An extension to overpartitions of Rogers-Ramanujan identities for even moduli

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We investigate class of well-poised basic hypergeometric series $\tilde{J}_{k,i}(a;x;q)$, interpreting these series as generating functions for overpartitions defined by multiplicity conditions. We also show how to interpret the $\tilde{J}_{k,i}(a;1;q)$ as generating functions for overpartitions whose successive ranks are bounded, for overpartitions that are invariant under a certain class of conjugations, and for special restricted lattice paths. We highlight the cases $(a,q) \to (1/q,q), (1/q,q^2)$, and $(0,q)$, where some of the functions $\tilde{J}_{k,i}(a;x;q)$ become infinite products. The latter case corresponds to Bressoud's family of Rogers-Ramanujan identities for even moduli.

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1 Introduction

Over the years, a great number of combinatorial identities [1, 2, 3, 4, 8, 10, 19, 23, 25] have been extracted from Andrews' functions [7, Ch. 7] $J_{k,i}(a;x;q)$, which are defined by

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) + axqH_{k,i-1}(a;xq;q), \tag{1.1}$$

where

$$H_{k,i}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+n-in} x^{kn} (1-x^i q^{2ni})(-1/a)_n(-axq^{n+1})_\infty}{(q)_n (xq^n)_\infty}. \tag{1.2}$$

Here we have employed the usual basic hypergeometric series notation [21]. Most recently [19], the first and third authors made a thorough combinatorial study of these functions, providing an interpretation of the general $J_{k,i}(a;x;q)$ in terms of overpartitions, which unified work of Andrews [4], Gordon [22], and the second author [23]. Moreover, it was shown that the $J_{k,i}(a;1;q)$ can be interpreted as generating functions for overpartitions with bounded successive ranks, for overpartitions with a specified Durfee dissection, and for certain restricted lattice paths. All of these interpretations generalized work of Andrews, Bressoud, and Burge on ordinary partitions [5, 6, 14, 15, 16].

In this paper we study a similar class of functions, which we call $\tilde{J}_{k,i}(a;x;q)$ and define by

$$\tilde{J}_{k,i}(a;x;q) = \tilde{H}_{k,i}(a;xq;q) + axq\tilde{H}_{k,i-1}(a;xq;q), \tag{1.3}$$

where

$$\tilde{H}_{k,i}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n-in} x^{(k-1)n} (1-x^i q^{2ni})(-x,-1/a)_n(-axq^{n+1})_\infty}{(q^2;q^2)_n (xq^n)_\infty}. \tag{1.4}$$

The $\tilde{J}_{k,i}(a;x;q)$ are the functions $F_{1,k,i}(-q,\infty;-1/a;x;q)$ in [11, eq. (2.1)]. Again the most natural combinatorial setting is that of overpartitions. We recall that an overpartition is a partition where the final occurrence of a part can be overlined [17]. For example there exist 8 overpartitions of 3:

$$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (1,1,\overline{1}).$$

Given an overpartition λ, let $f_\ell(\lambda)$ $(f_{\overline{\ell}}(\lambda))$ denote the number of occurrences of ℓ non-overlined (overlined) in λ. Let $V_\lambda(\ell)$ denote the number of overlined parts in λ less than or equal to ℓ. The following combinatorial interpretation of the general $\tilde{J}_{k,i}(a;x;q)$ is the principal result of the first half of this paper:

Theorem 1.1 *For $1 \leq i \leq k$ and $j \leq m$ define the function $c_{k,i}(j,m,n)$ to be the number of overpartitions λ of n with m parts, j of which are overlined, such that (i) $f_1(\lambda)+f_{\overline{1}}(\lambda) \leq i-1$, (ii) $f_\ell(\lambda)+f_{\ell+1}(\lambda)+f_{\overline{\ell+1}}(\lambda) \leq k-1$, and (iii) if λ is saturated at ℓ, that is, if the maximum in (ii) is achieved, then $\ell f_\ell(\lambda)+(\ell+1)f_{\ell+1}(\lambda)+(\ell+1)f_{\overline{\ell+1}}(\lambda) \equiv i-1+V_\lambda(\ell) \pmod 2$. Then*

$$\tilde{J}_{k,i}(a;x;q) = \sum_{j,m,n\geq 0} c_{k,i}(j,m,n)a^j x^m q^n. \tag{1.5}$$

It turns out that the $\tilde{J}_{k,i}(a;1;q)$ are infinite products for $(a,q) \to (0,q)$ and $(1/q,q^2)$, as well as for $(a,q) \to (1/q,q)$ when $i=1$, and hence we can deduce partition theorems from Theorem 1.1. In the case $(a,q) \to (0,q)$, the product is

$$\tilde{J}_{k,i}(0;1;q) = \frac{(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty},$$

and we have a new proof of Bressoud's Rogers-Ramanujan identities for even moduli [10]:

