*Fourth Colloquium on Mathematics and Computer Science* DMTCS proc. **AG**, 2006, 331–334

# *Conditioned Galton–Watson trees do not grow*

Svante Janson

*Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden*
svante.janson@math.uu.se *and* http://www.math.uu.se/∼svante/

An example is given which shows that, in general, conditioned Galton–Watson trees cannot be obtained by adding vertices one by one, while this can be done in some important but special cases, as shown by Luczak and Winkler.

**Keywords:** conditioned Galton–Watson trees, random trees, profile

## 1 Monotonicity of conditioned Galton–Watson trees?

A conditioned Galton–Watson tree is a random rooted tree that is (or has the same distribution as) the family tree of a Galton–Watson process with some given offspring distribution, conditioned on the total number of vertices.

We let $\xi$ be a random variable with the given offspring distribution; i.e., the number of offspring of each individual in the Galton–Watson process is a copy of $\xi$.

We let $\xi$ be fixed throughout the paper, and let $T_n$ denote the corresponding conditioned Galton–Watson tree with $n$ vertices. For simplicity, we consider only $\xi$ such that $\mathbb{P}(\xi = 0) > 0$ and $\mathbb{P}(\xi = 1) > 0$; then $T_n$ exists for all $n \geq 1$. Furthermore, we assume that $\mathbb{E}\,\xi = 1$ (the Galton–Watson process is critical) and $\sigma^2 := \mathrm{Var}(\xi) < \infty$.

The importance of this construction lies in that many combinatorially interesting random trees are of this type, for example the following:

(i) Random plane (= ordered) trees. $\xi \sim \mathrm{Ge}(1/2)$; $\sigma^2 = 2$.

(ii) Random unordered labelled trees (Cayley trees). $\xi \sim \mathrm{Po}(1)$; $\sigma^2 = 1$.

(iii) Random binary trees. $\xi \sim \mathrm{Bi}(2, 1/2)$; $\sigma^2 = 1/2$.

(iv) Random $d$-ary trees. $\xi \sim \mathrm{Bi}(d, 1/d)$; $\sigma^2 = 1 - 1/d$.

For further examples see e.g. Aldous (1) and Devroye (3); note also that the families of random trees obtained in this way are the same as the simply generated families of trees defined by Meir and Moon (9).

If we increase $n$, we get a new random tree that is in some sense larger, but the definition above gives no relation between, say, $T_n$ and $T_{n+1}$, since they are defined by two different conditionings. It is thus natural to ask whether $T_{n+1}$ is *stochastically larger* than $T_n$, i.e., whether there exists another construction (with the same distribution of each $T_n$) that further yields $T_n \subset T_{n+1}$, i.e., whether $(T_n)_{n\geq 1}$ has the following property:

**Property P1** *It is possible to define $T_n$ and $T_{n+1}$ on a common probability space such that $T_n \subset T_{n+1}$.*

Equivalently, Property P1 says that it is possible to add a new leaf to $T_n$ by some random procedure (depending on $n$ and $T_n$) such that the resulting tree has the distribution of $T_{n+1}$. It is thus immediately seen that Property P1 is equivalent to the following:

**Property P1′** *It is possible to construct $T_1, T_2, T_3, \ldots$ as a Markov chain where at each step a new leaf is added.*

This property was investigated by Luczak and Winkler (7), who showed that Properties P1 and P1′ indeed hold in the case of random binary trees, and more generally, for random $d$-ary trees, for any $d \geq 2$. The main purpose of this note is to give a simple counterexample (Section 3), showing that Property P1 does not hold for every $\xi$.

The question of whether Property P1 (or P1′) holds for all conditioned Galton–Watson trees has been considered by several people, and has been explicitly stated as an open problem at least in (5, Problem 1.15). The answer to this question is thus negative. The problem can be reformulated as follows.

**Problem 1** *For which conditioned Galton–Watson trees $(T_n)_n$ does Property P1 (or P1′) hold?*

In view of the result of Luczak and Winkler (7) just mentioned, it seems particularly interesting to study the cases of random plane trees and random labelled trees; as far as we know, the problem is still open for them.

It is well known that as $n \to \infty$, $T_n$ converges in the sense of finite-dimensional distributions (i.e., the distribution of the first $N$ generations for any fixed $N$) to an infinite random tree $T_\infty$ that is the family tree of the corresponding *size-biased* Galton–Watson process, see e.g. Kennedy (6), Aldous (1), Lyons, Pemantle and Peres (8). The size-biased Galton–Watson process is the same as the *Q-process* studied in (2, Section I.14); it can also be regarded as a branching process with two types: mortals with an offspring distribution $\xi$ and all children mortals, and immortals with the size-biased offspring distribution $\hat{\xi}$ with $\mathbb{P}(\hat{\xi} = j) = j\,\mathbb{P}(\xi = j)$ and exactly one immortal child (in a random position among its siblings); the process starts with a single immortal. (See also (4).) Note that the infinite random tree $T_\infty$ has exactly one infinite path from the root, with (finite) Galton–Watson trees attached to it.

