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Limit laws for a class of diminishing urn models.

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In this work we analyze a class of diminishing 2×2 Pólya-Eggenberger urn models with ball replacement matrix M given by $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$, $a, d \in \mathbb{N}$ and $c \in \mathbb{N}_0$. We obtain limit laws for this class of 2×2 urns by giving estimates for the moments of the considered random variables. As a special instance we obtain limit laws for the pills problem, proposed by Knuth and McCarthy, which corresponds to the special case $a = c = d = 1$. Furthermore, we also obtain limit laws for the well known sampling without replacement urn, $a = d = 1$ and $c = 0$, and corresponding generalizations, $a, d \in \mathbb{N}$ and $c = 0$.

Keywords: Pólya urns, diminishing urns, pills problem, sampling without replacement

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1 Introduction

1.1 Pólya-Eggenberger urn models

Pólya-Eggenberger urn models are defined as follows. We start with an urn containing n white balls and m black balls. The evolution of the urn occurs in discrete time steps. At every step a ball is drawn at random from the urn. The color of the ball is inspected and then the ball is reinserted into the urn. According to the observed color of the ball, balls are added/removed due to the following rules. If a white ball has been drawn, a white balls and b black balls are put into the urn, and if a black ball has been drawn, c white balls and d black balls are put into the urn. The values $a, b, c, d \in \mathbb{Z}$ are fixed integers and the urn model is specified by the 2×2 ball replacement matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This definition extends naturally also to higher dimensions. Most papers in the literature impose the so-called *tenability condition* on the ball replacement matrix, so that the process can be continued ad infinitum. However, in some applications, there are urn models with a very different nature, e. g. the OK Corral problem, $M = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$, see (FDP06), or the Cannibal urn problem, $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -2 \end{smallmatrix}\right)$, see (Pit87), which we will refer to as *diminishing urn models*. In this work we will analyze diminishing Pólya-Eggenberger urn models with ball replacement matrix M given by

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}, \quad \text{with } a, d \in \mathbb{N},\ c \in \mathbb{N}_0. \tag{1}$$

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Throughout this work $\mathbb{N}$ denotes the positive integers and $\mathbb{N}_0$ the non-negative integers.

We are interested in the distribution of the random variable $X_{n,m} = X_{n,m}(a, c, d)$, counting the number of white balls, when all black balls have been taken out of the urn, starting with n white and m black balls, respectively. The distribution of the random variable $X_{n,m}$ in the context of the evolution of the urn may be described as follows. We have a state space $\mathcal{S} := \{(i, j)| \, i, j \in \mathbb{N}_0\}$, where the evolution of the urn takes place. The evolution stops at *absorbing states* $\mathcal{A} := \{(0, j)|j \in \mathbb{N}_0\}$. No negative values of i and j should be reached, the state space $\mathcal{S}$ should not be left during the evolution of the urn. Then, the question is to determine the probability $\mathbb{P}\{X_{n,m} = k\}$, that a certain state $k \in \mathcal{A}$ is reached, starting with n white balls and m black balls. In the context of diminishing urns we call an urn *well defined*, if the evolution of the urn always ends in an absorbing state of $\mathcal{A}$, when starting at any point $(m, n) \in \mathcal{S}$, without ever leaving the state space. The urns considered here are not naturally well defined. Consider for example the evolution of the urn with ball replacement matrix $M = \left(\begin{smallmatrix} -5 & 0 \\ 1 & -1 \end{smallmatrix}\right)$, when there are less then 5 white balls. We will overcome this deficit by the introduction of extra rules in regions where the state space may be left in order to ensure that an absorbing state in $\mathcal{A}$ must be reached.

