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Elena Dyakonova

1Department of Discrete Mathematics, Steklov Mathematical Institute, Gubkin St. 8, 119991 Moscow, Russia

We study a multi-type branching process in i.i.d. random environment. Assuming that the associated random walk satisfies the Doney-Spitzer condition, we find the asymptotics of the survival probability at time \( n \) as \( n \to \infty \).

**Keywords:** branching processes in random environment, Doney-Spitzer condition, survival probability

**Introduction**

Branching processes in random environment constitute an important part of the theory of branching processes (see, for example, (1), (2), (3)-(7), (9)-(14)). A branching process in random environment was first considered by Smith and Wilkinson (10). The subsequent papers (2), (7), (11) investigated single- and multi-type Galton-Watson processes in random environment. The asymptotics of the survival probability of the critical branching processes in a random environment generated by a sequence of independent identically distributed random variables under the condition \( E X^2 < \infty \) for the increment \( X \) of the associated random walk was found in (6), (9), (11) for single-type processes, and in (4) for multi-type processes. Recent papers (1), (5), (12)-(14) study the survival probability for an extended class of the critical single-type branching processes in random environment where the case \( E X^2 = \infty \) is not excluded and, moreover, \( E X \) may not exist. The present paper investigates an extended class of multi-type critical branching processes in random environment whose associated random walks satisfy the Doney-Spitzer condition. In particular, we generalize some results established in (11) and (4) concerning the asymptotic behavior of survival probability.

Let \( Z(n) = (Z_1(n), \ldots, Z_p(n)) \), \( n = 0, 1, \ldots \), be a \( p \)-type Galton-Watson branching process in a random environment. This process can be described as follows.

Let \( \mathbb{N}_0 = \{ 0, 1, 2, \ldots \} \) and \( \mathbb{N}_0^p \) be the set of all vectors \( t = (t_1, \ldots, t_p) \) with non-negative integer coordinates. Denote by \( (\Delta_1, \mathcal{B}(\Delta_1)) \) a set of probability measures on \( \mathbb{N}_0^p \) with \( \sigma \)-algebra \( \mathcal{B}(\Delta_1) \) of Borel sets endowed with the metric of total variation, and by \( (\Delta, \mathcal{B}(\Delta)) \) the \( p \)-times product of the space \( (\Delta_1, \mathcal{B}(\Delta_1)) \) on itself. Let \( \mathcal{F} = (\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(p)}) \) be a random variable (random measure) taking values in \( (\Delta, \mathcal{B}(\Delta)) \). An infinite sequence \( \Pi = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots) \) of independent identically distributed copies of \( \mathcal{F} \) is said to form a random environment and we will say that \( \mathcal{F} \) generates \( \Pi \). A sequence of random \( p \)-dimensional vectors \( Z(0), Z(1), Z(2), \ldots \) with non-negative integer coordinates is called a \( p \)-type branching process in random environment \( \Pi \), if \( Z(0) \) is independent of \( \Pi \) and for all \( n \geq 0, z = (z_1, \ldots, z_p) \in \mathbb{N}_0^p \) and \( f_0, f_1, \ldots \in \Delta \)

\[
\mathcal{L}(Z(n+1) \mid Z(n) = (z_1, \ldots, z_p), \Pi = (f_1, f_2, \ldots)) = \mathcal{L}((\xi_{n,1}^{(1)} + \cdots + \xi_{n,1}^{(1)} + \xi_{n,1}^{(2)} + \cdots + \xi_{n,1}^{(2)} + \cdots + (\xi_{n,1}^{(p)} + \cdots + \xi_{n,1}^{(p)}), \ldots, (\xi_{n,p}^{(1)} + \cdots + \xi_{n,p}^{(1)} + \cdots + (\xi_{n,p}^{(p)} + \cdots + \xi_{n,p}^{(p)}),)
\]

