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On the maximum average degree and the incidence chromatic number of a graph

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We prove that the incidence chromatic number of every 3-degenerated graph \(G\) is at most \(\Delta(G) + 4\). It is known that the incidence chromatic number of every graph \(G\) with maximum average degree \(mad(G) < 3\) is at most \(\Delta(G) + 3\). We show that when \(\Delta(G) \geq 5\), this bound may be decreased to \(\Delta(G) + 2\). Moreover, we show that for every graph \(G\) with \(mad(G) < 22/9\) (resp. with \(mad(G) < 16/7\) and \(\Delta(G) \geq 4\)), this bound may be decreased to \(\Delta(G) + 2\) (resp. to \(\Delta(G) + 1\)).

**Keywords:** incidence coloring, \(k\)-degenerated graph, planar graph, maximum average degree

1 Introduction

The concept of incidence coloring was introduced by Brualdi and Massey \((3)\) in 1993.

Let \(G = (V(G), E(G))\) be a graph. An incidence in \(G\) is a pair \((v, e)\) with \(v \in V(G), e \in E(G)\), such that \(v\) and \(e\) are incident. We denote by \(I(G)\) the set of all incidences in \(G\). For every vertex \(v\), we denote by \(I_v\) the set of incidences of the form \((v, vw)\) and by \(A_v\) the set of incidences of the form \((w, wv)\). Two incidences \((v, e)\) and \((w, f)\) are adjacent if one of the following holds: (i) \(v = w\), (ii) \(e = f\) or (iii) the edge \(vw\) equals \(e\) or \(f\).

A \(k\)-incidence coloring of a graph \(G\) is a mapping \(\sigma\) of \(I(G)\) to a set \(C\) of \(k\) colors such that adjacent incidences are assigned distinct colors. The incidence chromatic number \(\chi_i(G)\) of \(G\) is the smallest \(k\) such that \(G\) admits a \(k\)-incidence coloring.

For a graph \(G\), let \(\Delta(G), \delta(G)\) denote the maximum and minimum degree of \(G\) respectively. It is easy to observe that for every graph \(G\) we have \(\chi_i(G) \geq \Delta(G) + 1\) (for a vertex \(v\) of degree \(\Delta(G)\) we must use \(\Delta(G)\) colors for coloring \(I_v\) and at least one additional color for coloring \(A_v\)). Brualdi and Massey proved the following upper bound:

**Theorem 1** (3) For every graph \(G\), \(\chi_i(G) \leq 2\Delta(G)\).

Guiduli (4) showed that the concept of incidence coloring is a particular case of directed star arboricity, introduced by Algor and Alon (1). Following an example from (1), Guiduli proved that there exist graphs \(G\) with \(\chi_i(G) \geq \Delta(G) + \Omega(\log \Delta(G))\). He also proved that for every graph \(G\), \(\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))\).

Concerning the incidence chromatic number of special classes of graphs, the following is known:
For every $n \geq 2$, $\chi_i(K_n) = n = \Delta(K_n) + 1$ (3).

For every $m \geq n \geq 2$, $\chi_i(K_{m,n}) = m + 2 = \Delta(K_{m,n}) + 2$ (3).

For every tree $T$ of order $n \geq 2$, $\chi_i(T) = \Delta(T) + 1$ (3).

For every Halin graph $G$ with $\Delta(G) \geq 5$, $\chi_i(G) = \Delta(G) + 1$ (8).

For every $k$-degenerated graph $G$, $\chi_i(G) \leq \Delta(G) + 2k - 1$ (5).

For every $K_4$-minor free graph $G$, $\chi_i(G) \leq \Delta(G) + 2$ and this bound is tight (5).

For every cubic graph $G$, $\chi_i(G) \leq 5$ and this bound is tight (5).

For every planar graph $G$, $\chi_i(G) \leq \Delta(G) + 7$ (5).

The maximum average degree of a graph $G$, denoted by $\text{mad}(G)$, is defined as the maximum of the average degrees $ad(H) = 2 \cdot |E(H)|/|V(H)|$ taken over all the subgraphs $H$ of $G$.

In this paper we consider the class of 3-degenerated graphs (recall that a graph $G$ is $k$-degenerated if $\delta(H) \leq k$ for every subgraph $H$ of $G$), which includes for instance the class of triangle-free planar graphs and the class of graphs with maximum average degree at most 3. More precisely, we shall prove the following:

1. If $G$ is a 3-degenerated graph, then $\chi_i(G) \leq \Delta(G) + 4$ (Theorem 2).
2. If $G$ is a graph with $\text{mad}(G) < 3$, then $\chi_i(G) \leq \Delta(G) + 3$ (Corollary 5).
3. If $G$ a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 5$, then $\chi_i(G) \leq \Delta(G) + 2$ (Theorem 8).
4. If $G$ is a graph with $\text{mad}(G) < 22/9$, then $\chi_i(G) \leq \Delta(G) + 2$ (Theorem 11).
5. If $G$ is a graph with $\text{mad}(G) < 16/7$ and $\Delta(G) \geq 4$, then $\chi_i(G) = \Delta(G) + 1$ (Theorem 13).

In fact we shall prove something stronger, namely that one can construct for these classes of graphs incidence colorings such that for every vertex $v$, the number of colors that are used on the incidences of the form $(v, uw)$ is bounded by some fixed constant not depending on the maximum degree of the graph.

More precisely, we define a $(k, \ell)$-incidence coloring of a graph $G$ as a $k$-incidence coloring $\sigma$ of $G$ such that for every vertex $v \in V(G)$, $|\sigma(A_v)| \leq \ell$.

We end this section by introducing some notation that we shall use in the rest of the paper.

Let $G$ be a graph. If $v$ is a vertex in $G$ and $uw$ is an edge in $G$, we denote by $N_G(v)$ the set of neighbors of $v$, by $d_G(v) = |N_G(v)|$ the degree of $v$, by $G \setminus v$ the graph obtained from $G$ by deleting the vertex $v$ and by $G \setminus vw$ the graph obtained from $G$ by deleting the edge $vw$.