The aim of this work is the derivation of limit laws of the random variables $X_{n,m}$ for diminishing urn models, when the urn evolves according to ball replacement matrix M given by (1). We will see that different limit laws arise according to the growth of n and m. In some cases we can extend our results to $a, d \in \mathbb{N}$ and $c \in \mathbb{Z}$. Note that choosing $c \in -\mathbb{N}$ imposes additional difficulties. We have to modify our state space $\mathcal{S}$ to $\mathcal{S} = \{(i, j)|i \in \mathbb{N}_0, j \in \mathbb{N}, -ci \le j\}$. The condition $(-c) \cdot i \le j$ is necessary to ensure that, starting with j white balls and i black balls, a point in $\mathcal{A}_1 = \{(0, n) : n \in \mathbb{N}_0\}$ can be reached. We also have to introduce a second absorbing region $\mathcal{A}_2 = \{(i, j)| - ci > j, \ i \in \mathbb{N}, j \in \mathbb{Z}\}$, with $X_{n,m} = 0$ for all $(i, j) \in \mathcal{A}_2$. Despite these additional restrictions we can obtain limit laws for $X_{n,m}$ if $m \in \mathbb{N}$ is fixed and n tends to infinity.

Due to their importance in applications, there is a huge literature on the stochastic behavior of urn models. There are quite recent very deep and general treatments by (Jan04; Jan05) and (FDP06; FGP05). We also refer to (JK77; KB97), which are the standard references for urn models.

A discussion of “diminishing urns” has been given in (HKP06), where a generating functions approach leading to partial differential equations has been used to establish exact and asymptotic results for the distribution of $X_{n,m}$ for several replacement matrices M. Here we choose a different approach dealing directly with recurrences for the s-th integer moments of $X_{n,m}$, which turns out to be suitable for a characterization of the behavior of the whole class of “2×2 triangular diminishing urn models”.

Our studies of the class of diminishing urns, with ball replacement matrix M given by (1), is motivated by the following problems.

1.2 The pills problem

The pills problem was originally proposed by (KM91), p. 264; the solution appeared in (Hes92). In a bottle there are n small pills and m large pills. The large pill is equivalent to two small pills. Every day a person chooses a pill at random. If a small pill is chosen, it is eaten up, if a large pill is chosen it is broken into two halves, one half is eaten and the other half which is now considered to be a small pill is returned to the bottle. The proposed problem was to find the expected number of small pills remaining when there are no more large pills left in the bottle. The pills problem corresponds to the derivation of the expected value of $X_{n,m}$ for a diminishing urn model with ball replacement matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$. The pills problem was revisited in (BP03), where it was shown how to derive higher moments of the random variable counting the number of small pills when all large pills have been consumed. In the recent work of (HKP06) the limit laws of the pills problem and a related model, $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -2 \end{smallmatrix}\right)$, were studied using generating functions. It was shown that the limit laws significantly differ for the two problems considered. The results of (HKP06) for these two specific urns were the primary motivation to analyze the whole class of urns with $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$.

1.3 Sampling without replacement

This fundamental urn model corresponds to the urn with ball replacement matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The distribution of the type of balls after t draws is very well known (see e. g. (FDP06)), but here we will focus on the limit laws of $X_{n,m}$. Note that this problem is often treated by introducing two absorbing axes, i.e., $\{(0,n) : n \in \mathbb{N}_0\} \cup \{(m,0) : m \in \mathbb{N}_0\}$, but we rather simply use the absorbing axis $\mathcal{A} = \{(0,n) : n \in \mathbb{N}_0\}$, which is fully sufficient. We will also derive limit laws for the generalization $M = \left(\begin{smallmatrix} -a & 0 \\ 0 & -d \end{smallmatrix}\right)$, which, best to our knowledge, has not been considered before.

1.4 Weighted lattice paths

It is useful to describe and visualize the evolution of an urn with ball replacement matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by "weighted paths", which is described here in the case of urns with two type of balls. If the urn contains n white balls and m black balls and we select a white ball (with probability $\frac{n}{m+n}$), then this corresponds to a step from $(m,n) \to (m+a, n+b)$, to which the weight $\frac{n}{m+n}$ is associated; and if we select a black ball (with probability $\frac{m}{m+n}$), then this corresponds to a step from $(m,n) \to (m+c, n+d)$ with weight $\frac{m}{m+n}$. The weight of a path after t successive draws consists of the product of the weights of every step. For a diminishing urn we obtain that the sum of the weights of all possible paths starting at state (m,n) and ending at the absorbing state $(i,j) \in \mathcal{A}$ (which did not pass another absorbing state earlier) gives then the required probability, that when starting at (m,n) we are ending at (i,j). Unfortunately, the expressions so obtained for the probability are, although exact, less useful for large n or m. An example for a weighted path corresponding to the evolution of a diminishing urn is given in Figure 1. The steps associated with ball replacement matrix $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$ are visualized in Figure 2.