where \( f_n = (f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(p)}) \in \Delta, \xi_{n,1}^{(1), (i)}, \xi_{n,2}^{(1), (i)}, \ldots, \xi_{n,2}^{(1), (i)}, i = 1, \ldots, p \), are independent \( p \)-dimensional random vectors, and for each \( i = 1, \ldots, p \) the random vectors \( \xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, \ldots, \xi_{n,2}^{(i)} \) are identically distributed according to the measure \( f_n^{(i)} \). Relation (1) defines a branching Galton-Watson process \( Z(n) \) in random environment which describes the evolution of a particle population \( Z(n) = (Z_1(n), \ldots, Z_p(n)) \), \( n = 0, 1, \ldots \), where \( Z_i(n), i = 1, \ldots, p \), is the number of type \( i \) particles in the \( n \)-th generation.
This population evolves as follows. If $F_n = f_n$ then each of the $Z_i(n)$ particles of type $i$ existing at the time $n$, produces offspring in accordance with the $p$-dimensional probability measure $f_{n(i)}$ independently of the reproduction of other particles. Thus, the $i$-th component of the vector $Z(n + 1) = (Z_1(n + 1), \ldots, Z_p(n + 1))$ is equal to the number of type $i$ particles among all direct descendants of the particles of the $n$-th generation. The distribution of $Z(0)$ will be specified later.

The main results

Let $J^p$ be the set of all column vectors $s = (s_1, \ldots, s_p)^T, 0 \leq s_i \leq 1, i = 1, \ldots, p$. For $s \in J^p$ and $t \in \mathbb{N}_0^p$ set $s^t = \prod_{i=1}^p s_i^t$. Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with $F = (F^{(1)}, \ldots, F^{(p)})$ generating functions $F^{(i)}(s)$ corresponding to $F(i), 1 \leq i \leq p$:

$$F^{(i)}(s) = \sum_{t \in \mathbb{N}_0^p} F^{(i)}(\{t\}) s^t, s \in J^p.$$ 

In a similar way we associate with the component $F_n = (F_n^{(1)}, \ldots, F_n^{(p)}), n \geq 0$, of the random environment $\Pi = (F_0, F_1, F_2, \ldots)$ a random vector $F_n(s) = (F_n^{(1)}(s), \ldots, F_n^{(p)}(s))^T, s \in J^p$, the components of which are multidimensional (random) generating functions $F_n^{(i)}(s)$, corresponding to $F^{(i)}(s), 1 \leq i \leq p$,

$$F_n^{(i)}(s) = \sum_{t \in \mathbb{N}_0^p} F_n^{(i)}(\{t\}) s^t.$$ 

Let $e_j, j = 1, \ldots, p$, be the $p$-dimensional row vector whose $j$-th component is equal to 1 and the others are zeros, $0 = (0, \ldots, 0)$ be the $p$-dimensional row vector all whose components are zeros, and let $1 = (1, \ldots, 1)^T$ be the $p$-dimensional column vector all whose components are equal to 1. For $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ we set $|x| = \sum_{i=1}^p |x_i|, |y| = \sum_{i=1}^p |y_i|, (x, y) = \sum_{i=1}^p x_i y_i$. Let $A = ||A(i, j)||_{i,j=1}^p$ be an arbitrary positive $p \times p$ matrix. Denote by $\rho(A)$ the Perron root of $A$ and by $v(A) = (v_1(A), \ldots, v_p(A))^T$ and $e(A) = (e_1(A), \ldots, e_p(A))$ the right and left eigenvectors of $A$ corresponding to the eigenvalue $\rho(A)$ and such that $|v(A)| = 1, (v(A), u(A)) = 1$.

For vector-valued generating functions $F(s)$ and $F_n(s)$ we introduce the mean matrices

$$M = M(F) = ||M(i, j)||_{i,j=1}^p = \left| \frac{\partial F^{(i)}(1)}{\partial s_j} \right|_{i,j=1}^p$$

and

$$M_n = M_n(F_n) = ||M_n(i, j)||_{i,j=1}^p = \left| \frac{\partial F_n^{(i)}(1)}{\partial s_j} \right|_{i,j=1}^p.$$ 

Let $C_{\alpha}, 0 < \alpha < 1$, be the class of all matrices $A = ||A(i, j)||_{i,j=1}^p$ such that

$$\alpha \leq \frac{A(i_1, j_1)}{A(i_2, j_2)} \leq \alpha^{-1}, 1 \leq i_1, i_2, j_1, j_2 \leq p.$$ 

One of our basic hypotheses is the following condition.

