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Queue Layouts of Graph Products and Powers†

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A k-queue layout of a graph $G$ consists of a linear order $\sigma$ of $V(G)$, and a partition of $E(G)$ into $k$ sets, each of which contains no two edges that are nested in $\sigma$. This paper studies queue layouts of graph products and powers.

Keywords: graph, queue layout, cartesian product, $d$-dimensional grid graph, $d$-dimensional toroidal grid graph, Hamming graph.

2000 MSC classification: 05C62 (graph representations)

1 Introduction

Let $G$ be a graph. (All graphs considered are finite, simple and undirected.) The vertex and edge sets of $G$ are denoted by $V(G)$ and $E(G)$, respectively. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The density of $G$ is $\eta(G) := |E(G)|/|V(G)|$.

A vertex ordering of $G$ is a bijection $\sigma : V(G) \to \{1, 2, \ldots, |V(G)|\}$. In a vertex ordering $\sigma$ of $G$, let $L_\sigma(e)$ and $R_\sigma(e)$ denote the endpoints of each edge $e \in E(G)$ such that $\sigma(L_\sigma(e)) < \sigma(R_\sigma(e))$. Where the vertex ordering $\sigma$ is clear from the context, we will abbreviate $L_\sigma(e)$ and $R_\sigma(e)$ by $L_e$ and $R_e$, respectively. For edges $e$ and $f$ of $G$ with no endpoint in common, there are the following three possible relations with respect to $\sigma$, as illustrated in Figure 1:

(a) $e$ and $f$ nest if $\sigma(L_e) < \sigma(L_f) < \sigma(R_f) < \sigma(R_e)$,
(b) $e$ and $f$ cross if $\sigma(L_e) < \sigma(L_f) < \sigma(R_e) < \sigma(R_f)$,
(c) $e$ and $f$ are disjoint if $\sigma(L_e) < \sigma(R_e) < \sigma(L_f) < \sigma(R_f)$.

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A queue in $\sigma$ is a set of edges $Q \subseteq E(G)$ such that no two edges in $Q$ are nested. Observe that when traversing $\sigma$ from left to right, the left and right endpoints of the edges in a queue are reached in first-in-first-out order—hence the name ‘queue’. Observe that $Q \subseteq E(G)$ is a queue if and only if for all edges $e, f \in Q$,

$$\sigma(L_e) \leq \sigma(L_f) \text{ and } \sigma(R_e) \leq \sigma(R_f) \quad ,$$

or $$\sigma(L_f) \leq \sigma(L_e) \text{ and } \sigma(R_f) \leq \sigma(R_e) .$$

A $k$-queue layout of $G$ is a pair

$$(\sigma, \{Q_1, Q_2, \ldots, Q_k\})$$

where $\sigma$ is a vertex ordering of $G$, and $\{Q_1, Q_2, \ldots, Q_k\}$ is a partition of $E(G)$, such that each $Q_i$ is a queue in $\sigma$. The queue-number of a graph $G$, denoted by $qn(G)$, is the minimum $k$ such that there is a $k$-queue layout of $G$.

Queue layouts were introduced by Heath et al. [15, 19]. Applications of queue layouts include sorting permutations [12, 20, 22, 24, 27], parallel process scheduling [3], matrix computations [23], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in the literature [7, 9, 10, 13, 25, 26]. Queue layouts of directed graphs [5, 11, 17, 18] and posets [16] have also been investigated.

Table 1 describes the best known upper bounds on the queue-number of various classes of graphs. Planar graphs are an interesting class of graphs for which it is not known whether the queue-number is bounded (see [6, 23]).

This paper studies queue layouts of graph products and graph powers. To prove optimality we use the following lower bound by Heath and Rosenberg [19]. See Pemmaraju [23] and Dujmović and Wood [9] for slightly more exact lower bounds.