Let $G$ be a graph and $\sigma'$ a partial incidence coloring of $G$, that is an incidence coloring only defined on some subset $I$ of $I(G)$. For every uncolored incidence $(v, vw) \in I(G) \setminus I$, we denote by $F^2_G(v, vw)$ the set of forbidden colors of $(v, vw)$, that is:

$$F^2_G(v, vw) = \sigma'(A_v) \cup \sigma'(I_v) \cup \sigma'(I_w).$$
We shall often say that we extend such a partial incidence coloring $\sigma'$ to some incidence coloring $\sigma$ of $G$. In that case, it should be understood that we set $\sigma(v, vw) = \sigma'(v, vw)$ for every incidence $(v, vw) \in I$.

We shall make extensive use of the fact that every $(k, \ell)$-incidence coloring may be viewed as a $(k', \ell)$-incidence coloring for any $k' > k$.

**Drawing convention.** In a figure representing a forbidden configuration, all the neighbors of “black” or “grey” vertices are drawn, whereas “white” vertices may have other neighbors in the graph.

## 2 3-degenerated graphs

In this section, we prove the following:

**Theorem 2** Every 3-degenerated graph $G$ admits a $(\Delta(G) + 4, 3)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 4$.

**Proof:** Let $G$ be a 3-degenerated graph. Observe first that if $\Delta(G) \leq 3$ then, by Theorem 1, $\chi_i(G) \leq 2\Delta(G) < \Delta(G) + 4 \leq 7$ and every $(\Delta(G) + 4)$-incidence coloring of $G$ is obviously a $(\Delta(G) + 4, 3)$-incidence coloring.

Therefore, we assume $\Delta(G) \geq 4$ and we prove the theorem by induction on the number of vertices of $G$. If $G$ has at most 5 vertices then $G \subseteq K_5$. Since for every $k > 0$, $\chi_i(K_n) = n$, we obtain $\chi_i(G) \leq \chi_i(K_5) = \Delta(K_5) + 1 = 5$, and every 5-incidence coloring of $G$ is obviously a $(\Delta(G) + 4, 3)$-incidence coloring. We assume now that $G$ has $n + 1$ vertices, $n \geq 5$, and that the theorem is true for all 3-degenerated graphs with at most $n$ vertices.

Let $v$ be a vertex of $G$ with minimum degree. Since $G$ is 3-degenerated, we have $d_G(v) \leq 3$. We consider three cases according to $d_G(v)$.

$d_G(v) = 1$:

Let $w$ denote the unique neighbor of $v$ in $G$ (see Figure 1(1)). Due to the induction hypothesis, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma'$. We extend $\sigma'$ to a $(\Delta(G) + 4, 3)$-incidence coloring of $G$. Since $|F_{G'}^v(w, vw)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G) - 1 + 3 = \Delta(G) + 2$, $\sigma'$ is a $(\Delta(G) + 4, 3)$-incidence coloring of $G$. Therefore, we assume $\Delta(G) \geq 4$ and we prove the theorem by induction on the number of vertices of $G$. If $G$ has at most 5 vertices then $G \subseteq K_5$. Since for every $k > 0$, $\chi_i(K_n) = n$, we obtain $\chi_i(G) \leq \chi_i(K_5) = \Delta(K_5) + 1 = 5$, and every 5-incidence coloring of $G$ is obviously a $(\Delta(G) + 4, 3)$-incidence coloring. We assume now that $G$ has $n + 1$ vertices, $n \geq 5$, and that the theorem is true for all 3-degenerated graphs with at most $n$ vertices.

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Let $v$ be a vertex of $G$ with minimum degree. Since $G$ is 3-degenerated, we have $d_G(v) \leq 3$. We consider three cases according to $d_G(v)$.
there is a color $a$ such that $a \notin F^\sigma_G(w, vw)$. We then set $\sigma(w, vw) = a$ and $\sigma(v, vw) = b$, for any color $b$ in $\sigma(A_w)$.

\[d_G(v) = 2:\]
Let $u, w$ be the two neighbors of $v$ in $G$ (see Figure 1 (2)). Due to the induction hypothesis, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma'$. We extend $\sigma'$ to a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma$ of $G$ as follows. We first set $\sigma(v, vu) = a$ for a color $a \in \sigma(A_u)$ (if $d_G(u) = 1$, we have the case 1). Now, if $|\sigma'(A_w)| \geq 2$, there is a color $b \in \sigma'(A_w) \setminus \{a\}$ and if $|\sigma'(A_w)| = 1$, since $|F^\sigma_G(v, vw)| = |\sigma'(I_w) \cup \{a\}| \leq \Delta(G) - 1 + 1 = \Delta(G)$, there is a color $b$ distinct from $a$ such that $b \notin F^\sigma_G(v, vw)$. We set $\sigma(v, vu) = b$.

We still have to color the two incidences $(u, uv)$ and $(w, vw)$. Since $a \in \sigma'(A_u)$, we have $|F^\sigma_G(u, uv)| = |\sigma'(I_u) \cup \sigma'(A_u) \cup \{a, b\}| \leq \Delta(G) - 1 + 3 + 2 - 1 = \Delta(G) + 3$. Therefore, there is a color $c$ such that $c \notin F^\sigma_G(u, uv)$. Similarly, since $b \in \sigma(A_w)$, we have $|F^\sigma_G(w, vw)| \leq \Delta(G) + 3$ and there exists a color $d$ such that $d \notin F^\sigma_G(w, vw)$. We can extend $\sigma'$ to a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma$ of $G$ by setting $\sigma(u, uv) = c$ and $\sigma(w, vw) = d$.

\[d_G(v) = 3:\]
Let $u_1, u_2$ and $u_3$ be the three neighbors of $v$ in $G$ (see Figure 1 (3)). Due to the induction hypothesis, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma'$.