Assumption A0. There exist a number $0 < \alpha < 1$ and a positive row vector $v = (v_1, \ldots, v_p), |v| = 1$, such that, with probability 1

$$M = M(F) \in C_{\alpha},$$

and

$$vM = \rho(M)v.$$ 

Set $\rho = \rho(M), \rho_n = \rho(M_n), n \geq 0$. It is not difficult to see that in our settings $X := \ln \rho, X_i := \ln \rho_{i-1}, i \geq 1$, are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk $S = (S_0, S_1, \ldots)$, where

$$S_n = X_1 + \cdots + X_n, n \geq 1, S_0 = 0.$$
**Assumption A1.** There exists a number $0 < a < 1$ such that
\[ P(S_n > 0) \rightarrow a, \; n \rightarrow \infty. \] (3)

Extending the known classification of single-type branching processes in random environment (see (1), (12)), we call a $p$-type branching process $Z(n), n \geq 0$, in random environment II critical if its associated random walk is of the oscillating type, i.e., \( \limsup_{n \rightarrow \infty} S_n = +\infty \) a.s. and \( \liminf_{n \rightarrow \infty} S_n = -\infty \) a.s. It is known that any random walk satisfying Assumption A1 oscillates. From now on we consider only critical $p$-type branching processes in random environment.

Let $0 =: \gamma_0 < \gamma_1 < ...$ be the strict descending ladder epochs of $S$. Put
\[ V(x) := \sum_{i=0}^{\infty} P(S_{\gamma_i} \geq -x), \; x \geq 0; \; V(x) = 0, \; x < 0. \]

Since $S$ is oscillating, the following relation holds (3):
\[ E V(x + X) = V(x), \; x \geq 0. \] (4)

For $d \in \mathbb{N}_0$ set
\[ O_d = \{ t = (t_1, ..., t_p) \in \mathbb{N}_0^p \mid t_i < d, \; i = 1, ..., p \}, \; U_d = \mathbb{N}_0^p \setminus O_d. \]

Introduce the random variable
\[ \kappa(d) = \sum_{t \in U_d} \sum_{i=1}^{p} v_i \sum_{j,k=1}^{p} F^{(i)}((t)) t_j t_k / \rho^2, \; d \in \mathbb{N}_0, \]

where $v = (v_1, ..., v_p)$ is from (2). Our next condition is connected with the random variable $\kappa(d)$, which is a generalization of the standardized truncated second moment of the reproduction law to the multi-type case.

**Assumption A2.** There exist $\varepsilon > 0$ and $d \in \mathbb{N}_0$ such that
\[ E (\ln^+ \kappa(d))^{1/a+\varepsilon} < \infty, \; E (V(X)(\ln^+ \kappa(d))^{1/a+\varepsilon}) < \infty. \]

Let $T = \min \{ n \geq 0 : Z(n) = 0 \}$ be the extinction moment for $Z(n)$. Introduce the random variables
\[ Q^{(i)}(n) = P(T > n | Z(0) = e_i, \Pi), \; Q(n) = (Q^{(1)}(n), ..., Q^{(p)}(n)), \]

and let
\[ q_i(k) = P(T > k | Z(0) = e_i) = E Q^{(i)}(k). \]

Note that under Assumptions A0 and A1 \( Q^{(i)}(n) \rightarrow 0 \) P-a.s. as $n \rightarrow \infty$ for all $1 \leq i \leq p$, since P-a.s.
\[ (v, Q(n)) \leq \min_{0 \leq k \leq n-1} |vM_0 \cdots M_k| \leq \exp \{ \min_{0 \leq k \leq n-1} S_k \} \rightarrow 0 \]
as $n \rightarrow \infty$. Denote by $u(n) = (u_1(n), ..., u_p(n))^T := u(M_0 \cdots M_n), n \geq 0$, the right eigenvector of the product $M_0 \cdots M_n$, corresponding to the Perron root $\rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n$. To investigate the asymptotic behavior of $q_i(k)$ and $Q^{(i)}(n)$ as $n \rightarrow \infty$ we need the following statement describing the behavior of $u(n)$.