**Lemma 1 ([19])** Every graph $G$ has queue-number $qn(G) > \eta(G)/2$.

This paper is organised as follows. In Section 2 we introduce the concepts of strict queue layout and strict queue-number. Many of the upper bounds on the queue-number that are presented in later sections will be expressed as functions of the strict queue-number. In Section 3 we prove bounds on the queue-number of the power of a graph in terms of the queue-number of the underlying graph. In Section 4 we define the graph products that will be studied in later sections. In Section 5 we study the queue-number of the cartesian product of graphs. Finally in Section 6 we study the queue-number of the direct and strong products of graphs.

‡ Dujmović and Wood [8] gave a simple proof of this result.
Tab. 1: Upper bounds on the queue-number.

<table>
<thead>
<tr>
<th>graph family</th>
<th>queue-number</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ vertices</td>
<td>$\left\lfloor \frac{n^2}{2} \right\rfloor$</td>
<td>Heath and Rosenberg [19]</td>
</tr>
<tr>
<td>$m$ edges</td>
<td>$e \sqrt{m}$</td>
<td>Dujmović and Wood [9]</td>
</tr>
<tr>
<td>tree-width $w$</td>
<td>$3^w \cdot 6(4^w-3w-1)/9-1$</td>
<td>Dujmović et al. [6]</td>
</tr>
<tr>
<td>tree-width $w$, max. degree $\Delta$</td>
<td>$36\Delta w$</td>
<td>Wood [29]</td>
</tr>
<tr>
<td>path-width $p$</td>
<td>$p$</td>
<td>Dujmović et al. [6]</td>
</tr>
<tr>
<td>band-width $b$</td>
<td>$\left\lfloor \frac{b^2}{2} \right\rfloor$</td>
<td>Heath and Rosenberg [19]</td>
</tr>
<tr>
<td>track-number $t$</td>
<td>$t-1$</td>
<td>Dujmović et al. [6]</td>
</tr>
<tr>
<td>2-trees</td>
<td>3</td>
<td>Rengarajan and Veni Madhavan [25]</td>
</tr>
<tr>
<td>$k$-ary butterfly</td>
<td>$\left\lfloor \frac{k^2}{2} \right\rfloor + 1$</td>
<td>Hasunuma [14]</td>
</tr>
<tr>
<td>$d$-ary de Bruijn</td>
<td>$d$</td>
<td>Hasunuma [14]</td>
</tr>
<tr>
<td>Halin</td>
<td>3</td>
<td>Ganley [13]</td>
</tr>
<tr>
<td>X-trees</td>
<td>2</td>
<td>Heath and Rosenberg [19]</td>
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<tr>
<td>outerplanar</td>
<td>2</td>
<td>Heath et al. [15]</td>
</tr>
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<td>arched levelled planar</td>
<td>1</td>
<td>Heath et al. [15]</td>
</tr>
<tr>
<td>trees</td>
<td>1</td>
<td>Heath and Rosenberg [19]</td>
</tr>
</tbody>
</table>

2 Strict Queue Layouts

Let $\sigma$ be a vertex ordering of a graph $G$. We say an edge $e$ is inside a distinct edge $f$, and $e$ and $f$ overlap, if

$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f).$$

A set of edges $Q \subseteq E(G)$ is a strict queue in $\sigma$ if no edge in $Q$ is inside another edge in $Q$. Alternatively, $Q$ is a strict queue in $\sigma$ if

$$\sigma(L_e) < \sigma(L_f) \quad \text{and} \quad \sigma(R_e) < \sigma(R_f),$$

or

$$\sigma(L_f) < \sigma(L_e) \quad \text{and} \quad \sigma(R_f) < \sigma(R_e).$$

Note that Equation (2) is obtained from Equation (1) by replacing “$\leq$” by “$<$”.

Hence a strict queue is a set of edges, no two of which are nested or overlapping, as illustrated in Figure 2. Note that edges forming a ‘butterfly’ can be in a single strict queue.

Fig. 2: Relationships between pairs of edges with a common endpoint in a vertex ordering.

