ is said to form a random environment and we will say that $F$ generates $\Pi$. A sequence of random $p$-dimensional vectors $Z(0), Z(1), Z(2), \ldots$ with non-negative integer coordinates is called a $p$-type branching process in random environment $\Pi$, if $Z(0)$ is independent of $\Pi$ and for all $n \geq 0$, $z = (z_1, ..., z_p) \in N_0^p$ and $f_0, f_1, \ldots \in \Delta$

$$
\mathcal{L}(Z(n+1) \mid Z(n) = (z_1, ..., z_p), \Pi = (f_1, f_2, \ldots)) = \\
\mathcal{L}(Z(n+1) \mid Z(n) = (z_1, ..., z_p), F_n = f_n) = \\
\mathcal{L}((\xi_{n,1}^{(1)} + \cdots + \xi_{n,z_1}^{(1)}), \cdots, (\xi_{n,1}^{(p)} + \cdots + \xi_{n,z_p}^{(p)})),
$$

where $f_0 = (f_0^{(1)}, f_0^{(2)}, ..., f_0^{(p)}) \in \Delta$, $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, ..., \xi_{n,z_i}^{(i)}$, $i = 1, ..., p$, are independent $p$-dimensional random vectors, and for each $i = 1, ..., p$ the random vectors $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, ..., \xi_{n,z_i}^{(i)}$ are identically distributed according to the measure $f_0^{(i)}$. Relation (1) defines a branching Galton-Watson process $Z(n)$ in random environment which describes the evolution of a particle population $Z(n) = (Z_1(n), ..., Z_p(n))$, $n = 0, 1, ..., Z_i(n), i = 1, ..., p$, is the number of type $i$ particles in the $n$-th generation.

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This population evolves as follows. If $F_n = f_n$ then each of the $Z_i(n)$ particles of type $i$ existing at the time $n$, produces offspring in accordance with the $p$–dimensional probability measure $f_n^{(i)}$ independently of the reproduction of other particles. Thus, the $i$–th component of the vector $Z(n + 1) = (Z_1(n + 1), ..., Z_p(n + 1))$ is equal to the number of type $i$ particles among all direct descendants of the particles of the $n$–th generation. The distribution of $Z(0)$ will be specified later.

### The main results

Let $J_p$ be the set of all column vectors $s = (s_1, ..., s_p)^T, 0 \leq s_i \leq 1, i = 1, \ldots, p$. For $s \in J_p$ and $t \in \mathbb{N}_0^p$ set $s^t = \prod_{i=1}^p s_i^t$. Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with $F = (F^{(1)}, \ldots, F^{(p)})$ generating functions $F$ a random $p$–dimensional column vector $F(s) = (F^{(1)}(s), \ldots, F^{(p)}(s))^T, s \in J_p$, whose components are $p$–dimensional (random) generating functions $F^{(i)}(s)$ corresponding to $F^{(i)}(s), 1 \leq i \leq p$:

$$F^{(i)}(s) = \sum_{t \in \mathbb{N}_0^p} F^{(i)}(\{t\}) s^t, s \in J_p.$$ 

In a similar way we associate with the component $F_n = (F_n^{(1)}, \ldots, F_n^{(p)}), n \geq 0$, of the random environment $\Pi = (F_0, F_1, F_2, \ldots)$ a random vector $F_n(s) = (F_n^{(1)}(s), \ldots, F_n^{(p)}(s))^T, s \in J_p$, the components of which are multidimensional (random) generating functions $F_n^{(i)}(s)$, corresponding to $F^{(i)}(s), 1 \leq i \leq p$,

$$F_n^{(i)}(s) = \sum_{t \in \mathbb{N}_0^p} F_n^{(i)}(\{t\}) s^t.$$ 

Let $e_j, j = 1, \ldots, p$, be the $p$–dimensional row vector whose $j$–th component is equal to 1 and the others are zeros, $0 = (0, \ldots, 0)$ be the $p$–dimensional row vector all whose components are zeros, and let $T = (1, \ldots, 1)^T$ be the $p$–dimensional column vector all whose components are equal to 1. For $x = (x_1, ..., x_p)$ and $y = (y_1, ..., y_p)^T$ we set $|x| = \sum_{i=1}^p |x_i|, |y| = \sum_{i=1}^p |y_i|$, $(x, y) = \sum_{i=1}^p x_i y_i$. Let $A = ||A(i, j)||_{i,j=1}^p$ be an arbitrary positive $p \times p$ matrix. Denote by $\rho(A)$ the Perron root of $A$ and by $v(A) = (u_1(A), \ldots, u_p(A))^T$ and $v(A) = (v_1(A), \ldots, v_p(A))$ the right and left eigenvectors of $A$ corresponding to the eigenvalue $\rho(A)$ and such that $|v(A)| = 1, \ v(A), u(A)) = 1$.

