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The number of spanning trees of finite Sierpiński graphs

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We show that the number of spanning trees in the finite Sierpiński graph of level \( n \) is given by

\[
\sqrt{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} (\sqrt{540})^n.
\]

The proof proceeds in two steps: First, we show that the number of spanning trees and two further quantities satisfy a 3-dimensional polynomial recursion using the self-similar structure. Secondly, it turns out, that the dynamical behavior of the recursion is given by a 2-dimensional polynomial map, whose iterates can be computed explicitly.

Keywords: combinatorial enumeration, spanning trees, finite Sierpiński graphs

1 Introduction

The enumeration of spanning trees in a finite graph ranges among the classical tasks of combinatorics and has been studied for more than 150 years. The number of spanning trees in a graph \( X \) is often called the complexity and denoted by \( \tau(X) \). Let us recall the famous Matrix-Tree Theorem of Kirchhoff [Kirchhoff(1847)]: The complexity \( \tau(X) \) of a graph \( X \) is equal to any cofactor of the Laplace matrix of \( X \), which is the degree matrix of \( X \) minus the adjacency matrix of \( X \). As a consequence the number of spanning trees in the complete graph with \( n \) vertices is given by \( 2^{n-2} \).

In the following the complexity of finite Sierpiński graphs is computed. These graphs are discrete analogs of the well-known Sierpiński gasket (see [Sierpinski(1915)]) and can be defined as follows: Denote by \( e_1 = (1, 0, 0) \), \( e_2 = (0, 1, 0) \), \( e_3 = (0, 0, 1) \) the canonical basis vectors of \( \mathbb{R}^3 \). For \( n = 0 \) the Sierpiński graph \( X_0 \) is given by \( V_{X_0} = \{e_1, e_2, e_3\} \) and \( E_{X_0} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\}\} \). For \( n > 0 \) the Sierpiński graph \( X_n \) is defined iteratively by

\[
V_{X_n} = (2^{n-1}e_1 + V_{X_{n-1}}) \cup (2^{n-1}e_2 + V_{X_{n-1}}) \cup (2^{n-1}e_3 + V_{X_{n-1}})
\]

and

\[
E_{X_n} = (2^{n-1}e_1 + E_{X_{n-1}}) \uplus (2^{n-1}e_2 + E_{X_{n-1}}) \uplus (2^{n-1}e_3 + E_{X_{n-1}}).
\]

The graph \( X_n \) is called Sierpiński graph of level \( n \); see Figure I for \( X_0 \), \( X_1 \), and \( X_2 \). A simple computation shows that \( |V_{X_n}| = \frac{3}{2}(3^n + 1) \) and \( |E_{X_n}| = 3^{n+1} \).

2 Main result

The complexity \( \tau(X_n) \) of the Sierpiński graph of level \( n \) is given by

\[
\tau(X_n) = \sqrt{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} (\sqrt{540})^n.
\]

The first numbers of this sequence are 3, 54, 524880, 803355125990400000, . . .

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The proof is based on the work in the paper [Teufl and Wagner(2006)]. For \( n \geq 0 \) and \( k \in \{1, 2, 3\} \) denote by \( \mathcal{F}_k(n) \) the set of spanning forests in \( X_n \) with \( k \) components, so that each component contains at least one vertex of \( B_n = 2^n \{e_1, e_2, e_3\} \), and set \( \mathcal{F}(n) = \mathcal{F}_1(n) \uplus \mathcal{F}_2(n) \uplus \mathcal{F}_3(n) \). For \( n \geq 0 \) and \( i \in \{1, 2, 3\} \) let \( X_n^i \) be the subgraph of \( X_n \), which is induced by \( 2^{n-1} e_i + V X_{n-1} \), and note that \( X_n^i \) is isomorphic to \( X_{n-1} \). The restriction of a spanning forest in \( \mathcal{F}_k(n) \) to \( X_n^j \) yields a spanning forest in \( \mathcal{F}_j(n-1) \) for some \( j \in \{1, 2, 3\} \). Hence each spanning forest in \( \mathcal{F}(n) \) can be decomposed into three forests in \( \mathcal{F}(n-1) \). For example, each spanning tree in \( \mathcal{F}_1(n) \) is composed of two trees in \( \mathcal{F}_1(n-1) \) and one forest in \( \mathcal{F}_2(n-1) \), see Figure 2.

Denote by \( c(n) \) the 3-dimensional counting vector
\[
c(n) = (c_1(n), c_2(n), c_3(n)) = (|\mathcal{F}_1(n)|, \frac{1}{3}|\mathcal{F}_2(n)|, |\mathcal{F}_3(n)|).
\]
Note that there are three possibilities to arrange the vertices of \( B_n \) in two components of a spanning forest in \( \mathcal{F}_2(n) \), and the factor \( \frac{1}{3} \) takes care of that. Furthermore, the complexity \( \tau(X_n) \) is given by \( \tau(X_n) = c_1(n) \).

The decomposition of forests in \( \mathcal{F}(n) \) into three forests in \( \mathcal{F}(n-1) \) implies that \( c(n) \) satisfies the polynomial recursion
\[
c(n) = Q(c(n-1))
\]
for \( n > 0 \), where \( Q \) is given by

\[
Q : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 6x_1^2x_2 \\ 7x_1x_2^2 + x_2^3x_3 \\ 14x_2^3 + 12x_1x_2x_3 \end{pmatrix}
\]

The initial vector is given by \( c(0) = (3, 1, 1) \).

Now define the map \( v : \mathbb{R}^2_+ \to \mathbb{R}^3 \) by

\[
v(a, b) = (3a, b, b^2/a),
\]

then \( c(0) = v(1, 1) \) and \( c(1) = v(18, 30) \). Finally, define \( T : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \) by

\[
T(a, b) = (18a^2b, 30ab^2),
\]

then the function \( Q \) maps the set \( v(\mathbb{R}^2_+) \) into itself and \( Q \circ v = v \circ T \).

Thus the vector \( c(n) \) is given by

\[
c(n) = v(T^n(1, 1)),
\]

where \( T^n \) is the \( n \)-fold iterate of \( T \). The \( n \)-fold iterate of \( T \) can be computed explicitly:

\[
T^n(a, b) = \left( \frac{3}{4} \right)^{-n/2} (\sqrt{540})^{3^n-1} a^{(3^n+1)/2} b^{(3^n-1)/2}
\]

for all \( n \geq 0 \). This implies the statement.

4 Conclusion

The ideas of this proof can be formalized and generalized to a large class of fractal-like graphs. In addition, it is also possible to count different combinatorial objects in those graphs; for example matchings (independent edge subsets), connected subsets, subtrees, colorings, factors, and vertex or edge coverings. Using a decomposition argument it is possible to obtain a multi-dimensional polynomial recursion. Of course, one cannot expect to get explicit formulas in general. However, the asymptotic behavior can be deduced under suitable conditions. See [Teufl and Wagner(2006)] for more information.

An interesting reformulation of the Matrix-Tree Theorem states, that the complexity of a graph \( X \) is given by the product of all non-zero eigenvalues of the Laplace matrix of \( X \) taking the multiplicities into account divided by the number of vertices. Note that the Dirichlet spectrum of the Laplace matrix of finite Sierpiński graphs \( X_n \) with respect to the boundary \( B_n \) is well understood, see [Shima(1991)]. However, it seems to be much more difficult to understand the normal spectrum.

References