A strict $k$-queue layout of $G$ is a pair $(\sigma, \{Q_1, Q_2, \ldots, Q_k\})$ where $\sigma$ is a vertex ordering of $G$, and $\{Q_1, Q_2, \ldots, Q_k\}$ is a partition of $E(G)$, such that each $Q_i$ is a strict queue in $\sigma$. We sometimes write
queue\(e\) = \(i\) for each edge \(e\) \(\in\) \(Q_i\). The \textit{strict-queue-number} of a graph \(G\), denoted by \(sqn(G)\), is the minimum \(k\) such that there is a strict \(k\)-queue layout of \(G\).

Heath and Rosenberg \cite{19} proved that a fixed vertex ordering of a graph \(G\) admits a \(k\)-queue layout of \(G\) if and only if it has no \((k + 1)\)-edge rainbow, where a \textit{rainbow} is a set of pairwise nested edges, as illustrated in Figure 3(a). Consider the analogous problem for strict queues: assign the edges of a graph \(G\) to the minimum number of strict queues given a fixed vertex ordering \(\sigma\) of \(G\). As illustrated in Figure 3(b), a \textit{weak rainbow} in \(\sigma\) is a set of edges \(R\) such that for every pair of edges \(e, f\) \(\in\) \(R\), \(e\) is inside \(f\) or \(f\) is inside \(e\).

\[\text{Fig. 3: (a) rainbow, (b) weak rainbow}\]

**Lemma 2** A vertex ordering of a graph \(G\) admits a strict \(k\)-queue layout of \(G\) if and only if it has no \((k + 1)\)-edge weak rainbow.

**Proof:** A strict \(k\)-queue layout has no \((k + 1)\)-edge weak rainbow since each edge of a weak rainbow must be in a distinct strict queue. Conversely, suppose we have a vertex ordering with no \((k + 1)\)-edge weak rainbow. For every edge \(e\) \(\in\) \(E(G)\), let \(\text{queue}(e)\) be one plus the maximum number of edges in a weak rainbow consisting of edges that are inside \(e\). If \(e\) is inside \(f\) then \(\text{queue}(e) < \text{queue}(f)\). Hence we have a valid strict queue assignment. The number of strict queues is at most \(k\).

\[\square\]

A \textit{linear forest} is a graph in which every component is a path. The \textit{linear arboricity} of a graph \(G\), denoted by \(la(G)\), is the minimum integer \(k\) such that \(E(G)\) can be partitioned in \(k\) linear forests; see \cite{1,2,30,31}. We have the following lower bounds on \(sqn(G)\).

**Lemma 3** The strict queue-number of every graph \(G\) satisfies:

(a) \(sqn(G) \geq la(G) > \eta(G)\),

(b) \(sqn(G) \geq la(G) \geq \Delta(G)/2\), and

(c) \(sqn(G) \geq \delta(G)\).

**Proof:** Say \(Q\) is a strict queue in a vertex ordering \(\sigma\) of \(G\). Every 2-edge path \((u, v, w)\) in \(Q\) has \(\sigma(u) < \sigma(v) < \sigma(w)\) (or \(\sigma(w) < \sigma(v) < \sigma(u)\)). Thus no vertex is incident to three edges in \(Q\), and \(Q\) induces a linear forest. Hence \(la(G) \leq sqn(G)\).

Since a linear forest in \(G\) has at most \(|V(G)| - 1\) edges, \(la(G) \geq |E(G)|/(|V(G)| - 1) > \eta(G)\). This proves (a). At most two edges incident to each vertex are a linear forest. Thus \(la(G) \geq \Delta(G)/2\). This proves (b).
In every vertex ordering of $G$, every edge incident to the first vertex is in a distinct strict queue. Hence $\text{sqn}(G) \geq \delta(G)$. This proves (c).