Corollary 1.2 (Bressoud) *For $k \geq 2$ and $1 \leq i \leq k-1$, let $\tilde{A}_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k$. Let $\tilde{B}_{k,i}(n)$ denote the number of partitions λ of n such that (i) $f_1(\lambda) \leq i-1$, (ii) $f_\ell(\lambda)+f_{\ell+1}(\lambda) \leq k-1$, and (iii) if $f_\ell(\lambda)+f_{\ell+1}(\lambda) = k-1$, then $\ell f_\ell(\lambda)+(\ell+1)f_{\ell+1}(\lambda) \equiv i-1 \pmod 2$. Then $\tilde{A}_{k,i}(n) = \tilde{B}_{k,i}(n)$.*

When $(a,q) \to (1/q,q^2)$, the product is

$$\tilde{J}_{k,i}(1/q;1;q^2) = \frac{(-q;q^2)_\infty(q^{2i-1},q^{4k-2i-1},q^{4k-2};q^{4k-2})_\infty}{(q^2;q^2)_\infty},$$

and the result is Bressoud's [11, eq. (3.9) and Theorem 2] mod $4k-2$ companion to Andrews' generalization of the Göllnitz-Gordon identities [4]:

Corollary 1.3 *For $1 \leq i \leq k-1$, let $\tilde{A}^2_{k,i}(n)$ denote the number of partitions of n where even parts are multiples of 4 not divisible by $8k-4$ and odd parts are not congruent to $\pm(2i-1)$ modulo $4k-2$, with parts congruent to $2k-1$ modulo $4k-2$ not repeatable. Let $\tilde{B}^2_{k,i}(n)$ denote the number of partitions λ of n such that (i) $f_1(\lambda)+f_2(\lambda) \leq i-1$, (ii) $f_{2\ell}(\lambda)+f_{2\ell+1}(\lambda)+f_{2\ell+2}(\lambda) \leq k-1$, and (iii) if the maximum in (ii) is achieved at ℓ, then $\ell f_{2\ell}(\lambda)+(\ell+1)f_{2\ell+2}(\lambda)+(\ell+1)f_{2\ell+1}(\lambda) \equiv i-1+V^o_\lambda(\ell) \pmod 2$. (Here $V^o_\lambda(\ell)$ is the number of odd parts of λ less than 2ℓ). Then $\tilde{A}^2_{k,i}(n) = \tilde{B}^2_{k,i}(n)$.*

Finally, when $(a,q) \to (1/q,q)$ and $i=1$, the product is

$$\tilde{J}_{k,1}(1/q;1;q) = \frac{(-q)_\infty(q,q^{2k-2},q^{2k-1};q^{2k-1})_\infty}{(q)_\infty},$$

and the result is an odd modulus companion to Theorem 1.2 of [23].

Corollary 1.4 *For $k \geq 2$, let $\tilde{A}^3_k(n)$ denote the number of overpartitions whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k-1$. Let $\tilde{B}^3_k(n)$ denote the number of overpartitions λ of n such that (i) $f_1(\lambda) = 0$, (ii) $f_\ell(\lambda)+f_{\overline{\ell}}(\lambda)+f_{\ell+1}(\lambda) \leq k-1$, and (iii) if the maximum in condition (ii) is achieved at ℓ, then $\ell f_\ell(\lambda)+\ell f_{\overline{\ell}}(\lambda)+(\ell+1)f_{\ell+1}(\lambda) \equiv V_\lambda(\ell) \pmod 2$. Then $\tilde{A}^3_k(n) = \tilde{B}^3_k(n)$.*

In the second half of the paper, we discuss three more combinatorial interpretations of the $\tilde{J}_{k,i}(a;1;q)$: one involving the theory of successive ranks for overpartitions as developed in [19], one involving a two-parameter generalization to overpartitions of Garvan's k-conjugation for partitions [20], and one involving a generalization of some lattice paths of Bressoud and Burge [14, 15, 16]. The following is the main theorem of this part, the combinatorial concepts being necessarily fully defined later in the paper. When $a=0$ and $X = C$, D, or E, we recover some of the main results of [14, 15, 16].

Theorem 1.5

Let $\tilde{B}_{k,i}(n,j)$ denote the number of overpartitions λ of n counted by $c_{k,i}(j,\ell(\lambda),n)$ where $\ell(\lambda)$ is the number of parts in λ (we thus have $\tilde{B}_{k,i}(n,j) = \sum_{m\geq j} c_{k,i}(j,m,n)$).

Let $\tilde{C}_{k,i}(n,j)$ denote the number of overpartitions of n with j overlined parts whose successive ranks lie in $[-i+2, 2k-i-2]$.

Let $\tilde{D}_{k,i}(n,j)$ denote the number of self-(k,i)-conjugate overpartitions of n with j overlined parts.