For vector–valued generating functions $F(s)$ and $F_n(s)$ we introduce the mean matrices

$$M = M(F) = ||M(i, j)||_{i,j=1}^p = \left| \frac{\partial F^{(i)}(T)}{\partial s_j} \right|_{i,j=1}^p$$

and

$$M_n = M_n(F_n) = ||M_n(i, j)||_{i,j=1}^p = \left| \frac{\partial F_n^{(i)}(T)}{\partial s_j} \right|_{i,j=1}^p.$$ 

Let $C_\alpha, 0 < \alpha < 1$, be the class of all matrices $A = ||A(i, j)||_{i,j=1}^p$ such that

$$\alpha \leq \frac{A(i_1, j_1)}{A(i_2, j_2)} \leq \alpha^{-1}, \ 1 \leq i_1, i_2, j_1, j_2 \leq p.$$ 

One of our basic hypotheses is the following condition.

**Assumption A0.** There exist a number $0 < \alpha < 1$ and a positive row vector $v = (v_1, \ldots, v_p), |v| = 1$, such that, with probability 1

$$M = M(F) \in C_\alpha,$$

and

$$v M = \rho(M)v.$$ 

Set $\rho = \rho(M), \rho_n = \rho(M_n), n \geq 0$. It is not difficult to see that in our settings $X := \ln \rho, X_i := \ln \rho_{i-1}, i \geq 1$, are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk $S = (S_0, S_1, \ldots)$, where

$$S_n = X_1 + \cdots + X_n, n \geq 1, \ S_0 = 0.$$
**Assumption A1.** There exists a number $0 < a < 1$ such that

$$
\mathbb{P}(S_n > 0) \to a, \ n \to \infty.
$$

(3)

Extending the known classification of single-type branching processes in random environment (see (1), (2), we call a $p$-type branching process $Z(n), n \geq 0$, in random environment II critical if its associated random walk is of the oscillating type, i.e., $\limsup_{n \to \infty} S_n = +\infty$ a.s. and $\liminf_{n \to \infty} S_n = -\infty$ a.s. It is known that any random walk satisfying Assumption A1 oscillates. From now on we consider only critical $p$-type branching processes in random environment.

Let $0 =: \gamma_0 < \gamma_1 < \ldots$ be the strict descending ladder epochs of $S$. Put

$$
V(x) := \sum_{i=0}^{\infty} \mathbb{P}(S_{\gamma_i} \geq -x), \ x \geq 0; \ V(x) = 0, \ x < 0.
$$

Since $S$ is oscillating, the following relation holds (3):

$$
\mathbb{E}V(x + X) = V(x), \ x \geq 0.
$$

(4)

For $d \in \mathbb{N}_0$ set

$$
O_d = \{t = (t_1, \ldots, t_p) \in \mathbb{N}_0^p \mid t_i < d, i = 1, \ldots, p\}, \ U_d = \mathbb{N}_0^p \setminus O_d.
$$

Introduce the random variable

$$
\kappa(d) = \sum_{t \in O_d} \sum_{i=1}^{p} v_i \sum_{j,k=1}^{p} \mathbb{P}^{(i)}((t)) t_j t_k / \rho^2, \ d \in \mathbb{N}_0,
$$

where $v = (v_1, \ldots, v_p)$ is from (2). Our next condition is connected with the random variable $\kappa(d)$, which is a generalization of the standardized truncated second moment of the reproduction law to the multi-type case.

**Assumption A2.** There exist $\varepsilon > 0$ and $d \in \mathbb{N}_0$ such that

$$
\mathbb{E}(\ln \kappa(d))^{1/\alpha + \varepsilon} < \infty, \ \mathbb{E} \left( V(X)(\ln \kappa(d))^{1/\alpha + \varepsilon} \right) < \infty.
$$

Let $T = \min\{n \geq 0 : Z(n) = 0\}$ be the extinction moment for $Z(n)$. Introduce the random variables

$$
Q^{(i)}(n) = \mathbb{P}(T > n | Z(0) = e_i, \Pi), \ Q(n) = (Q^{(1)}(n), \ldots, Q^{(p)}(n)),
$$

and let

$$
q_i(k) = \mathbb{P}(T > k | Z(0) = e_i) = \mathbb{E}Q^{(i)}(k).
$$

Note that under Assumptions A0 and A1, $Q^{(i)}(n) \to 0$ P-a.s. as $n \to \infty$ for all $1 \leq i \leq p$, since $\mathbb{P}$-a.s.

$$
(v, Q(n)) \leq \min_{0 \leq k \leq n-1} |vM_0 \cdots M_k| \leq \exp\{\min_{0 \leq k \leq n-1} S_k\} \to 0
$$

as $n \to \infty$. Denote by $u(n) = (u_1(n), \ldots, u_p(n))^T := u(M_0 \cdots M_n), n \geq 0$, the right eigenvector of the product $M_0 \cdots M_n$, corresponding to the Perron root $\rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n$. To investigate the asymptotic behavior of $q_i(n)$ and $Q^{(i)}(n)$ as $n \to \infty$ we need the following statement describing the behavior of $u(n)$.