If $\xi$ is such that Property P1′ holds, we can construct $T_n$, $n \geq 1$, such that $T_1 \subset T_2 \subset \dots$, and then evidently $T_n \to \bigcup_n T_n$ in the sense that the first $N$ generations of $T_n$ and $\bigcup_n T_n$ coincide for large $n$; in particular, $T_n \to \bigcup_n T_n$ in the sense of finite-dimensional distributions. Thus $\bigcup_n T_n \overset{\mathrm{d}}{=} T_\infty$, and we may assume that $T_\infty = \bigcup_n T_n$. Hence, Property P1 implies the following property:

**Property P2** *It is, for every $n \geq 1$, possible to define $T_n$ and $T_\infty$ on a common probability space such that $T_n \subset T_\infty$. In other words, each $T_n$ may be constructed by a suitable (random) pruning of $T_\infty$.*

Thus, by Luczak and Winkler (7), Property P2 holds for random binary and $d$-ary trees. On the other hand, our counterexample in Section 3 also fails to satisfy Property P2.

**Problem 2** *For which conditioned Galton–Watson trees $(T_n)_n$ does Property P2 hold?*

Again, this problem seems to be open for random plane trees and random labelled trees.

# 2 Monotonicity of the profile?

Properties P1 and P1′ are not only interesting in themselves, but also technically useful (when valid), For example, for any rooted tree $T$, let $W_k(T)$ denote the number of vertices in $T$ of distance $k$ from the root. The sequence $(W_k(T))_{k \geq 0}$ is known as the *profile* of the tree.

It is easy to see from the description of $T_\infty$ above that $\mathbb{E}\, W_k(T_\infty) = 1 + k\sigma^2$. (Use the fact that the expected number of mortal children of an immortal individual is $\mathbb{E}\,\hat{\xi} - 1 = \mathbb{E}\,\xi^2 - 1 = \sigma^2$.) Moreover, as $n \to \infty$, for each fixed $k \geq 0$,

$$\mathbb{E}\, W_k(T_n) \to \mathbb{E}\, W_k(T_\infty) = 1 + k\sigma^2. \tag{2.1}$$

If Property P1 holds, then also:

**Property P3** *For every $k \geq 0$ and $n \geq 1$, $\mathbb{E}\, W_k(T_n) \leq \mathbb{E}\, W_k(T_{n+1})$.*

Further, if any of Property P1, Property P2 or Property P3 holds, then, using (2.1), so does the following:

**Property P4** *For every $k \geq 0$ and $n \geq 1$, $\mathbb{E}\, W_k(T_n) \leq 1 + k\sigma^2$.*

A uniform estimate of this order, more precisely

$$\mathbb{E}\, W_k(T_n) \leq Ck, \qquad k \geq 1,\ n \geq 1. \tag{2.2}$$

for all $k, n \geq 1$ with a constant $C$ depending only on $\xi$, was needed in (5) and proved there (Theorem 1.13) by a more complicated argument. We will see that our counterexample in Section 3 fails also Property P4; thus another argument is indeed needed to prove (2.2) in general.

Our counterexample shows that Properties P3 and P4 fail for a certain $\xi$ and $n = 3$. An anonymous referee raised the following problem.

**Problem 3** *Do Properties P3 and P4 hold for large $n$?*

Note that Meir and Moon (9) gave explicit formulas for $\mathbb{E}\, W_k(T_n)$ for the cases of random labelled trees, plane trees and binary trees, which show that Properties P3 and P4 hold for these cases. (Actually, the binary trees considered in (9) are the "strict" or "complete" binary trees where all vertices have outdegree exactly 0 or 2; these are obtained as a conditioned Galton–Watson tree with $\mathbb{P}(\xi = 0) = \mathbb{P}(\xi = 2) = 1/2$. There is a simple correspondence between such binary trees with $2n + 1$ vertices and binary trees with $n$ vertices in our notation obtained by removing the $n + 1$ leaves (or external vertices), and it is easily seen that if the strict binary tree $\tilde{T}_{2n+1}$ corresponds to $T_n$, then $W_{k+1}(\tilde{T}_{2n+1}) = 2W_k(T_n)$ for all $k \geq 0$. Hence Properties P3 and P4 hold for both types of random binary trees.)


