The number of spanning trees of finite Sierpiński graphs

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We show that the number of spanning trees in the finite Sierpiński graph of level n is given by

$$\sqrt[4]{\frac{3}{20}}\left(\frac{5}{3}\right)^{-n/2}\left(\sqrt[4]{540}\right)^{3^n}.$$

The proof proceeds in two steps: First, we show that the number of spanning trees and two further quantities satisfy a 3-dimensional polynomial recursion using the self-similar structure. Secondly, it turns out, that the dynamical behavior of the recursion is given by a 2-dimensional polynomial map, whose iterates can be computed explicitly.

Keywords: combinatorial enumeration, spanning trees, finite Sierpiński graphs

1 Introduction

The enumeration of spanning trees in a finite graph ranges among the classical tasks of combinatorics and has been studied for more than 150 years. The number of spanning trees in a graph X is often called the *complexity* and denoted by $\tau(X)$. Let us recall the famous Matrix-Tree Theorem of Kirchhoff [Kirchhoff(1847)]: The complexity $\tau(X)$ of a graph X is equal to any cofactor of the Laplace matrix of X, which is the degree matrix of X minus the adjacency matrix of X. As a consequence the number of spanning trees in the complete graph with n vertices is given by n^{n-2}.

In the following the complexity of finite Sierpiński graphs is computed. These graphs are discrete analoga of the well-known Sierpiński gasket (see [Sierpinski(1915)]) and can be defined as follows: Denote by

$$\mathbf{e}_1 = (1,0,0), \qquad \mathbf{e}_2 = (0,1,0), \qquad \text{and} \qquad \mathbf{e}_3 = (0,0,1)$$

the canonical basis vectors of $\mathbb{R}^3$. For $n = 0$ the Sierpiński graph X_0 is given by $VX_0 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $EX_0 = \{\{\mathbf{e}_1, \mathbf{e}_2\}, \{\mathbf{e}_2, \mathbf{e}_3\}, \{\mathbf{e}_3, \mathbf{e}_1\}\}$. For $n > 0$ the Sierpiński graph X_n is defined iteratively by

$$VX_n = (2^{n-1}\mathbf{e}_1 + VX_{n-1}) \cup (2^{n-1}\mathbf{e}_2 + VX_{n-1}) \cup (2^{n-1}\mathbf{e}_3 + VX_{n-1})$$

and

$$EX_n = (2^{n-1}\mathbf{e}_1 + EX_{n-1}) \uplus (2^{n-1}\mathbf{e}_2 + EX_{n-1}) \uplus (2^{n-1}\mathbf{e}_3 + EX_{n-1}).$$

The graph X_n is called *Sierpiński graph of level* n; see Figure 1 for X_0, X_1, and X_2. A simple computation shows that $|VX_n| = \frac{3}{2}(3^n + 1)$ and $|EX_n| = 3^{n+1}$.

2 Main result

The complexity $\tau(X_n)$ of the Sierpiński graph of level n is given by

$$\tau(X_n) = \sqrt[4]{\frac{3}{20}}\left(\frac{5}{3}\right)^{-n/2}\left(\sqrt[4]{540}\right)^{3^n}.$$

The first numbers of this sequence are $3, 54, 524880, 803355125990400000, \ldots$

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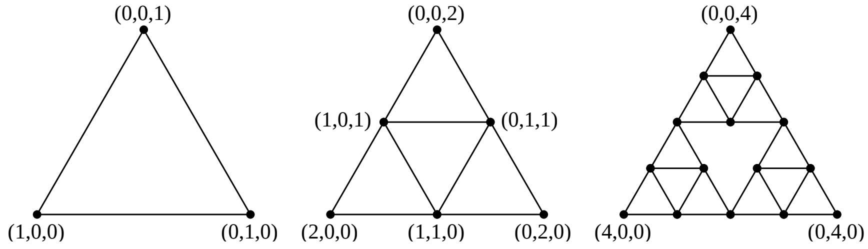

Fig. 1: The Sierpiński graphs X_0, X_1, and X_2.

Since every spanning tree is a subset of the edge set EX_n, it is natural to rewrite this formula in terms of $|EX_n|$: This yields

$$\tau(X_n) = \sqrt[4]{\frac{5}{12}}\ |EX_n|^{(1-2/d_s)/2} \left(\sqrt[12]{540}\right)^{|EX_n|}.$$

Here d_s is the spectral dimension of the Sierpiński gasket, which is given by

$$d_s = 2\,\frac{\log 3}{\log 5}.$$

The spectral dimension was first introduced using the integrated density of states of the Laplacian on the infinite Sierpiński graph. Later on, the exponent d_s was studied from several points of view, see for example [Barlow(1998), Kigami(2001)] and the references therein. Furthermore, note that the fraction $\frac{5}{3}$ is the so-called resistance scaling factor of the Sierpiński gasket.

We conjecture that the formula

$$\tau(X_n) = C|EX_n|^{\beta(1-2/d_s)}\,\alpha^{|EX_n|}$$

holds for a large class of sequences $(X_n)_{n\geq 0}$ of finite self-similar graphs, where $C > 0$, $\beta \geq 0$ and $\alpha \in (1,2)$ are some constants; see [Teufl and Wagner(2006)].