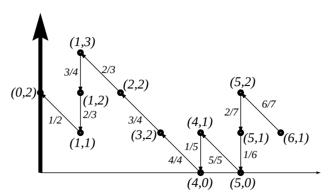

Fig. 1: An example of a weighted path from $(6,1)$ to the absorbing state $(0,2)$ for the pills problem with ball addition matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and the absorbing axis $\mathcal{A} = \{(0,n) : n \in \mathbb{N}_0\}$. The illustrated path has weight $\frac{6}{7}\frac{2}{7}\frac{1}{6}\frac{5}{5}\frac{1}{5}\frac{4}{4}\frac{3}{4}\frac{2}{4}\frac{3}{4}\frac{2}{3}\frac{1}{2} = \frac{3}{3920}$.

1.5 Goal

It will turn out in this paper that an elementary approach, which is inspired by the work of Brennan and Prodinger, is sufficient for the analysis of urn models with ball replacement matrix $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$. For $a \in \mathbb{N}$ and $c = p \cdot a$, $p \in \mathbb{N}_0$ we can provide exact formulas for the expectation. We will determine limit

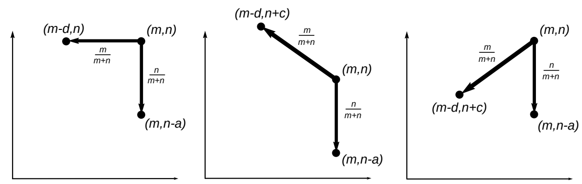

Fig. 2: The steps associated with $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, according to the cases $c = 0$, $c > 0$ and $c < 0$.

laws of the random variable $X_{n,m}$, with replacement matrix M as given in (1). As a byproduct we obtain limit laws for the pills problem and generalizations.

For m fixed and n tending to infinity we can show that $X_{n,m}/n$ tends to a limit law for arbitrary $c \in \mathbb{Z}$, which can be characterized as a Kumaraswamy distribution. Further, for $c \in \mathbb{N}_0$ we will show that for m tending to infinity and $n = n(m)$ the limit laws change according to the quotient a/d, with $a \in \mathbb{N}$ and $d \in \mathbb{N}$. We will encounter Weibull distributions as limit laws.

1.6 Notation

We denote with $X \overset{(d)}{=} Y$ the equality in distribution of the random variables X and Y and by $X_n \xrightarrow{(d)} X$ the weak convergence, i. e., the convergence in distribution, of the sequence of random variables X_n to a random variable X. Further $H_n := \sum_{k=1}^{n} 1/k$ denotes the n-th harmonic number.

2 Results

2.1 Exact results for the expected value

Theorem 1 *For $a \in \mathbb{N}$, $c = p \cdot a$, with $p \in \mathbb{N}_0$, the urn with ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$ is well defined when starting with $a \cdot n$ white and $d \cdot m$ black balls. The expectation of $X_{an,dm}$ is given by the exact formulas*

$$a/d = 1: \quad \mathbb{E}(X_{an,dm}) = \frac{an}{m+1} + paH_m,$$
$$a/d \neq 1: \quad \mathbb{E}(X_{an,dm}) = \frac{an}{\binom{m+\frac{a}{d}}{m}} + pa\Big(\frac{1}{1-\frac{a}{d}} \frac{m+\frac{a}{d}}{\binom{m+\frac{a}{d}}{m}} + \frac{\frac{a}{d}}{\frac{a}{d}-1}\Big).$$

2.2 Limiting distribution results for m fixed

Theorem 2 *The s-th moment of the random variable $X_{n,dm}$, counting the number of white balls, when all black balls have been removed, with ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, $a, d \in \mathbb{N}$, $c \in \mathbb{Z}$, starting with n white and $d \cdot m$ black balls, is for fixed $m \in \mathbb{N}$ and $n \to \infty$ asymptotically given by*

$$\mathbb{E}(X_{n,dm}^s) = \frac{n^s}{\binom{m+\frac{as}{d}}{m}} \Big(1 + \mathcal{O}(n^{-1})\Big).$$

The limiting distribution of $X_{n,dm}$ is characterized as follows:

$$\frac{X_{n,dm}}{n} \xrightarrow{(d)} X, \quad X \overset{(d)}{=} K(\frac{d}{a}, m),$$

where $K(r,u)$ denotes a Kumaraswamy distribution with parameters r and u.