Let $\tilde{E}_{k,i}(n,j)$ denote the number of special lattice paths of major index n with j South steps which start at $k-i$, whose height is less than k and where the peaks of coordinates $(x,k-1)$ are such that $x-u$ is congruent to $i-1$ modulo 2 (u is the number of South steps to the left of the peak).
Then for $X = B$, C, D, or E,

$$\sum_{n,j\geq 0} \tilde{X}_{k,i}(n,j)a^j q^n = \frac{(-aq)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1/a)_n(-1)^n a^n q^{(2k-1)\binom{n+1}{2}-in+n}}{(-aq)_n}. \tag{1.6}$$

Again, the right-hand side of (1.6) is in many cases an infinite product, and hence there are results like Corollaries 1.2 - 1.4 involving the functions $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$. However, we shall not highlight these corollaries.

The paper is organized as follows. In the next section we study the basic properties of the $\tilde{J}_{k,i}(a;x;q)$ and give proofs of Theorem 1.1 and Corollaries 1.2 - 1.4. In Section 3, we compute the generating function for the paths counted by $\tilde{E}_{k,i}(n,j)$ to show that they are in bijection with the overpartitions counted by $\tilde{B}_{k,i}(n,j)$. In Section 4, we present a direct bijection between the paths counted by $\tilde{E}_{k,i}(n,j)$ and the overpartitions counted by $\tilde{C}_{k,i}(n,j)$. In Section 5, we compute the generating function for the overpartitions counted by $\tilde{D}_{k,i}(n,j)$ to show that they are in bijection with the paths counted by $\tilde{E}_{k,i}(n,j)$. The techniques used in Sections 3,4, and 5 are very similar to [19]. We conclude in Section 6 with some suggestions for future research.

2 The $\tilde{J}_{k,i}(a;x;q)$ and the multiplicities

We begin by stating some facts about the functions $\tilde{H}_{k,i}(a;x;q)$ and $\tilde{J}_{k,i}(a;x;q)$ defined in the introduction.

Lemma 2.1

$$\begin{aligned} \tilde{H}_{k,0}(a;x;q) &= 0 && (2.7)\\ \tilde{H}_{k,-i}(a;x;q) &= -x^{-i}\tilde{H}_{k,i}(a;x;q) && (2.8)\\ \tilde{H}_{k,i}(a;x;q) - \tilde{H}_{k,i-2}(a;x;q) &= x^{i-2}(1+x)\tilde{J}_{k,k-i+1}(a;x;q). && (2.9)\end{aligned}$$

Now assume that $1 \leq i \leq k$. The following recurrences for the $\tilde{J}_{k,i}(a;x;q)$ are fundamental.

Theorem 2.2

$$\begin{aligned} \tilde{J}_{k,1}(a;x;q) &= \tilde{J}_{k,k}(a;xq;q) && (2.10)\\ \tilde{J}_{k,2}(a;x;q) &= (1+xq)\tilde{J}_{k,k-1}(a;xq;q) + axq\tilde{J}_{k,k}(a;xq;q) && (2.11)\\ \tilde{J}_{k,i}(a;x;q) - \tilde{J}_{k,i-2}(a;x;q) &= (xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+1}(a;xq;q) && (2.12)\\ &+ a(xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+2}(a;xq;q) \quad (3\leq i\leq k). \end{aligned}$$

See [18] for a proof of these results.

We now turn to the proof of Theorem 1.1. If we write

$$\tilde{J}_{k,i}(a;x;q) = \sum_{j,m,n\geq 0} b_{k,i}(j,m,n)a^j x^m q^n,$$

then the recurrences in Theorem 2.2 imply that

$$b_{k,1}(j,m,n) = b_{k,k}(j,m,n-m), \tag{2.13}$$

$$b_{k,2}(j,m,n) = b_{k,k-1}(j,m,n-m) + b_{k,k-1}(j,m-1,n-m) + b_{k,k}(j-1,m-1,n-m), \tag{2.14}$$

$$\begin{aligned} b_{k,i}(j,m,n) - b_{k,i-2}(j,m,n) &= b_{k,k-i+1}(j,m-i+2,n-m) + b_{k,k-i+1}(j,m-i+1,n-m) && (2.15)\\ &+ b_{k,k-i+2}(j-1,m-i+2,n-m) + b_{k,k-i+2}(j-1,m-i+1,n-m). \end{aligned}$$

We shall demonstrate that the $c_{k,i}(j,m,n)$ also satisfy these recurrences. In what follows we shall repeatedly employ a mapping $\lambda \to \widehat{\lambda}$, where $\widehat{\lambda}$ is obtained by removing the first column of the Ferrers diagram of λ. Before continuing, we make a couple of observations regarding this mapping. First, if λ satisfies condition (ii) in the statement of the theorem, so does $\widehat{\lambda}$. Second, if λ is an overpartition counted by $c_{k,i}(j,m,n)$ and $\widehat{\lambda}$ is saturated at ℓ, then λ was saturated at $\ell+1$, so we have

