Survival probability of a critical multi-type branching process in random environment

Elena Dyakonova[1†]

[1]*Department of Discrete Mathematics, Steklov Mathematical Institute, Gubkin St. 8, 119991 Moscow, Russia*

We study a multi-type branching process in i.i.d. random environment. Assuming that the associated random walk satisfies the Doney-Spitzer condition, we find the asymptotics of the survival probability at time n as $n \to \infty$.

Keywords: branching processes in random environment, Doney-Spitzer condition, survival probability

Introduction

Branching processes in random environment constitute an important part of the theory of branching processes (see, for example, (1), (2), (4)-(7), (9)-(14)). A branching process in random environment was first considered by Smith and Wilkinson (10). The subsequent papers (2), (7), (11) investigated single- and multi-type Galton-Watson processes in random environment. The asymptotics of the survival probability of the critical branching processes in a random environment generated by a sequence of independent identically distributed random variables under the condition $\mathbf{E}X^2 < \infty$ for the increment X of the associated random walk was found in (6), (9) for single-type processes, and in (4) for multi-type processes. Recent papers (1), (5), (12)-(14) study the survival probability for an extended class of the critical single-type branching processes in random environment where the case $\mathbf{E}X^2 = \infty$ is not excluded and, moreover, $\mathbf{E}X$ may not exist. The present paper investigates an extended class of multi-type critical branching processes in random environment whose associated random walks satisfy the Doney-Spitzer condition. In particular, we generalize some results established in (1) and (4) concerning the asymptotic behavior of survival probability.

Let $Z(n) = (Z_1(n), ..., Z_p(n)),\ n = 0, 1, ...,$ be a p-type Galton-Watson branching process in a random environment. This process can be described as follows.

Let $\mathbf{N}_0 = \{0, 1, 2, ...\}$ and $\mathbf{N}_0^p$ be the set of all vectors $t = (t_1, ..., t_p)$ with non-negative integer coordinates. Denote by $(\Delta_1, \mathcal{B}(\Delta_1))$ a set of probability measures on $\mathbf{N}_0^p$ with $\sigma-$algebra $\mathcal{B}(\Delta_1)$ of Borel sets endowed with the metric of total variation, and by $(\Delta, \mathcal{B}(\Delta))$ the $p-$times product of the space $(\Delta_1, \mathcal{B}(\Delta_1))$ on itself. Let $\mathbf{F} = (\mathbf{F}^{(1)}, ..., \mathbf{F}^{(p)})$ be a random variable (random measure) taking values in $(\Delta, \mathcal{B}(\Delta))$. An infinite sequence $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, ...)$ of independent identically distributed copies of $\mathbf{F}$ is said to form a random environment and we will say that $\mathbf{F}$ generates Π. A sequence of random $p-$dimensional vectors $Z\,(0)\,, Z\,(1)\,, Z\,(2)\,, ...$ with non-negative integer coordinates is called a $p-$type branching process in random environment Π, if $Z\,(0)$ is independent of Π and for all $n \geq 0, z = (z_1, ..., z_p) \in \mathbf{N}_0^p$ and $f_0, f_1, ... \in \Delta$

$$\begin{aligned}
&\mathcal{L}(Z\,(n+1)\ |\ Z(n) = (z_1, ..., z_p), \Pi = (f_1, f_2, ...)) \\
&\quad = \mathcal{L}(Z\,(n+1)\ |\ Z(n) = (z_1, ..., z_p), \mathbf{F}_n = f_n) \\
&\quad = \mathcal{L}((\xi_{n,1}^{(1)} + \cdots + \xi_{n,z_1}^{(1)}) + (\xi_{n,1}^{(2)} + \cdots + \xi_{n,z_2}^{(2)}) + \cdots + (\xi_{n,1}^{(p)} + \cdots + \xi_{n,z_p}^{(p)})),
\end{aligned} \tag{1}$$