**Theorem 1** If Assumption A0 is valid, then there exist a random vector $u = (u_1, ..., u_p)^T$ and a function $g(n) \geq 0, g(n) \rightarrow 0, n \rightarrow \infty$, such that with probability 1
\[ |u_i(n) - u_i| \leq g(n), \; i = 1, ..., p. \]

In addition,
\[ (v, u) = 1, \; \alpha \leq u_i \leq 1/\nu^*, \]
where $\nu^* = \min(v_1, ..., v_p)$ and $v = (v_1, ..., v_p)$ is from (2).

The following statement describes the behavior of $Q(n)$ as $n \rightarrow \infty$. 


Theorem 2 Assume Assumptions A0 and A1. Then $P$-a.s., as $n \to \infty$,

$$\frac{Q_i(n)}{(v_i Q(n))} \to u_i, \; i = 1, \ldots, p,$$

where $u = (u_1, \ldots, u_p)$ is from Theorem 1.

Now we are ready to formulate the main result of the paper.

Theorem 3 Assume Assumptions A0, A1, and A2. Then, as $n \to \infty$,

$$q_i(n) \sim c_i n^{-(1-a)s} l(n), \; c_i > 0, \; i = 1, \ldots, p,$$

where $l(n)$ is a function slowly varying at infinity.

Note that under our approach one of the key facts to prove Theorems 1, 2, 3 is convergence in distribution, as $n \to \infty$, of the products $\prod_{i=0}^{n} M_i^{-1}$ of random matrices to a limit matrix whose distribution is not concentrated at zero matrix. It is known (8) that for $p = 2$ the products $\prod_{i=0}^{n} M_i^{-1}$ of the positive bounded independent identically distributed $2 \times 2$ matrices $A_i = M_i^{-1}$ converges in distribution, as $n \to \infty$, to a limit matrix whose distribution is not concentrated at zero matrix if and only if all the matrices $A_i$ have a common positive right or left eigenvector. Hence, for the 2-type process $Z(n)$ our assumption on existence of a common positive left eigenvector of the matrices $M$ is essential indeed.

Observe also that Assumption A1 covers non-degenerate random walks with zero mean and finite variance of their increments, as well as all non-degenerate symmetric random walks. In these cases $a = 1/2$.

Another example when Assumption A1 is valid gives the random walk, whose increments have distribution belonging to the domain of attraction of a stable law.

In conclusion we give an example where Assumption A2 is fulfilled (given that Assumption A0 is valid)

Note that if the distribution of $X = \ln \rho$ has a regular varying tail then Assumptions A1 and A2 can be replaced by the following hypotheses (see (14) or (1)):

Assumption A1'. There exist constants $c_n, n \geq 0$, such that as $n \to \infty$ the scaled sums $c_n S_n$ converge weakly to a stable distribution $\mu$ with parameter $\beta \in (0, 2]$. The limit law $\mu$ is not one-sided, i.e., $0 < \mu(\mathbb{R}^+) < 1$.

Assumption A2'. There exist $\varepsilon > 0$ and $d \in \mathbb{N}_0$ such that

$$E(\ln^+ \kappa(d))^{\beta + \varepsilon} < \infty,$$

where $\beta$ is from Assumption A1'.

Note that Assumption A1' implies the validity of Assumption A1 with $a = \mu(\mathbb{R}^+)$, and Assumption A2 is stronger than Assumption A2' since $a \beta \leq 1$.

Corollary 1 Assume Assumptions A0 and A1'. Then the statement of Theorem 2 remains true.

Using the proof of Theorem 3 and results of paper (1), one can obtain also the following statement.

Theorem 4 Assume Assumptions A0, A1', and A2'. Then the statement of Theorem 3 remains true.

References


