Spanning trees of finite Sierpiński graphs
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To cite this version:

HAL Id: hal-01184698
https://hal.inria.fr/hal-01184698
Submitted on 17 Aug 2015

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We show that the number of spanning trees in the finite Sierpiński graph of level \( n \) is given by

\[
\sqrt{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} \left(\sqrt{540}\right)^n.
\]

The proof proceeds in two steps: First, we show that the number of spanning trees and two further quantities satisfy a 3-dimensional polynomial recursion using the self-similar structure. Secondly, it turns out, that the dynamical behavior of the recursion is given by a 2-dimensional polynomial map, whose iterates can be computed explicitly.

**Keywords:** combinatorial enumeration, spanning trees, finite Sierpiński graphs

## 1 Introduction

The enumeration of spanning trees in a finite graph ranges among the classical tasks of combinatorics and has been studied for more than 150 years. The number of spanning trees in a graph \( X \) is often called the **complexity** and denoted by \( \tau(X) \). Let us recall the famous Matrix-Tree Theorem of Kirchhoff [Kirchhoff(1847)]: The complexity \( \tau(X) \) of a graph \( X \) is equal to any cofactor of the Laplace matrix of \( X \), which is the degree matrix of \( X \) minus the adjacency matrix of \( X \). As a consequence the number of spanning trees in the complete graph with \( n \) vertices is given by \( n^{n-2} \).

In the following the complexity of finite Sierpiński graphs is computed. These graphs are discrete analogs of the well-known Sierpiński gasket (see Sierpinski(1915)) and can be defined as follows: Denote by \( e_1 = (1,0,0) \), \( e_2 = (0,1,0) \), and \( e_3 = (0,0,1) \) the canonical basis vectors of \( \mathbb{R}^3 \). For \( n = 0 \) the Sierpiński graph \( X_0 \) is given by \( V X_0 = \{e_1, e_2, e_3\} \) and \( EX_0 = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\}\} \). For \( n > 0 \) the Sierpiński graph \( X_n \) is defined iteratively by

\[
V X_n = (2^{n-1}e_1 + V X_{n-1}) \cup (2^{n-1}e_2 + V X_{n-1}) \cup (2^{n-1}e_3 + V X_{n-1})
\]

and

\[
EX_n = (2^{n-1}e_1 + EX_{n-1}) \uplus (2^{n-1}e_2 + EX_{n-1}) \uplus (2^{n-1}e_3 + EX_{n-1}).
\]

The graph \( X_n \) is called **Sierpiński graph of level \( n \)**; see Figure[1] for \( X_0, X_1, \) and \( X_2 \). A simple computation shows that \( |V X_n| = \frac{3}{2}(3^n + 1) \) and \( |EX_n| = 3^{n+1} \).

## 2 Main result

The complexity \( \tau(X_n) \) of the Sierpiński graph of level \( n \) is given by

\[
\tau(X_n) = \sqrt{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} \left(\sqrt{540}\right)^n.
\]

The first numbers of this sequence are 3, 54, 524880, 803355125990400000, . . .
Since every spanning tree is a subset of the edge set $EX_n$, it is natural to rewrite this formula in terms of $|EX_n|$: This yields

$$\tau(X_n) = \sqrt[3]{\frac{5}{12}} |EX_n|^{(1-2/d_s)/2} \left(\frac{12}{\sqrt{540}}\right)^{|EX_n|}.$$ 

Here $d_s$ is the spectral dimension of the Sierpiński gasket, which is given by

$$d_s = 2 \frac{\log 3}{\log 5}.$$ 

The spectral dimension was first introduced using the integrated density of states of the Laplacian on the infinite Sierpiński graph. Later on, the exponent $d_s$ was studied from several points of view, see for example [Barlow(1998), Kigami(2001)] and the references therein. Furthermore, note that the fraction $\frac{5}{2}$ is the so-called resistance scaling factor of the Sierpiński gasket.

We conjecture that the formula

$$\tau(X_n) = C |EX_n|^\beta (1-2/d_s) \alpha |EX_n|$$

holds for a large class of sequences $(X_n)_{n \geq 0}$ of finite self-similar graphs, where $C > 0$, $\beta \geq 0$ and $\alpha \in (1, 2)$ are some constants; see [Teufl and Wagner(2006)].