where $f_n = (f_n^{(1)}, f_n^{(2)}, ..., f_n^{(p)}) \in \Delta$, $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, ..., \xi_{n,z_i}^{(i)}, i = 1, ..., p$, are independent $p-$dimensional random vectors, and for each $i = 1, ..., p$ the random vectors $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, ..., \xi_{n,z_i}^{(i)}$ are identically distributed according to the measure $f_n^{(i)}$. Relation (1) defines a branching Galton-Watson process $Z(n)$ in random environment which describes the evolution of a particle population $Z(n) = (Z_1(n), ..., Z_p(n)),\ n = 0, 1, ...,$ where $Z_i(n), i = 1, ..., p$, is the number of type i particles in the $n-$th generation.

†Supported by RFBR grant 05-01-00035, grant Scientific School-4129.2006.1 and by the program "Contemporary Problems of Theoretical Mathematics" of the Russian Academy of Sciences.

This population evolves as follows. If $\mathbf{F}_n = f_n$ then each of the $Z_i(n)$ particles of type i existing at the time n, produces offspring in accordance with the $p-$dimensional probability measure $f_n^{(i)}$ independently of the reproduction of other particles. Thus, the i-th component of the vector $Z(n+1) = (Z_1(n+1), ..., Z_p(n+1))$ is equal to the number of type i particles among all direct descendants of the particles of the $n-$th generation. The distribution of $Z(0)$ will be specified later.

The main results

Let J^p be the set of all column vectors $s = (s_1, ..., s_p)^T, 0 \leq s_i \leq 1, i = 1, ..., p$. For $s \in J^p$ and $t \in \mathbf{N}_0^p$ set $s^t = \prod_{i=1}^p s_i^{t_i}$. Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with $\mathbf{F} = (\mathbf{F}^{(1)}, ..., \mathbf{F}^{(p)})$ generating Π a random $p-$dimensional column vector $F(s) = (F^{(1)}(s), ..., F^{(p)}(s))^T$, $s \in J^p$, whose components are $p-$dimensional (random) generating functions $F^{(i)}(s)$ corresponding to $\mathbf{F}^{(i)}, 1 \leq i \leq p$:

$$F^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}^{(i)}(\{t\}) s^t, s \in J^p.$$

In a similar way we associate with the component $\mathbf{F}_n = (\mathbf{F}_n^{(1)}, ..., \mathbf{F}_n^{(p)}), n \geq 0$, of the random environment $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, ...)$ a random vector $F_n(s) = (F_n^{(1)}(s), ..., F_n^{(p)}(s))^T$, $s \in J^p$, the components of which are multidimensional (random) generating functions $F_n^{(i)}(s)$, corresponding to $\mathbf{F}_n^{(i)}, 1 \leq i \leq p$,

$$F_n^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}_n^{(i)}(\{t\}) s^t.$$

Let $e_j, j = 1, ..., p$, be the $p-$dimensional row vector whose $j-$th component is equal to 1 and the others are zeros, $\overline{0} = (0, ..., 0)$ be the $p-$dimensional row vector all whose components are zeros, and let $\overline{1} = (1, ..., 1)^T$ be the $p-$dimensional column vector all whose components are equal to 1. For $x = (x_1, ..., x_p)$ and $y = (y_1, ..., y_p)^T$ we set $|x| = \sum_{i=1}^p |x_i|, |y| = \sum_{i=1}^p |y_i|, (x, y) = \sum_{i=1}^p x_i y_i$. Let $A = ||A(i,j)||_{i,j=1}^p$ be an arbitrary positive $p \times p$ matrix. Denote by $\rho(A)$ the Perron root of A and by $u(A) = (u_1(A), ..., u_p(A))^T$ and $v(A) = (v_1(A), ..., v_p(A))$ the right and left eigenvectors of A corresponding to the eigenvalue $\rho(A)$ and such that

$$|v(A)| = 1, \ (v(A), u(A)) = 1.$$

For vector-valued generating functions $F(s)$ and $F_n(s)$ we introduce the mean matrices

$$M = M(\mathbf{F}) = ||M(i,j)||_{i,j=1}^p = \left|\left| \frac{\partial F^{(i)}(\overline{1})}{\partial s_j} \right|\right|_{i,j=1}^p$$

and

$$M_n = M_n(\mathbf{F}_n) = ||M_n(i,j)||_{i,j=1}^p = \left|\left| \frac{\partial F_n^{(i)}(\overline{1})}{\partial s_j} \right|\right|_{i,j=1}^p.$$