Remark 1 *Kumaraswamy's double bounded distribution is in its simplest form defined on $[0,1]$. The probability density function $f_K(t)$ and the moments of a Kumaraswamy distributed random variable $K = K(r,u)$ are given by*

$$f_K(t) = rut^{r-1}(1-t^r)^{u-1}, \quad \mathbb{E}(K^s) = \frac{\Gamma(u+1)\Gamma(1+\frac{s}{r})}{\Gamma(u+1+\frac{s}{r})}, \quad s \in \mathbb{N}.$$

2.3 Limiting distribution results for m tending to infinity and $c \neq 0$

First we turn our attention to the case $c \in \mathbb{N}$ and $a/d \leq 1$. In this case the expected value of $X_{n,dm}$ converges to infinity as m tends to infinity regardless of the growth of $n = n(m)$.

Theorem 3 *The limiting distributions of $X_{n,dm}$, counting the number of white balls, when all black balls have been removed, starting with n white and $d \cdot m$ black balls, with ball replacement matrix $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$, $a, c, d \in \mathbb{N}$, $a/d \leq 1$, are for $m \to \infty$ and arbitrary $n = n(m)$, given as follows,*

$$\frac{X_{n,dm}}{g_{n,m}} \xrightarrow{(d)} X, \qquad X \overset{(d)}{=} W(\frac{d}{a}, 1), \tag{2}$$

where $W(r,u)$ denotes a Weibull distributed random variable with parameters r and u. The normalization values $g_{n,m}$ are given as follows,

$$g_{n,m} = \begin{cases} g_{n,m} = g_{n,m}(a,c,d) = \dfrac{n + m\frac{cd}{d-a}}{m^{\frac{a}{d}}}, & \text{for } a/d < 1, \\ g_{n,m} = \dfrac{n}{m} + c \log m, & \text{for } a/d = 1, \end{cases} \tag{3}$$

Remark 2 *The Weibull distribution is a continuous probability distribution. The probability density function $f_W(t)$ and the moments of a Weibull distributed random variable $W = W(r,u)$ are given by*

$$f_W(t) = \frac{r}{u}\Big(\frac{x}{u}\Big)^{r-1} e^{-(t/u)^r}, \quad t \geq 0, \quad \mathbb{E}(W^s) = u^s \Gamma(1 + \frac{s}{r}), \quad s \in \mathbb{N}.$$

For $a/d > 1$ we observe a different behavior of $X_{n,m}$, as suggested by closer inspection of the expected value $\mathbb{E}(X_{n,m})$ (and also the value $f_{n,m}^{[1]}$, as given by Lemma 2). In the cases $n \sim m^{a/d}$ and $n = o(m^{a/d})$, the expected value tends to a constant as $n = n(m)$ tends to infinity.

Note that for $c \neq p \cdot a$, $p \in \mathbb{N}$, the limit laws for $n \sim m^{a/d}$ and $n = o(m^{a/d})$ seem to be sensitive to the introduction of mandatory additional rules which are necessary in order to avoid leaving the state space $\mathcal{S}$. Therefore concerning the regions $n \sim m^{a/d}$ and $n = o(m^{a/d})$, with $n \to \infty$, we restrict ourselves to well-defined urns, $c = p \cdot a$, $p \in \mathbb{N}$, and start with $a \cdot n$ white balls and $d \cdot m$ black balls.

Theorem 4 *The limiting distribution of $X_{an,dm}$, with ball replacement matrix $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$, $a, d \in \mathbb{N}$, $c = p \cdot a$ with $p \in \mathbb{N}$, and $a/d > 1$, is for $m \to \infty$ and $n = n(m)$ given as follows,*

1. *For $m^{a/d} = o(n)$*

$$\frac{m^{\frac{a}{d}} X_{an,dm}}{an} \xrightarrow{(d)} X, \qquad X \overset{(d)}{=} W(\frac{d}{a}, 1),$$

where $W(r,u)$ denotes a Weibull distributed random variable with parameters r and u.