Obviously a proper edge $(\Delta(G) + 1)$-colouring \cite{28} can be combined with a $\text{sqn}(G)$-queue layout to obtain a strict queue layout.

**Lemma 4** Every graph $G$ has strict queue-number $\text{sqn}(G) \leq (\Delta(G) + 1) \cdot \text{qn}(G)$.

\section{Graph Powers}

Let $G$ be a graph, and let $d \in \mathbb{Z}^+$. The $d$-th power of $G$, denoted by $G^d$, is the graph with vertex set $V(G^d) = V(G)$, where $vw \in E(G^d)$ if and only if the distance between $v$ and $w$ in $G$ is at most $d$. The following general result is similar to a theorem of Dujmović and Wood \cite{19}.

**Theorem 1** For every graph $G$ and $d \in \mathbb{Z}^+$,

$$\text{qn}(G^d) \leq \frac{(2 \text{sqn}(G))^{d+1} - 1}{2 \text{sqn}(G) - 1} - \text{sqn}(G) - 1.$$ 

**Proof:** Let $\sigma$ be the vertex ordering in a strict $\text{sqn}(G)$-queue layout of $G$. Consider $\sigma$ to be a vertex ordering of $G^d$. For every pair of vertices $v, w \in V(G)$ with $\sigma(v) < \sigma(w)$ and at distance $\ell \leq d$, fix a path $P(vw)$ from $v$ to $w$ in $G$ with exactly $\ell$ edges. Suppose $P(vw) = (x_0, x_1, \ldots, x_\ell)$, where $v = x_0$ and $w = x_\ell$. For each $1 \leq i \leq \ell$, let $\text{dir}(x_{i-1}x_i)$ be ‘$+$’ if $\sigma(x_{i-1}) < \sigma(x_i)$, and ‘$-$’ otherwise. Let $f(vw)$ be the vector

$$f(vw) = \left[\text{queue}(x_{i-1}x_i), \text{dir}(x_{i-1}x_i) : 1 \leq i \leq \ell\right].$$

Consider two edges $vw, pq \in E(G^d)$ with $f(vw) = f(pq)$. Then $|P(vw)| = |P(pq)|$. Let $P(vw) = (x_0, x_1, \ldots, x_\ell)$ and $P(pq) = (y_0, y_1, \ldots, y_\ell)$. We have $\text{dir}(x_0x_1) = \text{dir}(y_0y_1)$ and $\text{queue}(x_0x_1) = \text{queue}(y_0y_1)$. Thus $x_0 \neq y_0$. Without loss of generality $\sigma(x_0) < \sigma(y_0)$. By Equation (2), $\sigma(x_1) < \sigma(y_1)$. In general, $\sigma(x_{i-1}) < \sigma(y_{i-1})$ implies $\sigma(x_i) < \sigma(y_i)$, since $\text{queue}(x_{i-1}x_i) = \text{queue}(y_{i-1}y_i)$ and $\text{dir}(x_{i-1}x_i) = \text{dir}(y_{i-1}y_i)$. By induction, $\sigma(x_i) < \sigma(y_i)$ for all $0 \leq i \leq \ell$. In particular, $\sigma(w) < \sigma(q)$. Thus $vw$ and $pq$ can be in the same strict queue. If we partition the edges of $G^d$ by the value of $f$ we obtain a strict queue layout of $G^d$. The number of queues is

$$\sum_{\ell=1}^{d} (2 \text{sqn}(G))^\ell = \frac{(2 \text{sqn}(G))^{d+1} - 1}{2 \text{sqn}(G) - 1} - 1.$$ 

Observe that for the edges of $G$ we have counted $2 \text{sqn}(G)$ queues. Of course we need only $\text{sqn}(G)$ queues. Thus the total number of queues is as claimed.