**Theorem 1** If Assumption A0 is valid, then there exist a random vector $u = (u_1, \ldots, u_p)^T$ and a function $g(n) \geq 0, g(n) \to 0, n \to \infty$, such that with probability 1

$$
|u_i(n) - u_i| \leq g(n), \ i = 1, \ldots, p.
$$

In addition,

$$
(v, u) = 1, \ \alpha \leq u_i \leq 1/v^*,
$$

where $v^* = \min(v_1, \ldots, v_p)$ and $v = (v_1, \ldots, v_p)$ is from (2).

The following statement describes the behavior of $Q(n)$ as $n \to \infty$. 


Theorem 2 Assume Assumptions A0 and A1. Then $P - a.s.$, as $n \to \infty$,
$$Q_i(n) \to u_i, \quad i = 1, \ldots, p,$$
where $u = (u_1, \ldots, u_p)$ is from Theorem 1.

Now we are ready to formulate the main result of the paper.

Theorem 3 Assume Assumptions A0, A1, and A2. Then, as $n \to \infty$,
$$q_i(n) \sim c_i n^{-(1-a)} l(n), \quad c_i > 0, \quad i = 1, \ldots, p,$$
where $l(n)$ is a function slowly varying at infinity.

Note that under our approach one of the key facts to prove Theorems 1 and 2 is convergence in distribution, as $n \to \infty$, of the products $\prod_{i=1}^{n} M_i \rho_i^{-1}$ of random matrices to a limit matrix whose distribution is not concentrated at zero matrix. It is known that for $p = 2$ the products $\prod_{i=1}^{n} M_i \rho_i^{-1}$ of the positive bounded independent identically distributed $2 \times 2$ matrices $A_i = M_i \rho_i^{-1}$ converges in distribution, as $n \to \infty$, to a limit matrix whose distribution is not concentrated at zero matrix if and only if all the matrices $A_i$ have a common positive right or left eigenvector. Hence, for the 2-type process $Z(n)$ our assumption on existence of a common positive left eigenvector of the matrices $M$ is essential indeed.

Observe also that Assumption A1 covers non-degenerate random walks with zero mean and finite variance of there increments, as well as all non-degenerate symmetric random walks. In these cases $a = 1/2$. Another example when Assumption A1 is valid gives the random walk, whose increments have distribution belonging to the domain of attraction of a stable law.

In conclusion we give an example where Assumption A2 is fulfilled (given that Assumption A0 is valid as well). Clearly, if the measure $F$ generating our random environment has a bounded support, i.e., if there exists a $p$-dimensional cube $B = [0, b]^p$, $b > 0$, such that $P(F(B) = 1) = 1$, then Assumption A2 holds since $\kappa(d) = 0$ $P$ - a.s. for $d > b$.

One can show that if $F$ satisfies Assumption A0 and
$$F(s) = 1 - \frac{M(1 - s)}{1 + \gamma(1 - s)}, \quad s \in J^p,$$
where the $p$-dimensional random row vector $\gamma$ with positive components and the random matrix $M$ are such that the components of the vector $y = (M\gamma)/|\gamma|$ are uniformly bounded from below then Assumption A2 holds true.

Note that if the distribution of $X = \ln \rho$ has a regular varying tail then Assumptions A1 and A2 can be replaced by the following hypotheses (see (13) or (14):

Assumption A1'. There exist constants $c_n, n \geq 0$, such that as $n \to \infty$ the scaled sums $c_n S_n$ converge weakly to a stable distribution $\mu$ with parameter $\beta \in (0, 2]$. The limit law $\mu$ is not one-side, i.e., $0 < \mu(\mathbb{R}^+) < 1$.

Assumption A2'. There exist $\varepsilon > 0$ and $d \in \mathbb{N}_0$ such that
$$E(\ln^+ \kappa(d))^{\beta + \varepsilon} < \infty,$$
where $\beta$ is from Assumption A1'.

Note that Assumption A1' implies the validity of Assumption A1 with $a = \mu(\mathbb{R}^+)$, and Assumption A2 is stronger than Assumption A2' since $a \beta \leq 1$.

Corollary 1 Assume Assumptions A0 and A1'. Then the statement of Theorem 3 remains true.

Using the proof of Theorem 3 and results of paper (1), one can obtain also the following statement.

Theorem 4 Assume Assumptions A0, A1', and A2'. Then the statement of Theorem 3 remains true.

References