Observe first that for every $i, 1 \leq i \leq 3$, since $|F^\sigma_G(v, vu_i)| \leq \Delta(G) - 1$ and since we have $\Delta(G) + 4$ colors, we have at least five colors which are not in $F^\sigma_G(v, vu_i)$. Moreover, if $|A_{u_i}| < 3$ then any of these five colors may be assigned to the incidence $(v, vu_i)$ whereas we have only three possible choices (among these five) if $|A_{u_i}| = 3$. In the following, we shall see that having only three available colors is enough, and therefore assume that $|\sigma'(A_{u_i})| = 3$ for every $i, 1 \leq i \leq 3$.

We define the sets $B$ and $B_{i,j}$ as follows:
- $\forall i, j, 1 \leq i, j \leq 3, i \neq j, B_{i,j} := (\sigma'(I_{u_i}) \cup \sigma'(A_{u_i})) \cap \sigma'(A_{u_j})$
- $B := \bigcup_{1 \leq i, j \leq 3} B_{i,j}, i \neq j$.

We consider now four subcases according to the degrees of $u_1, u_2$ and $u_3$:

1. $\forall i, 1 \leq i \leq 3, d_G(u_i) < \Delta(G)$.
   In this case, since we have 3 colors for the incidence $(v, vu_i)$ for every $i, 1 \leq i \leq 3$, we can find 3 distinct colors $a_1, a_2, a_3$ such that $a_i \notin F^\sigma_G(v, vu_i)$. We set $\sigma(v, vu_i) = a_i$ for every $i, 1 \leq i \leq 3$.
   We still have to color the three incidences $(u_i, u_i v), 1 \leq i \leq 3$. Since $a_i \in \sigma'(A_{u_i})$, we have $|F^\sigma_G(u_i, u_i v)| = |\sigma(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}| \leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3$ for every $i, 1 \leq i \leq 3$. So, there exist three colors $b_1, b_2, b_3$ such that $b_i \notin F^\sigma_G(u_i, u_i v), 1 \leq i \leq 3$. We can extend $\sigma'$ to a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma$ of $G$ by setting $\sigma(u_i, u_i v) = b_i$ for every $i, 1 \leq i \leq 3$.

2. Only one of the vertices $u_i$ is of degree $\Delta(G)$.
   We can suppose without loss of generality that $d_G(u_1), d_G(u_2) < \Delta(G)$ and $d_G(u_3) = \Delta(G)$.
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Since $|\sigma'(I_{u_i}) \cup \sigma'(A_{u_i})| = \Delta(G) + 1 + 3 = \Delta(G) + 2$ and $|\sigma'(A_{u_i})| = 3$, we have $B_{3,1} \neq \emptyset$. Let $a_1 \in B_{3,1}$. Since $|\sigma'(A_{u_i})| = 3$ for every $i$, $1 \leq i \leq 3$, there exist two distinct colors $a_2$ and $a_3$ distinct from $a_1$ such that $a_2 \in \sigma'(A_{u_2})$ and $a_3 \in \sigma'(A_{u_3})$. We set $\sigma(v, vu_i) = a_i$ for every $i$, $1 \leq i \leq 3$.

We still have to color the three incidences of form $(u_i, u_iv)$. Since $a_1 \in B_{3,1}$ and $a_3 \in \sigma'(A_{u_3})$ we have:

$$|F_G'(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta + 3$$

and since $a_i \in \sigma'(A_{u_i})$ for every $i = 1, 2$ we have:

$$|F_G'(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta + 3.$$ 

Therefore, there exist three colors $b_1, b_2, b_3$ such that $b_i \notin F_G'(u_i, u_iv) \cup \{a_1, a_2, a_3\}$, $1 \leq i \leq 3$. We can extend $\sigma'$ to a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma$ of $G$ by setting $\sigma(u_i, u_iv) = b_i$ for every $i$, $1 \leq i \leq 3$.

3. Only one vertex among the $u_i$’s is of degree less than $\Delta(G)$.

We can suppose without loss of generality that $d_G(u_1) < \Delta(G)$ and $d_G(u_2) = d_G(u_3) = \Delta(G)$.

Similarly to the previous case, we have $B_{2,1} \neq \emptyset$ and $B_{3,2} \neq \emptyset$. We consider two cases:

$B_{2,1} \neq B_{3,2}$

Let $a_1 \in B_{2,1}$, $a_2 \in B_{3,2} \setminus \{a_1\}$ and $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$. We set $\sigma(v, vu_i) = a_i$ for every $i$, $1 \leq i \leq 3$.

We still have to color the three incidences $(u_i, u_iv)$, $1 \leq i \leq 3$. Since $a_1 \in \sigma'(A_{u_1})$ we have:

$$|F_G'(u_i, u_1v)| = |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3$$

and since $a_i \in B_{i+1,i}$ for $i = 1, 2$ and $a_j \in \sigma'(A_{u_j})$ for $j = 2, 3$, we have:

$$|F_G'(u_i, u_1v)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 1 + 3 + 3 - 1 = \Delta(G) + 3.$$ 

Therefore, there exist three colors $b_1, b_2, b_3$ such that $b_i \notin F_G'(u_i, u_1v)$, $1 \leq i \leq 3$. We can extend $\sigma'$ to a $(\Delta(G) + 4, 3)$-incidence coloring $\sigma$ of $G$ by setting $\sigma(u_i, u_1v) = b_i$ for every $i$, $1 \leq i \leq 3$.

$B_{2,1} = B_{3,2}$

Let $a_1 \in B_{2,1} = B_{3,2}$, $a_2 \in \sigma'(A_{u_2}) \setminus \{a_1\}$ and $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$. We set $\sigma(v, vu_i) = a_i$ for every $i$, $1 \leq i \leq 3$. 