$$\begin{aligned} & \ell f_\ell(\widehat{\lambda}) + (\ell+1) f_{\ell+1}(\widehat{\lambda}) + (\ell+1) f_{\overline{\ell+1}}(\widehat{\lambda}) = \ell f_{\ell+1}(\lambda) + (\ell+1) f_{\ell+2}(\lambda) + (\ell+1) f_{\overline{\ell+2}}(\lambda) \\ = \ & (\ell+1) f_{\ell+1}(\lambda) + (\ell+2) f_{\ell+2}(\lambda) + (\ell+2) f_{\overline{\ell+2}}(\lambda) - (f_\ell(\widehat{\lambda}) + f_{\ell+1}(\widehat{\lambda}) + f_{\overline{\ell+1}}(\widehat{\lambda})) \\ \equiv \ & i - 1 + V_\lambda(\ell+1) - (f_\ell(\widehat{\lambda}) + f_{\ell+1}(\widehat{\lambda}) + f_{\overline{\ell+1}}(\widehat{\lambda})) \pmod 2 \\ \equiv \ & V_\lambda(\ell+1) + k - i \pmod 2 \end{aligned} \tag{2.16}$$

Finally, it is clear that

$$V_{\widehat{\lambda}}(\ell) \equiv \begin{cases} V_\lambda(\ell+1) \pmod 2, & \text{if } \overline{1} \notin \lambda \\ V_\lambda(\ell+1) + 1 \pmod 2, & \text{if } \overline{1} \in \lambda \end{cases} \tag{2.17}$$

We begin with (2.13). Given an overpartition λ counted by $c_{k,1}(j,m,n)$, $\widehat{\lambda}$ is an overpartition of $n-m$ with m parts, j of which are overlined. Since λ could have had at most $k-1$ twos, $\widehat{\lambda}$ has at most $k-1$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.16) and (2.17) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1) f_{\ell+1}(\widehat{\lambda}) + (\ell+1) f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k - 1 + V_{\widehat{\lambda}}(\ell) \pmod 2$. Thus $\widehat{\lambda}$ is an overpartition counted by $c_{k,k}(j,m,n-m)$. Since the mapping from λ to $\widehat{\lambda}$ is reversible, we have the recurrence (2.13) for the functions $c_{k,i}(j,m,n)$.

We turn to (2.14). Suppose now that λ is an overpartition counted by $c_{k,2}(j,m,n)$. Then λ has at most one 1. We consider three cases.

First, if λ has no ones, then it can have at most $k-2$ twos. For if λ had $k-1$ twos, then $1f_1(\lambda) + 2f_2(\lambda) + 2f_{\overline{2}}(\lambda) \equiv 0 \pmod 2$ violates condition (iii) in the definition of the $c_{k,2}(j,m,n)$. Hence $\widehat{\lambda}$ is an overpartition of $n-m$ into m parts, ℓ of which are overlined, and having at most $k-2$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.16) and (2.17) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1) f_{\ell+1}(\widehat{\lambda}) + (\ell+1) f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k - 2 + V_{\widehat{\lambda}}(\ell) \pmod 2$. Hence $\widehat{\lambda}$ is an overpartition counted by $c_{k,k-1}(j,m,n-m)$.

Second, if 1 occurs (non-overlined) in λ, then there can be up to $k-2$ twos, so $\widehat{\lambda}$ has at most $k-2$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.16) and (2.17) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1) f_{\ell+1}(\widehat{\lambda}) + (\ell+1) f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k - 2 + V_{\widehat{\lambda}}(\ell) \pmod 2$. Hence $\widehat{\lambda}$ is an overpartition counted by $c_{k,k-1}(j,m-1,n-m)$.

Third and finally, if $\overline{1}$ occurs in λ, then there can be at most $k-1$ twos, so $\widehat{\lambda}$ has at most $k-1$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.16) and (2.17) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1) f_{\ell+1}(\widehat{\lambda}) + (\ell+1) f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k - 1 + V_{\widehat{\lambda}}(\ell) \pmod 2$. Hence $\widehat{\lambda}$ is an overpartition counted by $c_{k,k}(j-1,m-1,n-m)$.

Since the mappings are reversible, we have the recurrence (2.14) for the functions $c_{k,i}(j,m,n)$.

The proof of the recurrence (2.15) is very similar to those of (2.13) and (2.14). See [18] for details.

To finalize the claim that the two families of functions are equal, we note that

$$b_{k,i}(j,m,n) = \begin{cases} 0, & \text{if } j < 0, m \leq 0 \text{ or } n \leq 0, \text{ and } (j,m,n) \neq (0,0,0) \\ 1, & \text{if } (j,m,n) = (0,0,0), \end{cases} \tag{2.18}$$

which is indeed also true for the $c_{k,i}(j,m,n)$. □

Before deducing Corollaries 1.2 - 1.4 we state a proposition which is a piece of Theorem 1.5 and from which it follows that several instances of the $\tilde{J}_{k,i}(a;1;q)$ are infinite products.