2. *For $n \sim m^{a/d}$ all moments of the random variable $X_{an,dm}$ converge, $\mathbb{E}(X_{an,dm}^s) \to \sum_{l=0}^{s} \Lambda_{s,l}$,*

$$\Lambda_{s,s} = a^s \Gamma(1+\frac{as}{d}), \quad \Lambda_{s,l} = \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{al}{d})}{k!d} \Big(\sum_{i=l+1}^{s} \binom{i}{l+1} (-a)^{i-l-1} \lambda_{s,i,k} + dk \binom{i}{l} c^{i-l} \lambda_{s,i,k-1} \Big),$$

for $1 \le l \le s-1$ and

$$\Lambda_{s,0} = \sum_{k=1}^{\infty} \sum_{i=1}^{s} c^i \lambda_{s,i,k},$$

with $\lambda_{s,k,m}$ essentially given by Lemma 1 on page 386 (setting $\lambda_{s,s,0} = a^s$).

3. *For $n = o(m^{a/d})$ all the moments of the random variable $X_{an,dm}$ converge, $\mathbb{E}(X_{a,d}^s) \to \Lambda_{s,0}$,*

$$\Lambda_{s,0} = \sum_{k=1}^{\infty} \sum_{i=1}^{s} c^i \lambda_{s,i,k},$$

with $\lambda_{s,k,m}$ essentially given by Lemma 1 on page 386 (setting $\lambda_{s,s,0} = a^s$).

For $c \neq p \cdot a$ and $a/d > 1$ and $m^{a/d} = o(n)$ we have the same results as in the case $c = p \cdot a$.

$$\frac{m^{\frac{a}{d}} X_{n,dm}}{n} \xrightarrow{(d)} X_{a,d}, \quad f_{X_{a,d}}(t) = \frac{d}{a} t^{\frac{d}{a}-1} e^{-t^{\frac{d}{a}}}, \quad t \ge 0.$$

Remark 3 *Up to now we have not been able to show that for the instances $n \sim m^{a/d}$ and $n = o(m^{a/d})$ the limits of the moments define unique distributions (by establishing suitable growth estimates on the moments and applying Carleman's criterion). Thus we state in Theorem 4 for case* 2. *and* 3. *only a moment's convergence result.*

2.4 The case $c = 0$

For $c = 0$ we do not have to separate between cases $a/d \le 1$ and $a/d > 1$. The limit laws are covered by our earlier results, namely Theorem 2 for m fixed and $n \to \infty$, and Theorem 4 for $m \to \infty$ and $n = m(n)$, where Theorem 4 stays valid for $a/d \le 1$, if $c = 0$. Concerning the region $n = o(m^{a/d})$ and $m \to \infty$, we can specify our earlier result as follows.

Corollary 1 *In the case $c = 0$ the random variable $X_{an,dm}$ is, for $n = o(m^{a/d})$ and $m \to \infty$, asymptotically zero,*

$$X_{an,dm} \xrightarrow{(d)} X, \quad \mathbb{P}\{X = 0\} = 1.$$

Unfortunately, we have not been able to show that for the remaining instance $n \sim m^{a/d}$ the limits of the moments define unique distributions. In the special case $a = d = 1$, the well known sampling without replacement urn, one can be much more precise by using exact results for the probabilities:

$$\mathbb{P}\{X_{n,m} = k\} = \frac{\binom{n+m-1-k}{m-1}}{\binom{n+m}{m}}.$$

An application of Stirling's formula for the Gamma function leads then to local limit theorems.

Corollary 2 *For the ordinary sampling urn with ball replacement matrix* $M = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$*, one obtains local limit laws for* $X_{n,m}$*. E. g., it holds that in the region* $n \sim m$ *and* $m \to \infty$*, the random variable* $X_{n,m}$ *is asymptotically geometrically distributed,*

$$\mathbb{P}\{X_{n,m} = k\} \to \frac{1}{2^{k+1}}, \quad k \in \mathbb{N} \cup \{0\}.$$

3 Sketch of the proof of the limit laws

For our proof of the limit laws for $X_{n,m}$, stated in Theorems 2 up to 4, with the method of moments we need a precise description of the structure of the moments.