We still have to color the three incidences \((u_i, u_iv), 1 \leq i \leq 3\). Since \(a_1 \in \sigma'(A_{u_i})\) we have:

\[
|F_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}|
\]

\[
\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3
\]

and since \(a_1 \in B_{2,1} = B_{3,2}\) and \(a_j \in \sigma'(A_{u_i})\) for \(j = 2, 3\), we have:

\[
|F_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}|
\]

\[
\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3.
\]

Therefore, there exist three colors \(b_1, b_2, b_3\) such that \(b_i \notin F_G(u_i, u_iv), 1 \leq i \leq 3\). We can extend \(\sigma'\) to a \((\Delta(G) + 4, 3)\)-incidence coloring \(\sigma\) of \(G\) by setting \(\sigma(u_i, u_iv) = b_i\) for every \(i, 1 \leq i \leq 3\).

4. \(d_G(u_1) = d_G(u_2) = d_G(u_3) = \Delta(G)\).

Similarly to the case (b) we have \(B_{ij} \neq \emptyset\) for every \(i, j, 1 \leq i, j \leq 3\) and thus \(|B| \geq 1\).

We prove first that in this case \(|B| \geq 2\). Suppose that \(|B| = |\{x\}| = 1\); in other words, \((\sigma'(I_{u_i}) \cup A_{u_i}) \cap A_{u_i}^j = \{x\}\) for every \(i, j, 1 \leq i, j \leq 3\). Thus we have:

\[
|\sigma'(A_{u_1}) \cup \sigma'(I_{u_i}) \cup \sigma'(A_{u_2}) \cup \sigma'(A_{u_3})| = \Delta(G) - 1 + 3 + 3 - 1 - 1
\]

\[
= \Delta(G) + 6. \tag{1}
\]

But the relation (1) is in contradiction with the fact that \(\sigma'\) is a \((\Delta(G) + 4, 3)\)-incidence coloring and we then get \(|B| \geq 2\).

Let \(a_1\) and \(a_2\) be two distinct colors in \(B\). We can suppose without loss of generality that \(a_1 \in B_{2,1}\) and \(a_2 \in B_{3,2}\).

We consider the two following subcases:

- \(B_{1,3} \setminus \{a_1, a_2\} \neq \emptyset\)

  Let \(a_3\) be a color in \(B_{1,3} \setminus \{a_1, a_2\}\). We set \(\sigma(v, vu_i) = a_i\) for every \(i, 1 \leq i \leq 3\).

  Since \(a_i \in B_{j,i} = (\sigma'(I_{u_i}) \cup \sigma'(A_{u_i})) \cap \sigma'(A_{u_i}), j = i + 1 \mod 3\), and \(a_i \in \sigma'(A_{u_i})\) for every \(i, 1 \leq i \leq 3\), we have:

  \[
  |F_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|
  \]

  \[
  \leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3.
  \]

  Therefore, there exist three colors \(b_1, b_2, b_3\) such that \(b_i \notin F_G(u_i, u_iv), 1 \leq i \leq 3\). We can extend \(\sigma'\) to a \((\Delta(G) + 4, 3)\)-incidence coloring \(\sigma\) of \(G\) by setting \(\sigma(u_i, u_iv) = b_i\) for every \(i, 1 \leq i \leq 3\).

- \(B_{1,3} \setminus \{a_1, a_2\} = \emptyset\)

  Since \(B_{1,3} \neq \emptyset\) we can suppose without loss of generality that \(a_2 \in B_{1,3}\). Let \(a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}\). We set \(\sigma(v, vu_i) = a_i\) for every \(i, 1 \leq i \leq 3\).

  Since \(a_i \in B_{j,i} = (\sigma'(I_{u_i}) \cup \sigma'(A_{u_i})) \cap \sigma'(A_{u_i}), j = i + 1 \mod 3\), and \(a_i \in \sigma'(A_{u_i})\) for \(i = 1, 2\), we have:

  \[
  |F_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|
  \]
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\[ \Delta(G) - 1 + 3 + 3 - 1 = \Delta(G) + 3 \]

and since \( a_2 \in \sigma'(I_{u_1}) \cup \sigma'(A_{u_1}) \) and \( a_1 \in \sigma'(A - u_1) \) we have:

\[ |F^o_G(u_1, u_1v)| = |\sigma'(I_{u_1}) \cup \sigma'(A_{u_1}) \cup \{a_1, a_2, a_3\}| \]

\[ \leq \Delta(G) - 1 + 3 + 3 - 1 = \Delta(G) + 3. \]

Therefore, there exist three colors \( b_1, b_2, b_3 \) such that \( b_i \notin F^o_G(u_i, u_iv), 1 \leq i \leq 3 \). We can extend \( \sigma' \) to a \((\Delta(G) + 4, 3)\)-incidence coloring \( \sigma \) of \( G \) by setting \( \sigma(u_i, u_iv) = b_i \) for every \( i, 1 \leq i \leq 3 \).

It is easy to check that in all cases we obtain a \((\Delta(G) + 4, 3)\)-incidence coloring of \( G \) and the theorem is proved.

Since every triangle free planar graph is 3-degenerated, we have:

**Corollary 3** For every triangle free planar graph \( G \), \( \chi_i(G) \leq \Delta(G) + 4 \).

### 3 Graphs with bounded maximum average degree

In this section we study the incidence chromatic number of graphs with bounded maximum average degree. The following result has been proved in (5).

**Theorem 4** Every \( k \)-degenerated graph \( G \) admits a \((\Delta(G) + 2k - 1, k)\)-incidence coloring.

Since every graph \( G \) with \( mad(G) < 3 \) is 2-degenerated, we get the following:

**Corollary 5** Every graph \( G \) with \( mad(G) < 3 \) admits a \((\Delta(G) + 3, 2)\)-incidence coloring. Therefore, \( \chi_i(G) \leq \Delta(G) + 3 \).

Concerning planar graphs, we have the following:

**Observation 6** For every planar graph \( G \) with girth at least 6, \( mad(G) < 2g/(g - 2) \).

Hence, we obtain:

**Corollary 7** Every planar graph \( G \) with girth \( g \geq 6 \) admits a \((\Delta(G) + 3, 2)\)-incidence coloring. Therefore, \( \chi_i(G) \leq \Delta(G) + 3 \).