Proposition 2.3 *We have*

$$\tilde{J}_{k,i}(a;1;q) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1/a)_n (-1)^n a^n q^{(2k-1)\binom{n+1}{2} - in + n}}{(-aq)_n}. \tag{2.19}$$

Corollary 2.4 *We have*

$$\tilde{J}_{k,i}(0;1;q) = \frac{(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty}, \tag{2.20}$$

$$\tilde{J}_{k,i}(1/q;1;q^2) = \frac{(-q;q^2)_\infty (q^{2i-1}, q^{4k-2i-1}, q^{4k-2}; q^{4k-2})_\infty}{(q^2;q^2)_\infty}, \tag{2.21}$$

and

$$\tilde{J}_{k,1}(1/q,1;q) = \frac{(-q)_\infty (q, q^{2k-2}, q^{2k-1}; q^{2k-1})_\infty}{(q)_\infty}. \tag{2.22}$$

We are now ready to prove Corollaries 1.2 - 1.4. In the following, we consider that λ is an overpartition of n with j overlined parts, hence it is counted in the coefficient of $q^n a^j$ of $\tilde{J}_{k,i}(a,1;q)$. This overpartition is such that (i) $f_1(\lambda) + f_{\overline{1}}(\lambda) \leq i-1$, (ii) $f_\ell(\lambda) + f_{\ell+1}(\lambda) + f_{\overline{\ell+1}}(\lambda) \leq k-1$, and (iii) if λ is saturated at ℓ, that is, if the maximum in (ii) is achieved, then $\ell f_\ell(\lambda) + (\ell+1) f_{\ell+1}(\lambda) + (\ell+1) f_{\overline{\ell+1}}(\lambda) \equiv i - 1 + V_\lambda(\ell) \pmod 2$.

For Corollary 1.2, we consider the functions $\tilde{J}_{k,i}(0;1;q)$. From Theorem 1.1 we easily see that the coefficient of q^n in $J_{k,i}(0;1;q)$ is $\tilde{B}_{k,i}(n)$. On the other hand, from (2.20) , this coefficient is also $\tilde{A}_{k,i}(n)$. □

For Corollary 1.3, we use the functions $\tilde{J}_{k,i}(1/q;1;q^2)$. A little thought reveals that the coefficient of q^n in $\tilde{J}_{k,i}(1/q;1;q^2)$ is $\tilde{B}^2_{k,i}(n)$. Rewriting of the product in (2.21) as

$$(q^2;q^4)_\infty (q^{8k-4};q^{8k-4})_\infty (q^{2i-1},q^{4k-2i-1};q^{4k-2})_\infty (-q^{2k-1};q^{4k-2})_\infty \prod_{n \not\equiv 2k-1 \pmod{4k-2}} \frac{1}{(1-q^n)}$$

shows that this coefficient is also $\tilde{A}^2_{k,i}(n)$. □

Finally, for Corollary 1.4, we use the functions $\tilde{J}_{k,1}(1/q;1;q)$. Again it may readily be seen that the coefficient of q^n therein is $\tilde{B}^3_k(n)$. On the other hand, from (2.22), this coefficient is also $\tilde{A}^3_k(n)$. □

3 Lattice Paths

We study paths in the first quadrant that use four kinds of unitary steps:

- North-East $NE : (x,y) \to (x+1,y+1)$,
- South-East $SE : (x,y) \to (x+1,y-1)$,
- South $S : (x,y) \to (x,y-1)$,
- East $E : (x,0) \to (x+1,0)$.

The *height* of a vertex corresponds to its y-coordinate. A South step can only appear after a North-East step and an East step can only appear at height 0. The paths must end with a North-East or South step. A *peak* is a vertex preceded by a North-East step and followed by a South step (in which case it will be called a *NES peak*) or by a South-East step (in which case it will be called a *NESE peak*). If the path ends with a North-East step, its last vertex is also a NESE peak. The *major index* of a path is the sum of the x-coordinates of its peaks (see Figure 1 for an example). When the paths have no South steps, this is the definition of the paths in [14].

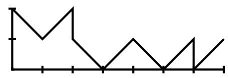

Fig. 1: This path has four peaks : two NES peaks (located at $(2,2)$ and $(6,1)$) and two NESE peaks (located at $(4,1)$ and $(7,1)$). Its major index is $2+4+6+7=19$.

Let k and i be positive integers with $i \leq k$. Let $\tilde{E}_{k,i}(n,j)$ be the number of paths of major index n with j South steps which satisfy the following *special* (k,i)*-conditions*: (i) the paths start at height $k-i$, (ii) their height is less than k, (iii) every peak of coordinates $(x,k-1)$ satisfies $x - u \equiv i-1 \pmod 2$ where u is the number of South steps to the left of the peak.