3.1 The elementary approach: a recursive description of the moments

We consider the urn model with ball replacement matrix $M = \left(\begin{smallmatrix} -a & 0 \\ c & -d \end{smallmatrix}\right)$, with arbitrary but fixed parameters $a, d \in \mathbb{N}$ and $c \in \mathbb{N}_0$. Let $X_{n,m} = X_{n,m}(a, c, d)$ denote the random variable counting the number of white balls when the number of black balls reaches zero. By definition the probability generating function $h_{n,m}(v) = \sum_{k \geq 0} \mathbb{P}\{X_{n,m} = k\} v^k$ satisfies the following recurrence.

$$h_{n,m}(v) = \frac{n}{n+m} h_{n-a,m}(v) + \frac{m}{n+m} h_{n+c,m-d}(v), \tag{4}$$

with initial values $h_{n,0}(v) = v^n$, $n \in \mathbb{N}_0$. It is natural to start with n white balls and $d \cdot m$ black balls in order to secure that the urn is well defined with respect to m and to avoid specifications of more initial values. Therefore we will restrict ourselves to the random variable $X_{n,dm}$. Note that by construction the number of black balls can never be below zero in this case.

When relying on the so-called method of moments, we have to give precise estimates for the ordinary moments $e^{[s]}_{n,m} = \mathbb{E}(X^s_{n,dm})$ of $X_{n,dm}$. The moments satisfy for $n \geq a$ the recurrence

$$e^{[s]}_{n,m} = \frac{n}{n+dm} e^{[s]}_{n-a,m} + \frac{dm}{n+dm} e^{[s]}_{n+c,m-1}, \tag{5}$$

with initial values $e^{[s]}_{n,0} = n^s$, $n \in \mathbb{N}_0$.

We have to circumvent situations where the number of white balls may drop below zero. Hence the most natural choices for the parameters a and c are $c = p \cdot a$ with $p \in \mathbb{N}_0$. In these cases we start with $a \cdot n$ white balls in order to obtain well defined urns. Then, the recurrence for the moments holds for $n \geq 1$ and one obtains in principle exact results for arbitrarily high moments, especially the expectation and the variance.

If $a > 1$ and $c \neq p \cdot a$ the urn is no longer well defined. Therefore we have to introduce extra rules in order to specify what to do in case of $0 \leq n < a$. We define that any layer $0, 1, \ldots, a-1$ is *reflective* in the sense of weighted paths: from points (i, j) with $0 \leq j \leq a-1$ only the points $(i-d, j+c)$ can be reached (in the context of the pills problem it means that only large pills will be consumed below a certain amount a of small pills). This means we have to deal with a reflective strip $\mathcal{R}$ consisting of points $\mathcal{R} = \{(i, j) | \, 0 \leq j \leq a-1, i \in \mathbb{N}_0\}$.

Due to the reflective region the initial values get more complicated and we cannot hope to explicitly obtain the moments $\mathbb{E}(X^s_{n,dm})$ anymore. Therefore we introduce values $f^{[s]}_{n,m}$, which will be used to approximate the moments $e_{n,m}^{[s]} := \mathbb{E}(X_{n,dm}^s)$ for all $s \in \mathbb{N}$. The values $f_{n,m}^{[s]}$ are defined by the recurrence

$$(n + md)f_{n,m}^{[s]} = nf_{n-a,m}^{[s]} + mdf_{n+c,m-1}^{[s]}, \tag{6}$$

for $m > 1, n \in \mathbb{Z}$, with $f_{n,0}^{[s]} := n^s$. It is crucial that we allow n to be negative in order to deal with the reflective strip $\mathcal{R}$ in the case of $a > 1$ and $c \neq p \cdot a$. The key point is the observation that $f_{n,m}^{[s]}$ can be obtained by the *Ansatz* $f_{n,m}^{[s]} = \sum_{k=0}^{s} \lambda_{s,k,m} n^s$, e. g. $f_{n,m}^{[s]}$ can be expressed as a polynomial of degree s in n. For $c = p \cdot a$, $p \in \mathbb{N}_0$, and starting with $a \cdot n$ white balls, the sequence $f_{n,m}^{[s]}$ is exactly the moment sequence $e_{n,m}^{[s]} \equiv f_{n,m}^{[s]}$, whereas for $c \neq p \cdot a$ we will use $f_{n,m}^{[s]}$ as an approximate value for $e_{n,m}^{[s]}$. A similar approach was used by Brennan and Prodinger for the calculation of the expectation and the variance in the case $a = d = 1$ and some other cases.