**Proof:** By Observation 6 we have \( mad(G) < 2g/(g - 2) \leq (2 \times 6)/(6 - 2) = 3 \) and we get the result from Corollary 5.

If the graph has maximum degree at least 5, the previous result can be improved:

**Theorem 8** Every graph \( G \) with \( mad(G) < 3 \) and \( \Delta(G) \geq 5 \) admits a \((\Delta(G) + 2, 2)\)-incidence coloring. Therefore, \( \chi_i(G) \leq \Delta(G) + 2 \).

**Proof:** Suppose that the theorem is false and let \( G \) be a minimal counter-example (with respect to the number of vertices). We first show that \( G \) must avoid all the configurations depicted in Fig. 2.
(1) Let $w$ denote the unique neighbor of $v$ in $G$. Due to the minimality of $G$, the graph $G' = G \setminus w$ admits a $(\Delta(G) + 2, 2)$-incidence coloring $\sigma'$. We extend $\sigma'$ to a $(\Delta(G) + 2, 2)$-incidence coloring $\sigma$ of $G$. Since $|F^\sigma_G(w, vw)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$ and since we have $\Delta(G) + 2$ possible colors, there is a color $a$ such that $a \notin F^\sigma_G(w, vw)$. We set $\sigma(w, vw) = a$ and $\sigma(v, vw) = b$, for any color $b$ in $\sigma'(A_w)$.

(2) Let $w_1$, $w_2$ denote the two neighbors of $v$ in $G$. Due to the minimality of $G$, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 2, 2)$-incidence coloring $\sigma'$. We extend $\sigma'$ to a $(\Delta(G) + 2, 2)$-incidence coloring $\sigma$ of $G$.

Since $|F^\sigma_G(w_1, w_1v)| = |\sigma'(I_{w_1}) \cup \sigma'(A_{w_1})| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$ and since we have $\Delta(G) + 2$ possible colors, there is a color $a$ such that $a \notin F^\sigma_G(w_1, w_1v)$. We set $\sigma(w_1, w_1v) = a$. If $|\sigma'(A_{w_2}) \setminus \{a\}| \geq 1$ then there is a color $b \in \sigma'(A_{w_2}) \setminus \{a\}$ and if $\sigma'(A_{w_2}) = \{a\}$, since $|F^\sigma_G(v, vw_2)| = |\sigma'(I_{w_2}) \cup \{a\}| \leq 3 + 1 = 4 \leq \Delta(G) - 1$, there is a color $b$ such that $b \notin F^\sigma_G(v, vw_2)$. We set $\sigma(v, vw_2) = b$.

Now, if $|\sigma'(A_{w_1}) \setminus \{b\}| \geq 1$ then there is a color $c \in \sigma'(A_{w_1}) \setminus \{b\}$ and if $\sigma'(A_{w_1}) = \{b\}$, since $|F^\sigma_G(v, vw_1)| = |\sigma'(I_{w_1}) \cup \{b\}| \leq \Delta(G) + 1$, there is a color $c$ such that $c \notin F^\sigma_G(v, vw_1)$. We set $\sigma(v, vw_1) = c$.

Since $|F^\sigma_G(w_2, w_2v)| = |\sigma'(I_{w_2}) \cup \sigma'(A_{w_2}) \cup \{c\}| \leq 3 + 2 + 1 = 6 \leq \Delta(G) + 1$, there is a color $d$ such that $d \notin F^\sigma_G(w_2, w_2v)$, and we set $\sigma(w_2, w_2v) = d$.

(3) Let $u_i$, $1 \leq i \leq 5$, denote the five neighbors of $v$ and $w_i$ denote the other neighbor of $u_i$ in $G$ (see Figure 2(3)). Due to the minimality of $G$, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 2, 2)$-incidence coloring $\sigma$. We extend $\sigma'$ to a $(\Delta(G) + 2, 2)$-incidence coloring $\sigma$ of $G$.

Let $a_i = \sigma'(w_i, w_iu_i), 1 \leq i \leq 5$. Since we have $\Delta(G) + 2 \geq 7$ colors, there is a color $x$ distinct from $a_i$ for every $i, 1 \leq i \leq 5$.

Since $|F^\sigma_G(u_i, u_iw_i)| = |\sigma'(I_{u_i})| \leq \Delta(G)$ we have two possible colors for the incidence $(u_i, u_iw_i)$ for every $i, 1 \leq i \leq 5$. So, we can suppose that $\sigma'(u_i, u_iw_i) \neq x$ for every $i, 1 \leq i \leq 5$. We set $\sigma(u_i, u_iw_i) = x$ for every $i, 1 \leq i \leq 5$.

Since $F^\sigma_G(v, uu_i) = \{x, \sigma'(u_i, u_iw_i)\}$ for every $i, 1 \leq i \leq 5$, and since we have at least 7 colors, there is 5 distinct colors $c_1, c_2, \ldots, c_5$ such that $c_i \notin \{x, \sigma'(u_i, u_iw_i)\}, 1 \leq i \leq 5$, and we set $\sigma(v, uu_i) = c_i$ for every $i, 1 \leq i \leq 5$. 

Fig. 2: Forbidden configurations for the proof of Theorem [8]
It is easy to check that in every case we have obtained a \((\Delta(G) + 2, 2)\)-incidence coloring of \(G\), which contradicts our assumption.

We now associate with each vertex \(v\) of \(G\) an initial charge \(d(v) = d_G(v)\), and we use the following discharging procedure: each vertex of degree at least 5 gives \(1/2\) to each of its 2-neighbors.