\subsection{Powers of Paths and Cycles}

In a vertex ordering $\sigma$ of a graph $G$, the width of an edge $e$ is $\sigma(R_e) - \sigma(L_e)$. The bandwidth of $\sigma$ is the maximum width of an edge of $G$. The bandwidth of $G$, denoted by $\text{bw}(G)$, is the minimum bandwidth of a vertex ordering of $G$. Alternatively, $\text{bw}(G) = \min\{k : G \subseteq P^k\}$ for every $n$-vertex graph $G$. 
Heath and Rosenberg [19] observed that edges whose widths differ by at most one are not nested. Thus $qn(G) \leq \lceil bw(G)/2 \rceil$, as mentioned in Table 1. In a vertex ordering, edges with the same width are not nested or overlapping, and thus form a strict queue. The next lemma follows.

**Lemma 5** Every graph $G$ has strict queue-number $sqn(G) \leq bw(G)$.

We have the following results that give more precise bounds on the queue-number and strict-queue-number of powers of paths and cycles than Theorem 1.

**Lemma 6** The $k$-th power of a path $P_n$ ($n \geq k+1$) has queue-number $qn(P^k_n) = \lceil k/2 \rceil$ and strict queue-number $sqn(P^k_n) = k$.

**Proof:** The bandwidth of a graph $G$ can be thought of as the minimum integer $k$ such that $G \subseteq P^k_n$. Thus the upper bound is nothing more than the result $qn(G) \leq \lceil bw(G)/2 \rceil$ of Heath and Rosenberg [19]. The lower bound follows since $P^k_n$ contains a $(k+1)$-clique, which contains $\lceil k/2 \rceil$ pairwise nested edges in any vertex ordering, all of which must be assigned to distinct queues.

The natural vertex-ordering of $P^k_n$ has no $(k+1)$-edge weak rainbow. Thus $sqn(P^k_n) \leq k$ by Lemma 2. The lower bound follows since $P^k_n$ contains a $(k+1)$-clique, which contains a $k$-edge weak rainbow in any vertex ordering.

A graph is unicyclic if every connected component has at most one cycle. Heath and Rosenberg [19] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle has a 1-queue layout. More generally,

**Lemma 7** The $k$-th power of a cycle $C_n$ ($n \geq 2k$) has queue-number $\frac{k}{2} < qn(C^k_n) \leq k$, and strict queue-number $sqn(C^k_n) = 2k$.

**Proof:** Observe that $\delta(C^k_n) = \Delta(C^k_n) = 2k$ and $\eta(C^k_n) = k$. Thus the claimed lower bounds follow from Lemmata 1 and 3. For the upper bounds, say $C_n = (v_1, v_2, \ldots, v_n)$. By considering the vertex ordering

$$(v_1, v_n; v_2, v_{n-1}; \ldots; v_i, v_{n-i+1}; \ldots; v_{[n/2]}, v_{[n/2]}) \tag{3}$$

we see that $C^k_n \subseteq P^{2k}_n$. The result follows from Lemma 6.

### 4 Graph Products

Let $G_1$ and $G_2$ be graphs. Below we define a number of graph products whose vertex set is $V(G_1) \times V(G_2) = \{(a, v) : a \in V(G_1), v \in V(G_2)\}$.

We classify a potential edge $(a, v)(b, w)$ as follows:

- **$G_1$-edge:** $ab \in E(G_1)$ and $v = w$.
- **$G_2$-edge:** $a = b$ and $vw \in E(G_2)$.
- **direct edge:** $ab \in E(G_1)$ and $vw \in E(G_2)$.
The cartesian product $G_1 \square G_2$ consists of the $G_1$-edges and the $G_2$-edges. The direct product $G_1 \times G_2$ consists of the direct edges. The strong product $G_1 \boxtimes G_2$ consists of the $G_1$-edges, the $G_2$-edges, and the direct edges. That is, $G_1 \boxtimes G_2 = (G_1 \square G_2) \cup (G_1 \times G_2)$. Note that other names abound for these graph products. Our notation is taken from the survey by Klavžar [21]. Assuming isomorphic graphs are equal, each of the above three products are associative, and for instance, $G_1 \square G_2 \square \cdots \square G_d$ is well-defined. Figure 4 illustrates these three types of graph products.