Let $\mathcal{C}_\alpha, 0 < \alpha < 1$, be the class of all matrices $A = ||A(i,j)||_{i,j=1}^p$ such that

$$\alpha \leq \frac{A(i_1, j_1)}{A(i_2, j_2)} \leq \alpha^{-1}, \ 1 \leq i_1, i_2, j_1, j_2 \leq p.$$

One of our basic hypotheses is the following condition.

Assumption A0. There exist a number $0 < \alpha < 1$ and a positive row vector $v = (v_1, ..., v_p), |v| = 1$, such that, with probability 1

$$M = M(\mathbf{F}) \in \mathcal{C}_\alpha,$$

and

$$vM = \rho(M)v. \tag{2}$$

Set $\rho = \rho(M), \rho_n = \rho(M_n), n \geq 0$. It is not difficult to see that in our settings $X := \ln \rho$, $X_i := \ln \rho_{i-1}, i \geq 1$, are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk $S = (S_0, S_1, ...)$, where

$$S_n = X_1 + \cdots + X_n, \ n \geq 1, \ S_0 = 0.$$

Assumption A1. There exists a number $0 < a < 1$ such that

$$\mathbf{P}(S_n > 0) \to a,\ n \to \infty. \tag{3}$$

Extending the known classification of single-type branching processes in random environment (see (1), (12)) , we call a $p-$type branching process $Z(n), n \geq 0$, in random environment Π critical if its associated random walk is of the oscillating type, i.e., $\limsup_{n\to\infty} S_n = +\infty$ a.s. and $\liminf_{n\to\infty} S_n = -\infty$ a.s. It is known that any random walk satisfying Assumption A1 oscillates. From now on we consider only critical p-type branching processes in random environment.

Let $0 =: \gamma_0 < \gamma_1 < ...$ be the strict descending ladder epochs of S. Put

$$V(x) := \sum_{i=0}^{\infty} \mathbf{P}(S_{\gamma_i} \geq -x),\ x \geq 0;\ V(x) = 0,\ x < 0.$$

Since S is oscillating, the following relation holds (3):

$$\mathbf{E}V(x + X) = V(x),\ x \geq 0. \tag{4}$$

For $d \in \mathbf{N}_0$ set

$$O_d = \{t = (t_1, ..., t_p) \in \mathbf{N}_0^p \mid t_i < d, i = 1, ..., p\},\ U_d = \mathbf{N}_0^p \setminus O_d.$$

Introduce the random variable

$$\kappa(d) = \sum_{t\in U_d} \sum_{i=1}^{p} v_i \sum_{j,k=1}^{p} \mathbf{F}^{(i)}(\{t\}) t_j t_k/\rho^2,\ d \in \mathbf{N}_0,$$

where $v = (v_1, ..., v_p)$ is from (2). Our next condition is connected with the random variable $\kappa(d)$, which is a generalization of the standardized truncated second moment of the reproduction law to the multi-type case.

Assumption A2. There exist $\varepsilon > 0$ and $d \in \mathbf{N}_0$ such that

$$\mathbf{E}(\ln^+ \kappa(d))^{1/a+\varepsilon} < \infty,\ \mathbf{E}\left(V(X)(\ln^+ \kappa(d))^{1/a+\varepsilon}\right) < \infty.$$