Let $\tilde{\mathcal{E}}_{k,i}(a,q)$ be the generating function for those paths, that is $\tilde{\mathcal{E}}_{k,i}(a,q) = \sum_{n,j} \tilde{E}_{k,i}(n,j) a^j q^n$. Let $\tilde{\mathcal{E}}_{k,i}(N)$ be the generating function for paths counted by $\tilde{\mathcal{E}}_{k,i}(a,q)$ which have N peaks. Moreover, for $0 \leq i < k$, let $\tilde{\Gamma}_{k,i}(N)$ be the generating function for paths obtained by deleting the first NE step of a path which is counted in $\tilde{\mathcal{E}}_{k,i+1}(N)$ and begins with a NE step. Then

Proposition 3.1

$$\begin{aligned}
\tilde{\mathcal{E}}_{k,i}(N) &= q^N \tilde{\mathcal{E}}_{k,i+1}(N) + q^N \tilde{\Gamma}_{k,i-1}(N); \quad 1 \leq i < k && (3.23)\\
\tilde{\Gamma}_{k,i}(N) &= q^N \tilde{\Gamma}_{k,i-1}(N) + (a + q^{N-1}) \tilde{\mathcal{E}}_{k,i+1}(N-1); \quad 0 < i < k && (3.24)\\
\tilde{\mathcal{E}}_{k,k}(N) &= q^N \tilde{\mathcal{E}}_{k,k-1}(N) + q^N \tilde{\Gamma}_{k,k-1}(N) && (3.25)\\
\tilde{\mathcal{E}}_{k,i}(0) = 1 \qquad & \tilde{\Gamma}_{k,0}(N) = 0 && (3.26)
\end{aligned}$$

Proof: We prove that result by induction on the length of the path. If the path is not empty, then we take off its first step. When we do this, we increase or decrease i by 1 and thus change the parity of $i-1$; moreover, all the peaks are shifted by 1, so the parity of $x-u-i$ is not changed (if the step we remove is a South step, the peaks are not shifted but u decreases by 1 for all peaks, so the result is the same). The case $i=k$ needs further explanation. For these paths the fact that every peak of coordinates $(x,k-1)$ satisfies $x-u\equiv k-1 \pmod 2$ is equivalent to the fact that every peak of coordinates $(x,k-1)$ has an even number of East steps to its left. Therefore the paths counted in $\tilde{\mathcal{E}}_{k,k}(N)$ that start with an East step where this step is deleted are in bijection with the paths counted in $\tilde{\mathcal{E}}_{k,k-1}(N)$ (see [18] for details). Moreover it is easy to see that the paths counted in $\tilde{\mathcal{E}}_{k,k}(N)$ that start with a North-East step where this step is deleted are the paths counted in $\tilde{\Gamma}_{k,k-1}(N)$. □

These recurrences uniquely define the series $\tilde{\mathcal{E}}_{k,i}(N)$ and $\tilde{\Phi}_{k,i}(N)$. We get that

Theorem 3.2

$$\tilde{\mathcal{E}}_{k,i}(N) = a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N} (-1)^n \frac{q^{(k-1)n^2+(k-i)n}}{(q)_{N-n}(q)_{N+n}} \tag{3.27}$$

$$\tilde{\Phi}_{k,i}(N) = a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{(k-1)n^2+(k-i-1)n}}{(q)_{N-n-1}(q)_{N+n}} \tag{3.28}$$

The proof is omitted. It uses simple algebraic manipulation to prove that these generating functions satisfy the recurrence relations of Proposition 3.1.

We recall a proposition proved in [19] that will enable us to compute $\tilde{\mathcal{E}}_{k,i}(a,q)$ from the recurrences:

Proposition 3.3 *For any $n\in\mathbb{Z}$*

$$\sum_{N\geq |n|} \frac{(-aq)_n(-q^n/a)_{N-n} q^{\binom{N+1}{2}-\binom{n+1}{2}} a^{N-n}}{(q)_{N+n}(q)_{N-n}} = \frac{(-aq)_\infty}{(q)_\infty}.$$

From (3.27), summing on N using Proposition 3.3 (see [19] for details), we get Equation (1.6) and Theorem 1.5 for $X=E$.

For the work in Section 5, we'll need the definition of the relative height of a peak. This notion was defined by Bressoud in [14]. The definition we use is a simpler version taken from [9].

Definition 3.4 ([9]) *The relative height of a peak (x,y) is the largest integer h for which we can find two vertices on the path, $(x',y-h)$ and $(x'',y-h)$, such that $x'<x<x''$ and such that between these two vertices there are no peaks of height $>y$ and every peak of height y has weight $\geq x$.*

For the paths corresponding to overpartitions, i.e. the paths counted by $\tilde{E}_{k,i}(n,j)$, we have to modify the definition of the relative height a little bit to take into account the NES peaks, for which we can have $x''=x$.

Definition 3.5 *The relative height of a peak (x,y) is the largest integer h for which we can find two vertices on the path, $(x',y-h)$ and $(x'',y-h)$, such that $x'<x\leq x''$ and such that between these two vertices there are no peaks of height $>y$ and every peak of height y has weight $\leq x$.*

See [19] for examples.