The following lemma is well-known and easily proved.

**Lemma 8** For all graphs $G_1$ and $G_2$, the density satisfies

\( (a) \quad \eta(G_1 \square G_2) = \eta(G_1) + \eta(G_2), \)
\( (b) \quad \eta(G_1 \times G_2) = 2\eta(G_1) \cdot \eta(G_2), \)
\( (c) \quad \eta(G_1 \boxtimes G_2) = 2\eta(G_1) \cdot \eta(G_2) + \eta(G_1) + \eta(G_2). \)

## 5 The Cartesian Product

We have the following bounds on the queue-number of a cartesian product. In a vertex ordering $\sigma$ of a graph product, we abbreviate $\sigma((v, a))$ by $\sigma(v, a)$.

**Theorem 2** For all graphs $G$ and $H$,

\[ \text{qn}(G \square H) \leq \text{sqn}(G) + \text{qn}(H). \]
Furthermore, if for some constant $c$ we have $\text{sqn}(G) \leq c \cdot \eta(G)$ and $\text{qn}(H) \leq c \cdot \eta(H)$, then

\[ \text{qn}(G \square H) \geq \frac{1}{2c} (\text{sqn}(G) + \text{qn}(H)) . \]

**Proof:** First we prove the upper bound. Let $\sigma$ be the vertex ordering in a strict $\text{sqn}(G)$-queue layout of $G$. Let $\pi$ be the vertex ordering in a $\text{qn}(H)$-queue layout of $H$. Let $\phi$ be the vertex ordering of $G \square H$ in which $\phi(v, a) < \phi(w, b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$.

For all edges $e$ of $G$ and for all vertices $a$ of $H$, we have $\phi(L_e, a) < \phi(R_e, a)$. Similarly, for all edges $e$ of $H$ and for all vertices $v$ of $G$, we have $\phi(v, L_e) < \phi(v, R_e)$.

Consider two $G$-edges $(L_e, a)(R_e, a)$ and $(L_f, b)(R_f, b)$ of $G \square H$, for which $e$ and $f$ are in the same strict queue of $G$. By Equation (2), without loss of generality, $\sigma(L_e) < \sigma(L_f)$ and $\sigma(R_e) < \sigma(R_f)$. Thus $\phi(L_e, a) < \phi(L_f, b)$ and $\phi(R_e, a) < \phi(R_f, b)$. Hence for each strict queue in $G$, the corresponding $G$-edges of $G \square H$ form a strict queue in $\phi$.

Consider two $H$-edges $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ of $G \square H$, for which $e$ and $f$ are in the same queue of $H$. By Equation (1), without loss of generality, $\pi(L_e) \leq \pi(L_f)$ and $\pi(R_e) \leq \pi(R_f)$. First suppose that $\sigma(v) \leq \sigma(w)$. Then $\phi(v, L_e) \leq \phi(w, L_f)$ and $\phi(v, R_e) \leq \phi(w, R_f)$. Thus $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ are not nested in $\phi$. Now suppose that $\sigma(w) < \sigma(v)$. Then $\phi(w, L_f) < \phi(w, R_f) < \phi(v, L_e) < \phi(v, R_e)$. Thus $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ are disjoint. Thus for each queue in $H$, the corresponding $H$-edges of $G \square H$ form a queue in $\phi$. Therefore $\phi$ admits a $(\text{sqn}(G) + \text{qn}(H))$-queue layout of $G \square H$.

Now we prove the lower bound. By Lemmata [1] and [8](a), $\text{qn}(G \square H) > \eta(G \square H)/2 = (\eta(G) + \eta(H))/2$. The result follows since $\eta(G) \geq 1/2 \text{sqn}(G)$ and $\eta(H) \geq 1/2 \text{qn}(H)$. □

Theorem 2 has the following immediate corollary.