**Fig. 1:** The trees with three vertices

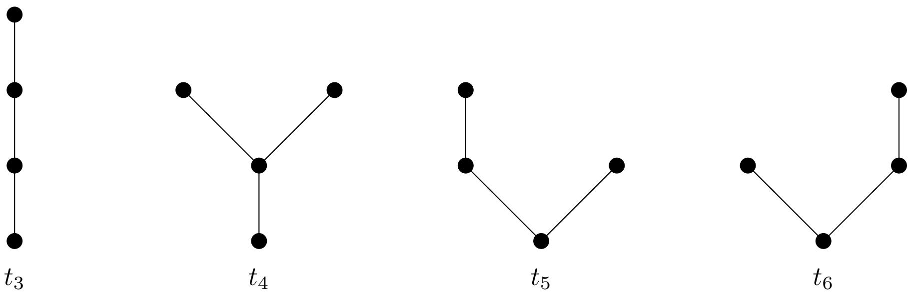


**Fig. 2:** The trees with four vertices

# 3 A counterexample

Let $\varepsilon > 0$ be a small number and let the offspring distribution be given by

$$\mathbb{P}(\xi = 0) = \frac{1-\varepsilon}{2}, \qquad \mathbb{P}(\xi = 1) = \varepsilon, \qquad \mathbb{P}(\xi = 2) = \frac{1-\varepsilon}{2}.$$

We have $\mathbb{E}\,\xi = 1$ and $\sigma^2 := \operatorname{Var}\xi = 1 - \varepsilon$. Let $\mathcal{T}$ be the (unconditional) Galton–Watson tree with this offspring distribution.

For $n = 3$ we have the two possible trees in Figure 1. The corresponding probabilities are, with $p_j := \mathbb{P}(\xi = j)$,

$$\mathbb{P}(\mathcal{T} = t_1) = p_1^2 p_0 = \varepsilon^2 \frac{1-\varepsilon}{2} = \frac{1}{2}\varepsilon^2 + O(\varepsilon^3),$$
$$\mathbb{P}(\mathcal{T} = t_2) = p_2 p_0^2 = \Big(\frac{1-\varepsilon}{2}\Big)^3 = \frac{1}{8} + O(\varepsilon),$$

and thus, conditioning on $|\mathcal{T}| = 3$, i.e. on $\mathcal{T} \in \{t_1, t_2\}$,

$$\mathbb{P}(T_3 = t_1) = \frac{\mathbb{P}(\mathcal{T} = t_1)}{\mathbb{P}(\mathcal{T} = t_1) + \mathbb{P}(\mathcal{T} = t_2)} = 4\varepsilon^2 + O(\varepsilon^3),$$
$$\mathbb{P}(T_3 = t_2) = 1 - 4\varepsilon^2 + O(\varepsilon^3).$$

For $n = 4$ we similarly have the four possible trees in Figure 2 and

$$\mathbb{P}(\mathcal{T} = t_3) = p_1^3 p_0 = \varepsilon^3 \frac{1-\varepsilon}{2} = \frac{1}{2}\varepsilon^3 + O(\varepsilon^4),$$
$$\mathbb{P}(\mathcal{T} = t_4) = \mathbb{P}(\mathcal{T} = t_5) = \mathbb{P}(\mathcal{T} = t_6) = p_1 p_2 p_0^2 = \varepsilon\Big(\frac{1-\varepsilon}{2}\Big)^3 = \frac{1}{8}\varepsilon + O(\varepsilon^2),$$

and thus, conditioning on $|\mathcal{T}| = 4$,

$$\mathbb{P}(T_4 = t_3) = O(\varepsilon^2)$$
$$\mathbb{P}(T_4 = t_4) = \mathbb{P}(T_4 = t_5) = \mathbb{P}(T_4 = t_6) = \frac{1}{3} + O(\varepsilon^2).$$

In particular,

$$\mathbb{E}\,W_1(T_3) = 2 + O(\varepsilon^2),$$
$$\mathbb{E}\,W_1(T_4) = \frac{5}{3} + O(\varepsilon^2),$$

and thus $\mathbb{E}\,W_1(T_3) > \mathbb{E}\,W_1(T_4)$ if $\varepsilon$ is small enough, so Property P3 fails. (An exact calculation shows that $0 < \varepsilon < 1/3$ is enough.)

By (2.1), $\mathbb{E}\,W_1(T_\infty) = 1 + \sigma^2 = 2 - \varepsilon$, and thus Property P4 too fails for $k = 1$, $n = 3$ and small $\varepsilon$ ($0 < \varepsilon < 1/5$). Consequently, Properties P1 and P2 too fail.

## Acknowledgements

This research was done during a workshop at Bellairs Research Institute in Barbados, March 2006. I thank the participants for their interest and comments.

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