We shall prove that the modernized degree \(d^*(v)\) of each vertex of \(G\) is at least 3 which contradicts \(\text{mad}(G) < 3\) (since \(\sum_{u \in G} d^*(u) = \sum_{u \in G} d(u)\)). Let \(v\) be a vertex of \(G\); we consider the possible cases for old degree \(d_G(v)\) of \(v\) (since \(G\) does not contain the configuration 2(1), we have \(d_G(v) \geq 2\)):

\(d_G(v) = 2\).

Since \(G\) does not contain the configuration 2(2) the two neighbors of \(v\) are of degree at least 5. Therefore, \(v\) receives \(1/2\) from each of its neighbors so that \(d^*(v) = 2 + 1/2 + 1/2 = 3\).

\(3 \leq d_G(v) \leq 4\).

In this case we have \(d^*(v) = d_G(v) \geq 3\).

\(d_G(v) = 5\).

Since \(G\) does not contain the configuration 2(3) at least one of the neighbors of \(v\) is of degree at least 3 and \(v\) gives at most \(4 \times 1/2 = 2\). We obtain \(d^*(v) \geq 5 - 2 = 3\).

\(d_G(v) = k \geq 6\).

In this case \(v\) gives at most \(k \times 1/2\) so that \(d^*(v) \geq k - k/2 = k/2 \geq 6/2 = 3\).

Therefore, every vertex in \(G\) gets a modernized degree of at least 3 and the theorem is proved. \(\square\)

**Remark 9** The previous result also holds for graphs with maximum degree 2 and for graphs with maximum degree 3 (by the result from [6]) but the question remains open for graphs with maximal degree 4.

As previously, for planar graphs we obtain:

**Corollary 10** Every planar graph \(G\) of girth \(g \geq 6\) with \(\Delta(G) \geq 5\) admits a \((\Delta(G) + 2, 2)\)-incidence coloring. Therefore, \(\chi_i(G) \leq \Delta(G) + 2\).

For graphs with maximum average degree less than \(22/9\), we have:

**Theorem 11** Every graph \(G\) with \(\text{mad}(G) < 22/9\) admits a \((\Delta(G) + 2, 2)\)-incidence coloring. Therefore, \(\chi_i(G) \leq \Delta(G) + 2\).

**Proof:** It is enough to consider the case of graphs with maximum degree at most 4, since for graphs with maximum degree at least 5 the theorem follows from Theorem 8. Suppose that the theorem is false and let \(G\) be a minimal counter-example (with respect to the number of vertices and edges). Observe first that we have \(\Delta(G) \geq 3\) since otherwise we obtain by Theorem 1 that \(\chi_i(G) \leq 2\Delta(G) \leq \Delta(G) + 2\) and every \((\Delta(G) + 2)\)-incidence coloring of \(G\) is obviously a \((\Delta(G) + 2, 2)\)-incidence coloring.

We first show that \(G\) cannot contain any of the configurations depicted in Figure 3.

(1) This case is similar to case 1 of Theorem 8.
(2) Let \( x \) (resp. \( y \)) denote the other neighbor of \( u \) (resp. \( v \)) in \( G \). Due to the minimality of \( G \), the graph \( G' = G \setminus uv \) admits a \((\Delta(G) + 2, 2)\)-incidence coloring \( \sigma' \). We extend \( \sigma' \) to a \((\Delta(G) + 2, 2)\)-incidence coloring \( \sigma \) of \( G \).

Suppose \( \sigma'(u,ux) = a, \sigma'(v,vy) = b, \sigma'(x,xu) = c \) and \( \sigma'(y,yv) = d \).

Suppose first that \(|\{a, b, c, d\}| = 4\). In that case, we set \( \sigma(u,uv) = d \) and \( \sigma(v,vu) = c \).

Now, if \(|\{a, b, c, d\}| \leq 3\), we set \( \sigma(u,uv) = e \) and \( \sigma(v,vu) = f \) for any \( e, f \notin \{a, b, c, d\} \).

(3) Let \( u_1, u_2 \) and \( u_3 \) denote the three neighbors of \( v \) and \( w_i \) denotes the other neighbor of \( u_i, 1 \leq i \leq 3 \), in \( G \). Due to the minimality of \( G \), the graph \( G' = G \setminus v \) admits a \((\Delta(G) + 2, 2)\)-incidence coloring \( \sigma' \). We extend \( \sigma' \) to a \((\Delta(G) + 2, 2)\)-incidence coloring \( \sigma \) of \( G \).

Suppose that \( a_i = \sigma'(w_i, w_iu_i) \), \( 1 \leq i \leq 3 \). Since we have \( \Delta(G) + 2 \geq 5 \) colors, there is a color \( x \) distinct from \( a_i \) for every \( i, 1 \leq i \leq 3 \).

Since \( F_{G'}(u_i, u_iw_i) = |\sigma'(I_{w_i})| \leq \Delta(G) \) we have at least two colors for the incidence \((u_i, u_iw_i)\) for every \( i, 1 \leq i \leq 3 \). Thus, we can suppose \( \sigma'(u_i, u_iw_i) \neq x \) for every \( i, 1 \leq i \leq 3 \). We then set \( \sigma(u_i, u_iw_i) = x \) for every \( i, 1 \leq i \leq 3 \).

Since \( F_{G'}(v, uv_i) = \{x, \sigma'(u_i, u_iw_i)\} \) for every \( i, 1 \leq i \leq 3 \), and since we have at least 5 colors, there are 3 distinct colors \( c_1, c_2 \) and \( c_3 \) such that \( c_i \notin \{x, \sigma'(u_i, u_iw_i)\}, 1 \leq i \leq 3 \). We then set \( \sigma(v, uv_i) = c_i \) for every \( i, 1 \leq i \leq 3 \).

Therefore, in all cases we obtain a \((\Delta(G) + 2, 2)\)-incidence coloring of \( G \), which contradicts our assumption.

We now associate with each vertex \( v \) of \( G \) an initial charge \( d(v) = d_G(v) \), and we use the following discharging procedure: each vertex of degree at least 3 gives 2/9 to each of its 2-neighbors.