Proposition 3.6 *For $n_1\geq n_2\geq\cdots\geq n_{k-1}$,*

$$\frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

is the generating function for the paths (counted by major index and number of south steps) satisfying the special (k,i)-conditions and having n_j peaks of relative height $\geq j$ for $1\leq j\leq k-1$.

Proof: It is similar to that of Proposition 6.1 of [19]. See [18] for details. □

4 Successive Ranks

The Frobenius representation of an overpartition [17, 24] of n is a two-rowed array $\begin{pmatrix} a_1 & a_2 & \dots & a_N \\ b_1 & b_2 & \dots & b_N \end{pmatrix}$ where $(a_1, \dots, a_N)$ is a partition into distinct nonnegative parts and $(b_1, \dots, b_N)$ is an overpartition into nonnegative parts where the first occurrence of a part can be overlined and $N + \sum(a_i + b_i) = n$.

We call that the Frobenius representation of an overpartition because it is in bijection with overpartitions. We say that the *generalized Durfee square* of an overpartition has size N if N is the largest integer such that the number of overlined parts plus the number of non-overlined parts greater or equal to N is greater than or equal to N.

We now define the successive ranks.

Definition 4.1 *[19] The* successive ranks *of an overpartition can be defined from its Frobenius representation. If an overpartition has Frobenius representation* $\begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \end{pmatrix}$ *then its* i*th successive rank* r_i *is* $a_i - b_i$ *minus the number of non-overlined parts in* $\{b_{i+1}, \dots, b_N\}$.

For example, the successive ranks of $\begin{pmatrix} 7 & 4 & 2 & 0 \\ \overline{3} & 3 & 1 & \overline{0} \end{pmatrix}$ are $(2, 0, 1, 0)$.

We now state the main result of this section, which implies Theorem 1.5 for $X = C$.

Proposition 4.2 *There exists a one-to-one correspondence between the paths of major index* n *with* j *South steps, counted by* $\tilde{E}_{k,i}(n, j)$ *and the overpartitions of* n *with* j *non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in* $[-i + 2, 2k - i - 2]$*, counted by* $\tilde{C}_{k,i}(n, j)$*. This correspondence is such that the paths have* N *peaks if and only if the Frobenius representation of the overpartition has* N *columns.*

See [18] for the proof.

5 Generalized self-conjugate overpartitions

In this section we prove Theorem 1.5 for $X = D$. We define an operation for overpartitions called k-conjugation, where $k \geq 2$ is an integer. From the Frobenius representation of an overpartition π, we use Algorithm III of [24] to get three partitions λ_1, λ_2 and μ as described in the following paragraph.

Let N be the number of columns of the Frobenius representation. We get λ_1, which is a partition into N nonnegative parts, by removing a staircase from the top row (*i.e.* we remove 0 from the smallest part, 1 from the next smallest, and so on). We get λ_2 (which is a partition into N nonnegative parts) and μ (which is a partition into distinct nonnegative parts less than N) as follows. First, we initialize λ_2 to the bottom row. Then, if the mth part of the bottom row is overlined, we remove the overlining of the mth part of λ_2, we decrease the $m - 1$ first parts of λ_2 by one and we add a part $m - 1$ to μ (see [18] for an example).

Let λ'_1 (resp. λ'_2) be the conjugate of λ_1 (resp. of λ_2). λ'_1 and λ'_2 are thus partitions into parts less than or equal to N. We now consider two regions. The first region is the portion of λ'_2 below its $(k - 2)$-th Durfee square (for $k = 2$, this region is λ'_2). Recall that the Durfee square of a partition is the largest square contained in its diagram and that the i^{th} Durfee square is the Durfee square of the partition that is under the $(i - 1)^{st}$ Durfee square [6]. The second region consists of the parts of λ'_1 which are less than or equal to the size of the $(k - 2)$-th Durfee square of λ'_2 (for $k = 2$, this region is λ'_1).

Definition 5.1 *The* k-conjugation *consists of interchanging these two regions (if* λ'_2 *has less than* $k - 2$ *Durfee squares, the* k*-conjugation is the identity).*

Remark 5.2 *For* $k = 2$*, we just swap* λ'_1 *and* λ'_2 *(which boils down to swapping* λ_1 *and* λ_2*) and we get the* F*-conjugation defined by Lovejoy [24].*

Remark 5.3 *If there are no overlined parts, we get the* k*-conjugation for partitions defined by Garvan [20].*

Definition 5.4 *We say that an overpartition is* self-k-conjugate *if it is fixed by* k*-conjugation.*

Proposition 5.5 *The generating function for self-*k*-conjugate overpartitions is*

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2} + n_2^2 + \cdots + n_{k-1}^2} (-1/a)_{n_1} a^{n_1}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-2} - n_{k-1}} (q^2; q^2)_{n_{k-1}}}$$

where n_1 *is the number of columns of the Frobenius symbol and* $n_2, \dots, n_{k-1}$ *are the sizes of the* $k - 2$ *first successive Durfee squares of* λ'_2.