3 Proof

The proof is based on the work in the paper [Teufl and Wagner(2006)]. For $n \geq 0$ and $k \in \{1,2,3\}$ denote by $\mathcal{F}_k(n)$ the set of spanning forests in X_n with k components, so that each component contains at least one vertex of $B_n = 2^n\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and set $\mathcal{F}(n) = \mathcal{F}_1(n) \uplus \mathcal{F}_2(n) \uplus \mathcal{F}_3(n)$. For $n \geq 0$ and $i \in \{1,2,3\}$ let X_n^i be the subgraph of X_n, which is induced by $2^{n-1}\mathbf{e}_i + VX_{n-1}$, and note that X_n^i is isomorphic to X_{n-1}. The restriction of a spanning forest in $\mathcal{F}_k(n)$ to X_n^i yields a spanning forest in $\mathcal{F}_j(n-1)$ for some $j \in \{1,2,3\}$. Hence each spanning forest in $\mathcal{F}(n)$ can be decomposed into three forests in $\mathcal{F}(n-1)$. For example, each spanning tree in $\mathcal{F}_1(n)$ is composed of two trees in $\mathcal{F}_1(n-1)$ and one forest in $\mathcal{F}_2(n-1)$, see Figure 2.

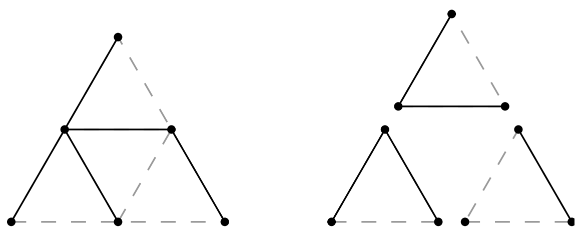

Fig. 2: The decomposition of a spanning tree of X_1.

Denote by $c(n)$ the 3-dimensional counting vector

$$c(n) = (c_1(n), c_2(n), c_3(n)) = \left(|\mathcal{F}_1(n)|, \tfrac{1}{3}|\mathcal{F}_2(n)|, |\mathcal{F}_3(n)|\right).$$

Note that there are three possibilities to arrange the vertices of B_n in two components of a spanning forest in $\mathcal{F}_2(n)$, and the factor $\frac{1}{3}$ takes care of that. Furthermore, the complexity $\tau(X_n)$ is given by $\tau(X_n) = c_1(n)$.

The decomposition of forests in $\mathcal{F}(n)$ into three forests in $\mathcal{F}(n-1)$ implies that $c(n)$ satisfies the polynomial recursion

$$c(n) = Q(c(n-1))$$

for $n > 0$, where Q is given by

$$Q : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 6x_1^2 x_2 \\ 7x_1 x_2^2 + x_1^2 x_3 \\ 14x_2^3 + 12x_1 x_2 x_3 \end{pmatrix}$$

The initial vector is given by $c(0) = (3, 1, 1)$.

Now define the map $v : \mathbb{R}_+^2 \to \mathbb{R}^3$ by

$$v(a, b) = (3a, b, b^2/a),$$

then $c(0) = v(1, 1)$ and $c(1) = v(18, 30)$. Finally, define $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ by

$$T(a, b) = (18a^2 b, 30ab^2),$$

then the function Q maps the set $v(\mathbb{R}_+^2)$ into itself and $Q \circ v = v \circ T$.

Thus the vector $c(n)$ is given by

$$c(n) = v(T^n(1, 1)),$$

where T^n is the n-fold iterate of T. The n-fold iterate of T can be computed explicitly:

$$T^n(a, b) = \begin{pmatrix} \left(\frac{5}{3}\right)^{-n/2} \left(\sqrt[4]{540}\right)^{3^n - 1} a^{(3^n+1)/2} b^{(3^n-1)/2} \\ \left(\frac{5}{3}\right)^{n/2} \left(\sqrt[4]{540}\right)^{3^n - 1} a^{(3^n-1)/2} b^{(3^n+1)/2} \end{pmatrix}$$

for all $n \geq 0$. This implies the statement.

4 Conclusion

The ideas of this proof can be formalized and generalized to a large class of fractal-like graphs. In addition, it is also possible to count different combinatorial objects in those graphs; for example matchings (independent edge subsets), connected subsets, subtrees, colorings, factors, and vertex or edge coverings. Using a decomposition argument it is possible to obtain a multi-dimensional polynomial recursion. Of course, one cannot expect to get explicit formulas in general. However, the asymptotic behavior can be deduced under suitable conditions. See [Teufl and Wagner(2006)] for more information.

An interesting reformulation of the Matrix-Tree Theorem states, that the complexity of a graph X is given by the product of all non-zero eigenvalues of the Laplace matrix of X taking the multiplicities into account divided by the number of vertices. Note that the Dirichlet spectrum of the Laplace matrix of finite Sierpiński graphs X_n with respect to the boundary B_n is well understood, see [Shima(1991)]. However, it seems to be much more difficult to understand the normal spectrum.

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