Let $T = \min\{n \geq 0 : Z(n) = \overline{0}\}$ be the extinction moment for $Z(n)$. Introduce the random variables

$$Q^{(i)}(n) = \mathbf{P}(T > n | Z(0) = e_i, \Pi),\ Q(n) = (Q^{(1)}(n), ..., Q^{(p)}(n)),$$

and let

$$q_i(k) = \mathbf{P}(T > k | Z(0) = e_i) = \mathbf{E}Q^{(i)}(k).$$

Note that under Assumptions A0 and A1 $Q^{(i)}(n) \to 0$ $\mathbf{P}-$a.s. as $n \to \infty$ for all $1 \leq i \leq p$, since $\mathbf{P}-$a.s.

$$(v, Q(n)) \leq \min_{0\leq k\leq n-1} |vM_0 \cdots M_k| \leq \exp\{\min_{0\leq k\leq n-1} S_k\} \to 0$$

as $n \to \infty$. Denote by $u(n) = (u_1(n), ..., u_p(n))^T := u(M_0 \cdots M_n), n \geq 0$, the right eigenvector of the product $M_0 \cdots M_n$, corresponding to the Perron root $\rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n$. To investigate the asymptotic behavior of $q_i(n)$ and $Q^{(i)}(n)$ as $n \to \infty$ we need the following statement describing the behavior of $u(n)$.

Theorem 1 *If Assumption A0 is valid, then there exist a random vector $u = (u_1, ..., u_p)^T$ and a function $g(n) \geq 0, g(n) \to 0, n \to \infty$, such that with probability 1*

$$|u_i(n) - u_i| \leq g(n),\ i = 1, ..., p.$$

In addition,

$$(v, u) = 1,\ \alpha \leq u_i \leq 1/v^*,$$

where $v^* = \min(v_1, ..., v_p)$ and $v = (v_1, ..., v_p)$ is from (2).

The following statement describes the behavior of $Q(n)$ as n$\to \infty$.

Theorem 2 *Assume Assumptions A0 and A1. Then* $\mathbf{P}-$*a.s., as* $n \to \infty$,

$$\frac{Q_i(n)}{(v, Q(n))} \to u_i, \; i = 1, ..., p,$$

where $u = (u_1, ..., u_p)$ *is from Theorem 1.*

Now we are ready to formulate the main result of the paper.

Theorem 3 *Assume Assumptions A0, A1, and A2. Then, as* $n \to \infty$,

$$q_i(n) \sim c_i n^{-(1-a)} l(n), \; c_i > 0, \; i = 1, ..., p,$$

where $l(n)$ *is a function slowly varying at infinity.*

Note that under our approach one of the key facts to prove Theorems 1, 2, 3 is convergence in distribution, as $n \to \infty$, of the products $\prod_{i=0}^{n} M_i \rho_i^{-1}$ of random matrices to a limit matrix whose distribution is not concentrated at zero matrix. It is known (8) that for $p = 2$ the products $\prod_{i=0}^{n} M_i \rho_i^{-1}$ of the positive bounded independent identically distributed 2×2 matrices $A_i = M_i \rho_i^{-1}$ converges in distribution, as $n \to \infty$, to a limit matrix whose distribution is not concentrated at zero matrix if and only if all the matrices A_i have a common positive right or left eigenvector. Hence, for the 2-type process $Z(n)$ our assumption on existence of a common positive left eigenvector of the matrices M is essential indeed.

Observe also that Assumption A1 covers non-degenerate random walks with zero mean and finite variance of there increments, as well as all non-degenerate symmetric random walks. In these cases $a = 1/2$. Another example when Assumption A1 is valid gives the random walk, whose increments have distribution belonging to the domain of attraction of a stable law.

In conclusion we give an example where Assumption A2 is fulfilled (given that Assumption A0 is valid as well). Clearly, if the measure $\mathbf{F}$ generating our random environment has a bounded support, i.e., if there exists a $p-$dimensional cube $B = [0, b]^p, b > 0$, such that $\mathbf{P}(\mathbf{F}(B) = 1) = 1$, then Assumption A2 holds since $\kappa(d) = 0$ $\mathbf{P}$ -a.s. for $d > b$.