Proof: We decompose a self-k-conjugate overpartition in the following way :

- μ (region IV in Figure 2), which is counted by $a^{n_1}(-1/a)_{n_1}$;
- the staircase of the top row and the part n_1 (region III), which are counted by $q^{\binom{n_1+1}{2}}$;
- the $k-2$ Durfee squares of λ_2' (region V), which are counted by $q^{n_2^2+\cdots+n_{k-1}^2}$;
- the regions between the Durfee squares of λ_2' (region VI), which are counted by $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q$;
- the parts in λ_1' which are $> n_{k-1}$ and of course $\leq n_1$ (region I): they are counted by
$$\frac{1}{(1-q^{n_{k-1}+1})\cdots(1-q^{n_1})} = \frac{(q)_{n_{k-1}}}{(q)_{n_1}};$$
- the two identical regions (regions II and VII), which are counted by $\frac{1}{(q^2;q^2)_{n_{k-1}}}$.

Summing on $n_1, n_2, \ldots, n_{k-1}$, we get the generating function :

$$\sum_{n_1 \geq n_2 \geq \ldots \geq n_{k-1} \geq 0} (-1/a)_{n_1} a^{n_1} q^{\binom{n_1+1}{2}} q^{n_2^2+\cdots+n_{k-1}^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \frac{(q)_{n_{k-1}}}{(q)_{n_1}} \frac{1}{(q^2;q^2)_{n_{k-1}}}$$
$$= \sum_{n_1 \geq n_2 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2}(-1/a)_{n_1} a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

□

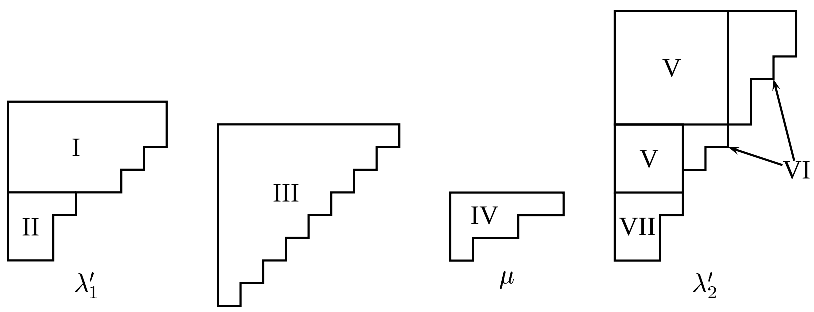

Fig. 2: Decomposition of a self-k-conjugate overpartition (in this example, $k = 4$).

Corollary 5.6 *When there are no overlined parts, $a \to 0$ and we get the generating function for self-k-conjugate partitions [20].*

Definition 5.7 *Let i and k be integers with $1 \leq i \leq k$. We say that an overpartition is* self-(k,i)-conjugate *if it is obtained by taking a self-k-conjugate overpartition and adding a part n_j (n_j is the size of the $(j-1)$-th successive Durfee square of λ_2') to λ_2' for $i \leq j \leq k-1$ (if $i = k$, no parts are added).*

Remember that we denote by $\tilde{D}_{k,i}(n,j)$ the number of self-(k,i)-conjugate overpartitions with j overlined parts (or, equivalently, the number of self-(k,i)-conjugate overpartitions whose Frobenius representation has j non-overlined parts in its bottom row).

Proposition 5.8
$$\tilde{\mathcal{E}}_{k,i}(a,q) = \sum_{n,j} \tilde{D}_{k,i}(n,j) a^j q^n.$$

See [18] for the proof.

6 Concluding Remarks

We would like to mention that the $J_{k,i}(a;x;q)$ and $\tilde{J}_{k,i}(a;x;q)$ can be embedded in a family of functions that satisfy recurrences like those in Lemma 2.1 and are sometimes infinite products when $x=1$. For $m \geq 1$ we define

$$J_{k,i,m}(a;x;q) = H_{k,i,m}(a;xq;q) + axqH_{k,i-1,m}(a;xq;q), \tag{6.29}$$

where

$$H_{k,i,m}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+n-in-(m-1)\binom{n}{2}} x^{n(k-m-1)}(1-x^i q^{2ni})(-1/a)_n(-axq^{n+1})_\infty (x^m;q^m)_n}{(q^m;q^m)_n (x)_\infty}. \tag{6.30}$$

The case $m=1$ gives the $J_{k,i}(a;x;q)$ and $m=2$ corresponds to the $\tilde{J}_{k,i}(a;x;q)$. Equations (2.7) and (2.8) of Lemma 2.1 are true for the $H_{k,i,m}(a;x;q)$, and following the proof of (2.9), one may show that

$$H_{k,i,m}(a;x;q) - H_{k,i-m,m}(a;x;q) = x^{i-m}(1+x+x^2+\cdots+x^{m-1})J_{k,k-i+1,m}(a;x;q).$$

It would certainly be worth investigating what kinds of combinatorial identities are stored in these general series.

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