We shall prove that the modernized degree \( d^* \) of each vertex of \( G \) is at least 22/9 which contradicts the assumption \( mad(G) < 22/9 \). Let \( v \) be a vertex of \( G \); we consider the possible cases for old degree \( d_G(v) \) of \( v \) (since \( G \) does not contain the configuration \([3][1]\)), we have \( d_G(v) \geq 2\):

\[
d_G(v) = 2.
\]

Since \( G \) does not contain the configuration \([3][2]\) the two neighbors of \( v \) are of degree at least 3. Therefore, \( v \) receives then \( 2/9 \) from each of its neighbors so that \( d^*(v) = 2 + 2/9 + 2/9 = 22/9 \).
Corollary 12

Every planar graph $G$ therefore, every vertex in $d$, $\chi$, $G$ theorem 13

Let $F(2)$

(1) $(v)$ in Figure 4.

$G$, with respect to the number of vertices. We first show that coloring. Therefore, $\chi$, $G$ proof:

Since for every graph $G$, the graph $G'$ $\chi_i(G) = \Delta(G) + 2$, $G$ 

Finally, for graphs with maximum average degree less than $16/7$, we have:

Theorem 13

Every graph $G$ with $\text{mad}(G) < 16/7$ and $\Delta(G) \geq 4$ admits a $(\Delta(G) + 1, 1)$-incidence coloring. Therefore, $\chi_i(G) = \Delta(G) + 1$.

Proof: Since for every graph $G$, $\chi_i(G) \geq \Delta(G) + 1$, it is enough to prove that $G$ admits a $(\Delta(G) + 1, 1)$-incidence coloring. Suppose that the theorem is false and let $G$ be a minimal counter-example (with respect to the number of vertices). We first show that $G$ cannot contain any of the configurations depicted in Figure 4.

(1) This case is similar to case 1 of Theorem 8.

(2) Let $u$, $i = 1, 2$, be the two neighbors of $v$ and $w_i$ denote the other neighbor of $u_i$ in $G$. Due to the minimality of $G$, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 1, 1)$-incidence coloring $\sigma'$. We extend $\sigma'$ to a $(\Delta(G) + 1, 1)$-incidence coloring $\sigma$ of $G$.

Suppose that $\sigma'(w_1, w_1 u_1) = a$, $\sigma'(u_1, u_1 w_1) = b$, $\sigma'(w_2, w_2 w_2) = c$ and $\sigma'(u_2, u_2 w_2) = d$. Since $|F_G^c(u_1, w_1 u_1) \cup \{c\}| = |\sigma'(I_{w_1}) \cup \sigma'(A_{w_1}) \cup \{c\}| \leq \Delta(G) - 2 + 1 + 1 = \Delta(G)$, we can suppose that $a \neq c$. We then set $\sigma(v, vu_1) = a$ and $\sigma(v, vu_2) = c$.

Now, since $F_G^c(u_1, u_1 v) \cup F_G^c(u_2, u_2 v) = \{a, b, c, d\}$ and since we have at least $\Delta(G) + 1 \geq 5$ colors, there is a color $x$ such that $x \notin \{a, b, c, d\}$. We then set $\sigma(u_1, u_1 v) = \sigma(u_2, u_2 v) = x$. 

Fig. 4: Forbidden configurations for the proof of Theorem 13

$d_G(v) = 3$.

Since $G$ does not contain the configuration, $v$ is adjacent to at most two 2-vertices and $v$ gives at most $2 \times 2/9 = 4/9$. We obtain $d^*(v) \geq 3 - 4/9 = 23/9 \geq 22/9$.

$d_G(v) = 4$.

In this case, $v$ gives at most $4 \times 2/9 = 8/9$ so that $d^*(v) \geq 4 - 8/9 = 28/9 \geq 22/9$.

Therefore, every vertex in $G$ gets a modernized degree of at least 3 and the theorem is proved. 

By considering cycles of length $\ell \not\equiv 0 \pmod{3}$, we get that the upper bound of Theorem 11 is tight.

As previously, for planar graphs we obtain:

Corollary 12

Every planar graph $G$ of girth $g \geq 11$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

Finally, for graphs with maximum average degree less than $16/7$, we have:
(3) Let $u_i$, $1 \leq i \leq 3$ be the three neighbors of $v$, $x_i$ denote the other neighbor of $u_i$ and $w_i$ denote the other neighbor of $x_i$ in $G$. Due to the minimality of $G$, the graph $G' = G \setminus \{v, u_1, u_2, u_3\}$ admits a $(\Delta(G) + 1, 1)$-incidence coloring $\sigma'$. We extend $\sigma'$ to a $(\Delta(G) + 1, 1)$-incidence coloring $\sigma$ of $G$.

Suppose that $\sigma'(w_i, w_i; x_i) = a_i$ and $\sigma'(x_i, x_i; w_i) = b_i$ for every $i, 1 \leq i \leq 3$. Since $|F_G'(w_i, w_i; x_i) \cup \{b_i\}| = |\sigma'(I_{w_i}) \setminus \{a_i\} \cup \{b_i, b_1\}| \leq 2 + 2 = 4$ for $i = 2, 3$, and since we have $\Delta(G) + 1 \geq 5$ colors, we can suppose that $a_2 \neq b_1 \neq a_3$. We then set $\sigma(u_i, u_i x_i) = a_i$ and $\sigma(u_i, u_i v) = b_i$ for every $i, 1 \leq i \leq 3$.

Since $F_G'(v, v u_1) \cup F_G'(x_j, x_j u_j) = \{b_1, b_i, a_j\}$ for $j = 2, 3$, there are two distinct colors $c_2$ and $c_3$ such that $c_j \notin \{b_1, b_j, a_j\}, j = 2, 3$. We set $\sigma(v, v u_j) = \sigma(x_j, x_j u_j) = c_j, j = 2, 3$.