One can show that if $\mathbf{F}$ satisfies Assumption A0 and

$$F(s) = \overline{1} - \frac{M(\overline{1} - s)}{1 + \gamma(\overline{1} - s)}, \; s \in J^p,$$

where the $p-$dimensional random row vector γ with positive components and the random matrix M are such that the components of the vector $y = (M\overline{1})/|\gamma|$ are uniformly bounded from below then Assumption A2 holds true.

Note that if the distribution of $X = \ln \rho$ has a regular varying tail then Assumptions A1 and A2 can be replaced by the following hypotheses (see (13) or (1)):

Assumption A1'. There exist constants $c_n, n \geq 0$, such that as $n \to \infty$ the scaled sums $c_n S_n$ converge weakly to a stable distribution μ with parameter $\beta \in (0, 2]$. The limit law μ is not one-side, i.e., $0 < \mu(\mathbb{R}^+) < 1$.

Assumption A2'. There exist $\varepsilon > 0$ and $d \in \mathbf{N}_0$ such that

$$\mathbf{E}(\ln^+ \kappa(d))^{\beta+\varepsilon} < \infty,$$

where β is from Assumption A1'.

Note that Assumption A1' implies the validity of Assumption A1 with $a = \mu(\mathbb{R}^+)$, and Assumption A2 is stronger than Assumption A2' since $a\beta \leq 1$.

Corollary 1 *Assume Assumptions A0 and A1' . Then the statement of Theorem 2 remains true.*

Using the proof of Theorem 3 and results of paper (1), one can obtain also the following statement.

Theorem 4 *Assume Assumptions A0, A1' , and A2'. Then the statement of Theorem 3 remains true.*

References

[1] Afanasyev V.I., Geiger J., Kersting G., and Vatutin V.A. *Criticality for branching processes in random environment.* Ann. Prob., 33(2005), 2, p. 645-673.

[2] Athreya K.B. and Karlin S. *On branching processes with random environments: I,II.* Ann. Math. Statist., 42 (1971), 5, p. 1499-1520; 42(1971), 6, p. 1843-1858.

[3] Bertoin J. and Doney R.A. *On conditioning a random walk to stay nonnegtive.* Ann. Prob., 22 (1994), p. 2152-2167.

[4] Dyakonova E.E. *Asymptotic behavior of the probability of non-extinction for a multitype branching process in random environment.* Discrete Math. Appl., 9 (1999), 2, p. 119-136.

[5] Dyakonova E.E., Geiger J., and Vatutin V.A. *On the survival probability and a functional limit theorem for branching processes in random environment.* Markov Process. Related Fields, 10 (2004), 2, p. 289-306.

[6] Geiger J. and Kersting G. *The survival probability of a critical branching process in random environment.* Theory Probab. Appl., 45(2000), 3, p. 607-615.

[7] Kaplan N. *Some results about multidimentional branching processes with random environments.* Ann. Prob., 2 (1974), 3, p. 441-455.

[8] Kesten H. and Spitzer F. *Convergence in distribution of products of random matrices.* Z. Wahrsch. verw. Geb., 67 (1984), 4, p. 363-386.

[9] Kozlov M.V. *On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment.* Theory Probab. Appl., 21 (1976), 4, p. 791-804.

[10] Smith W.L. and Wilkinson W. *On branching processes in random environment.* Ann. Math. Statist., 40 (1969), 3, p. 814-827.

[11] Tanny D. *On multitype branching processes in a random environment.* Adv. Appl. Prob., 13 (1981), 3, p. 464-497.

[12] Vatutin V.A. and Dyakonova E.E. *Galton-Watson branching processes in random environments, I: limit theorems.* Theory Probab. Appl., 48(2003), 2, p. 314-336.

[13] Vatutin V.A. and Dyakonova E.E. *Galton-Watson branching processes in random environments, II: joint distributions.* Theory Probab. Appl., 49 (2004), 2, p. 275-308.

[14] Vatutin V.A. and Dyakonova E.E. *Branching processes in random environment and bottlenecks in the evolution of populations.* Theory Probab. Appl., 51(2006), 1.