### 3 Proof

The proof is based on the work in the paper [Teufl and Wagner(2006)]. For $n \geq 0$ and $k \in \{1, 2, 3\}$ denote by $F_k(n)$ the set of spanning forests in $X_n$ with $k$ components, so that each component contains at least one vertex of $B_n = 2^n \{e_1, e_2, e_3\}$, and set $F(n) = F_1(n) \cup F_2(n) \cup F_3(n)$. For $n \geq 0$ and $i \in \{1, 2, 3\}$ let $X_n^i$ be the subgraph of $X_n$, which is induced by $2^{n-1} e_i + V X_{n-1}$, and note that $X_n^i$ is isomorphic to $X_{n-1}$. The restriction of a spanning forest in $F_k(n)$ to $X_n^i$ yields a spanning forest in $F_{k-1}(n-1)$ for some $j \in \{1, 2, 3\}$. Hence each spanning forest in $F(n)$ can be decomposed into three forests in $F(n-1)$. For example, each spanning tree in $F_1(n)$ is composed of two trees in $F_2(n-1)$ and one forest in $F_2(n-1)$, see Figure 2.

Denote by $c(n)$ the 3-dimensional counting vector

$$c(n) = (c_1(n), c_2(n), c_3(n)) = (|F_1(n)|, \frac{1}{2}|F_2(n)|, |F_3(n)|).$$

Note that there are three possibilities to arrange the vertices of $B_n$ in two components of a spanning forest in $F_2(n)$, and the factor $\frac{1}{2}$ takes care of that. Furthermore, the complexity $\tau(X_n)$ is given by $\tau(X_n) = c_1(n)$.

The decomposition of forests in $F(n)$ into three forests in $F(n-1)$ implies that $c(n)$ satisfies the polynomial recursion

$$c(n) = Q(c(n-1))$$
for $n > 0$, where $Q$ is given by

$$Q : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 6x_1^2 x_2 \\ 7x_1 x_2^2 + x_2^2 x_3 \\ 14x_2^3 + 12x_1 x_2 x_3 \end{pmatrix}$$

The initial vector is given by $c(0) = (3, 1, 1)$.

Now define the map $v : \mathbb{R}_+^2 \to \mathbb{R}^3$ by

$$v(a, b) = (3a, b, b^2/a),$$

then $c(0) = v(1, 1)$ and $c(1) = v(18, 30)$. Finally, define $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ by

$$T(a, b) = (18a^2 b, 30ab^2),$$

then the function $Q$ maps the set $v(\mathbb{R}_+^2)$ into itself and $Q \circ v = v \circ T$.

Thus the vector $c(n)$ is given by

$$c(n) = v(T^n(1, 1)),$$

where $T^n$ is the $n$-fold iterate of $T$. The $n$-fold iterate of $T$ can be computed explicitly:

$$T^n(a, b) = \begin{pmatrix} \frac{3^n}{4} \left( \sqrt[3]{60} \right)^{3n-1} a^{(3n+1)/2} b^{(3n-1)/2} \\ \frac{3^n}{4} \left( \sqrt[3]{60} \right)^{3n-1} a^{(3n-1)/2} b^{(3n+1)/2} \end{pmatrix}$$

for all $n \geq 0$. This implies the statement.

## 4 Conclusion

The ideas of this proof can be formalized and generalized to a large class of fractal-like graphs. In addition, it is also possible to count different combinatorial objects in those graphs; for example matchings (independent edge subsets), connected subsets, subtrees, colorings, factors, and vertex or edge coverings. Using a decomposition argument it is possible to obtain a multi-dimensional polynomial recursion. Of course, one cannot expect to get explicit formulas in general. However, the asymptotic behavior can be deduced under suitable conditions. See [Teufl and Wagner (2006)] for more information.

An interesting reformulation of the Matrix-Tree Theorem states, that the complexity of a graph $X$ is given by the product of all non-zero eigenvalues of the Laplace matrix of $X$ taking the multiplicities into account divided by the number of vertices. Note that the Dirichlet spectrum of the Laplace matrix of finite Sierpiński graphs $X_n$ with respect to the boundary $B_n$ is well understood, see [Shima (1991)]. However, it seems to be much more difficult to understand the normal spectrum.

## References