Now, since $F_G'(v, v u_1) \cup F_G'(x_1, x_1 u_1) = \{a_1, b_1, c_2, c_3\}$ and since we have at least 5 colors, there is a color $c_1$ such that $c_1 \notin \{a_1, b_1, c_2, c_3\}$. We then set $\sigma(v, v u_1) = \sigma(x_1, x_1 u_1) = c_1$.

Therefore, in all cases we obtain a $(\Delta(G) + 1, 1)$-incidence coloring of $G$, which contradicts our assumption.

We now associate with each vertex $v$ of $G$ an initial charge $d(v) = d_G(v)$, and we use the following discharging procedure:

(R1) each vertex of degree 3 gives $2/7$ to each of its 2-neighbors which has a 2-neighbor adjacent to a 3-vertex and gives $1/7$ to its other 2-neighbors.

(R2) each vertex of degree at least 4 gives $2/7$ to each of its 2-neighbors and gives $1/7$ to each 2-vertex which is adjacent to one of its 2-neighbors.

We shall prove that the modernized degree $d^*$ of each vertex of $G$ is at least $16/7$ which contradicts the assumption $\text{mod}(G) < 16/7$. Let $v$ be a vertex of $G$, we consider the possible cases for old degree $d_G(v)$ of $v$ (since $G$ does not contain the configuration 4(1), we have $d_G(v) \geq 2$):

$d_G(v) = 2$. In this case we consider five subcases:

1. $v$ has two 2-neighbors, say $z_1$ and $z_2$. Let $y_i$ be the other neighbor of $z_i$, $i = 1, 2$, in $G$. Since $G$ does not contain the configuration 4(2), $y_i$ is of degree $\Delta(G) \geq 4$ for $i = 1, 2$. Each $y_i, i = 1, 2$, gives $1/7$ to $v$ so that $d^*(v) = 2 + 1/7 + 1/7 = 16/7$.

2. $v$ is adjacent to a 3-vertex $z_1$ and a 2-vertex which is itself adjacent to a 3-vertex. In this case $v$ receives $2/7$ from $z_1$ and we have $d^*(v) = 2 + 2/7 = 16/7$.

3. $v$ is adjacent to a 3-vertex $z_1$ and a 2-vertex which is itself adjacent to a vertex $z_2$ of degree at least 4. In this case $v$ receives $1/7$ from $z_1$ and $1/7$ from $z_2$ so that $d^*(v) = 2 + 1/7 + 1/7 = 16/7$.

4. $v$ is adjacent to two 3-vertices that both gives $1/7$ to $v$ so that $d^*(v) = 2 + 1/7 + 1/7 = 16/7$.

5. One of the two neighbors of $v$ is of degree at least 4. In this case $v$ receives at least $2/7$ so that $d^*(v) \geq 2 + 2/7 = 16/7$.

$d_G(v) = 3$.

Let $u_1$, $u_2$ and $u_3$ be the three neighbors of $v$. We consider two subcases according to the degrees of $u_i$'s.
1. One of the $u_i$'s is of degree at least 3, say $u_1$. In this case $v$ gives at most $2/7$ to $u_2$ and $2/7$ to $u_3$ so that $d^*(v) \geq 3 - 2/7 - 2/7 = 17/7 \geq 16/7$.

2. All the $u_i$'s are of degree 2. Let $x_i$ be the other neighbor of $u_i$, $1 \leq i \leq 3$.

(a) One of the $x_i$'s is of degree at least 3, say $x_1$. In this case $v$ gives $1/7$ to $u_1$, at most $2/7$ to $u_2$ and at most $2/7$ to $u_3$. We then have $d^*(v) \geq 3 - 1/7 - 2/7 - 2/7 = 16/7$.

(b) All the $x_i$'s are of degree 2. Let $w_i$ be the other neighbor of $x_i$ in $G$, $1 \leq i \leq 3$. Since $G$ does not contain the configuration $(4)(2)$ we have $d_G(w_i) \geq 3$ for every $i$, $1 \leq i \leq 3$, and since $G$ does not contain the configuration $(4)(3)$, at most one of the $w_i$'s, $1 \leq i \leq 3$, can be of degree 3. Thus, we can suppose without loss of generality that $d_G(w_1)$ and $d_G(w_2) \geq 4$. In this case, $v$ gives $1/7$ to $w_1$, $1/7$ to $w_2$ and at most $2/7$ to $w_3$. We then have $d^*(v) \geq 3 - 1/7 - 1/7 - 2/7 = 17/7 \geq 16/7$.

$d_G(v) = k \geq 4$.

In this case, $v$ gives at most $k \times (2/7 + 1/7) = 3k/7$ so that $d^*(v) \geq k - 3k/7 = 4k/7 \geq 16/7$.

Therefore, every vertex in $G$ gets a modernized degree of at least $16/7$ and the theorem is proved. \hfill \square

Considering the lower bound discussed in Section 1, we get that the upper bound of Theorem 13 is tight.

**Remark 14** For every graph $G$, the square of $G$, denoted by $G^2$, is the graph obtained from $G$ by linking any two vertices at distance at most 2. It is easy to observe that providing a $(k, 1)$-incidence coloring of $G$ is the same as providing a proper $k$-vertex-colouring of $G^2$, for every $k$ (by identifying for every vertex $v$ the color of $A_v$ in $G$ with the color of $v$ in $G^2$). By considering the cycle $C_4$ on four vertices (note that $C_4^2 = K_4$) we get that the previous result cannot be extended to the case $\Delta = 2$. Consider now the graph $H$ obtained from the cycle $C_5$ on five vertices by adding one pending edge with a new vertex. Since $H^2$ contains a subgraph isomorphic to $K_5$, we similarly get that the previous result cannot be extended to the case $\Delta = 3$.

As previously, for planar graphs we obtain:

**Corollary 15** Every planar graph $G$ of girth $g \geq 16$ and with $\Delta(G) \geq 4$ admits a $(\Delta(G) + 1, 1)$-incidence coloring. Therefore, $\chi_i(G) = \Delta(G) + 1$. 

Maximum average degree and incidence chromatic number

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References