Lemma 1 *The values $f_{n,m}^{[s]}$, determined by* (6)*, satisfy the expansion $f_{n,m}^{[s]} = \sum_{k=0}^{s} \lambda_{s,k,m} n^k$, where*

$$\lambda_{s,s,m} = \frac{1}{\binom{m+\frac{as}{d}}{m}}, \quad \lambda_{s,l,m} = \sum_{k=0}^{m-1} \frac{\binom{m}{k}}{\binom{m+\frac{al}{d}}{k}} \mu_{s,l,m-k}, \quad \textit{for } 1 \leq l \leq s-1, \tag{7}$$

with

$$\mu_{s,l,m} := \frac{1}{md + al} \sum_{k=l+1}^{s} \binom{k}{l-1} (-a)^{k-l-1} \lambda_{s,k,m} + \frac{dm}{dm + al} \sum_{k=l+1}^{s} \binom{k}{l} c^{k-l} \lambda_{s,k,m-1}. \tag{8}$$

For $l = 0$ we have

$$\lambda_{s,0,m} = \sum_{k=0}^{m-1} \mu_{s,0,k}, \quad \textit{with } \mu_{s,0,m} := \sum_{k=1}^{s} \lambda_{s,k,m} c^k. \tag{9}$$

The initial values are given by $\lambda_{s,s,0} = 1$ and $\lambda_{s,l,0} = 0$ for $0 \leq l \leq s-1$.

Proof: We use the *Ansatz* $f_{n,m}^{[s]} = \sum_{k=0}^{s} \lambda_{s,k,m} n^s$ to determine the sequence $f_{n,m}^{[s]}$. From the recurrence (6) we get

$$(n + dm) \sum_{k=0}^{s} \lambda_{s,k,m} n^k = n \sum_{k=0}^{s} \lambda_{s,k,m} (n-a)^k + dm \sum_{k=0}^{s} \lambda_{s,k,m-1} (n+c)^k. \tag{10}$$

Hence we obtain from (10) a system of recurrences by comparison of coefficients of n^l, with $l = 0, .., s+1$, from which one can deduce the stated results for $\lambda_{s,l,m}$. For example $\lambda_{s,s,m}$ is determined by the following recurrence,

$$dm\lambda_{s,s,m} + \lambda_{s,s-1,m} = -sa\lambda_{s,s,m} + \lambda_{s,s-1,m} + dm\lambda_{s,s,m-1}.$$

We have

$$dm\lambda_{s,s,m} = dm\lambda_{s,s,m-1} - as\lambda_{s,s,m}, \quad (dm + as)\lambda_{s,s,m} = dm\lambda_{s,s,m-1},$$

and consequently,

$$\lambda_{s,s,m} = \frac{dm}{dm + as} \lambda_{s,s,m-1}, \quad \lambda_{s,s,m} = \frac{m!}{(m + \frac{as}{d})^{\underline{m}}} = \frac{1}{\binom{m+\frac{as}{d}}{m}}.$$

□

Next we will provide explicit expressions for $f^{[1]}_{n,m}$, which are obtained by straight forward computations using Lemma 1.

Lemma 2 *The values $f^{[1]}_{n,m}$ are given by*

$$f^{[1]}_{n,m} = \frac{n}{\binom{m+\frac{a}{d}}{m}} + \frac{c}{1-\frac{a}{d}}\Big(\frac{m+\frac{a}{d}}{\binom{m+\frac{a}{d}}{m}} - \frac{a}{d}\Big), \quad \textit{for } \frac{a}{d}\neq 1,$$
$$f^{[1]}_{n,m} = \frac{n}{m+1} + cH_m, \quad \textit{for } \frac{a}{d} = 1. \tag{11}$$

Lemma 2 has the consequence that for $a/d = 1$ there is a transition of the behavior of $f^{[1]}_{n,m}$. For $a/d \leq 1$ we always have $f^{[1]}_{n,m} \to \infty$ for $\max\{n, m\} \to \infty$, since at least one of the two terms converge to infinity. For $a/d > 1$ this condition is not true anymore. For $m \to \infty$ we have the expansion

$$f^{[1]}_{n,m} = \frac{\Gamma(1+\frac{a}{d})n}{m^{\frac{a}{d}}} + \frac{c\Gamma(1+\frac{a}{d})}{(1-\frac{a}{d})m^{\frac{a}{d}-1}} + \mathcal{O}(1). \tag{12}$$

Hence for $a/d > 1$ and $m = o(n^{\frac{a}{d}})$ the value $f^{[1]}_{n,m}$ tends to infinity and we can normalize by multiplying with $m^{\frac{a}{d}}/n$, whereas for $m \sim n^{\frac{a}{d}}$ or $n = o(m^{\frac{a}{d}})$ it holds that $f^{[1]}_{n,m}$ remains bounded.

Let $\Delta^{[s]}_{n,m} := e^{[s]}_{n,m} - f^{[s]}_{n,m}$ denote the difference between the s-th moment of $X_{n,dm}$ and the approximative value $f^{[s]}_{n,m}$ for $a \in \mathbb{N}$. By definition $\Delta^{[s]}_{n,m}$ satisfies a recurrence relation of the form (6) and $\Delta^{[s]}_{n,0} = 0$ for all $n \geq 0$ and $s \geq 1$. For $c = p \cdot a$ the values $\Delta^{[s]}_{n,m} \equiv 0$.

The limiting distributions are obtained as follows. Lemma 1 provides an asymptotic expansion of $f^{[s]}_{n,m}$ of the form $f^{[s]}_{n,m} \sim f^{[s]} \cdot g^s_{n,m}$, with $f^{[s]}$ being independent of n and m. By using

$$\frac{e^{[s]}_{n,m}}{g^s_{n,m}} = \frac{\Delta^{[s]}_{n,m}}{g^s_{n,m}} + \frac{f^{[s]}_{n,m}}{g^s_{n,m}}, \tag{13}$$

and suitable growth estimates on $\Delta^{[s]}_{n,m}$, we obtain $\frac{e^{[s]}_{n,m}}{g^s_{n,m}} \sim \frac{f^{[s]}_{n,m}}{g^s_{n,m}} \sim f^{[s]}$. The last step is to prove that the sequence $(f^{[s]})_{s\in\mathbb{N}}$ defines a unique distribution, using Carlemans Criterion. The complete proofs will be contained in the full version of this work, which will also include the case m fixed and $c \in -\mathbb{N}$, and other extensions of this approach. We want to remark that for $c \in -\mathbb{N}$ the moments of $X_{n,dm}$ do not have the same regular form: e. g. $M = \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$ and $\mathbb{E}(X_{n,1}) = \frac{n}{2} - 1 + \frac{1}{n+1}$.

4 Conclusion

With our methods a full characterization of the limiting distribution of $X_{n,m}$ should be possible. For the readers convenience we have collected our findings in the following table. Note that the distributions marked with an asterisk are conjectural.

	$n \to \infty$: m fixed	$m \to \infty$: $m = o(n^{\frac{a}{d}})$	$m \to \infty$: $m \sim n^{\frac{a}{d}}$	$m \to \infty$: $n = o(m^{\frac{a}{d}})$
$a/d \leq 1$ and $c \in \mathbb{N}$	Kuramaswamy	Weibull	Weibull	Weibull
$a/d > 1$ and $c \in \mathbb{N}$	Kuramaswamy	Weibull	Discrete*	Discrete*
$a, d \in \mathbb{N}$ and $c = 0$	Kuramaswamy	Weibull	Discrete*	Degenerate

The authors are currently trying to identify the distributions for the regions where the moments of $X_{n,m}$ converge. Furthermore it seems possible to extend the analysis of this class of diminishing urns to more general urn models. The authors are trying to extend this study to three types of balls.

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