**Corollary 1** For all graphs $G_1, G_2, \ldots, G_d$,

\[ \text{qn}(G_1 \square G_2 \square \cdots \square G_d) \leq \text{qn}(G_1) + \sum_{i=2}^{d} \text{sqn}(G_i) . \]

□

### 5.1 Grids

A $d$-dimensional grid is a graph $P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d}$, for all $n_i \geq 1$. Heath and Rosenberg [19] determined the queue-number of every 2-dimensional grid.

**Lemma 9 ([19])** Every 2-dimensional grid has queue-number one. □

A generalised $d$-dimensional grid is a graph $G = P_{n_1}^k \square P_{n_2}^k \square \cdots \square P_{n_d}^k$, for all $k \geq 1$ and $n_i \geq k + 1$. Now $P_{n}^k$ has $kn - k(k + 1)/2$ edges. Thus $\eta(P_{n}^k) = k - \frac{k(k+1)}{2n}$. By Lemma 8(a),

\[ \eta(G) = \sum_{i=1}^{d} \left( k - \frac{k(k + 1)}{2n_i} \right) = dk - \frac{k}{2} k(k + 1) \sum_{i=1}^{d} \frac{1}{n_i} . \tag{4} \]

Lemma 9 generalises as follows.
Theorem 3 For all \( d \geq 2 \), the queue-number of a \( d \)-dimensional grid \( G = P_{n_1} \sqcap P_{n_2} \sqcap \cdots \sqcap P_{n_d} \) satisfies:

\[
\frac{d}{4} \leq \frac{1}{2} \left( d - \sum_{i=1}^{d} \frac{1}{n_i} \right) < qn(G) \leq d - 1.
\]

Proof: The lower bound follows from Lemma 1 and Equation (4) with \( k = 1 \).

For the upper bound, we have \( qn(P_{n_1} \sqcap P_{n_2}) = 1 \) by Lemma 9. Obviously \( qn(P_{n_i}) = 1 \) for all \( i \geq 3 \). Thus \( qn(G) \leq d - 1 \) by Corollary 1.

We now give an alternative proof of the upper bound using a different construction. The graph \( G \) can be thought of as having vertex set \( \{(x_1, x_2, \ldots, x_d) : 1 \leq x_i \leq n_i, 1 \leq i \leq d\} \), where two vertices \((x_1, x_2, \ldots, x_d)\) and \((y_1, y_2, \ldots, y_d)\) are adjacent if and only if \(|x_i - y_i| = 1\) for some \( i \), and \( x_j = y_j \) for all \( j \neq i \). We say this edge is in the \( i\)-th dimension. For all \( s \geq 0 \), let \( V_s \) be the set of vertices

\[
V_s = \{(x_1, x_2, \ldots, x_d) : \sum_{i=1}^{d} x_i = s\}.
\]

Order the vertices \((V_0, V_1, \ldots)\), where each \( V_s \) is ordered lexicographically. If \( vw \) is an edge then \( v \) and \( w \) differ in exactly one coordinate, and \( v \in V_s \) and \( w \in V_{s+1} \) for some \( s \). Thus if two edges \( vw \) and \( pq \) are nested then \( v, p \in V_s \) and \( w, q \in V_{s+1} \) for some \( s \). Let \( Q_i \) be the set of edges in the \( i\)-th dimension. Consider two edges \( e \) and \( f \) in \( Q_i \). Say

\[
e = (x_1, x_2, \ldots, x_d)(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_d),
\]

and

\[
f = (y_1, y_2, \ldots, y_d)(y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_d).
\]

Without loss of generality \((x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)\), which implies that

\[
(x_1, \ldots, x_{i-1}, x_i + j, x_{i+1}, \ldots, x_d) \prec (y_1, \ldots, y_{i-1}, y_i + j, y_{i+1}, \ldots, y_d).
\]

Thus \( e \) and \( f \) are not nested, and \( Q_i \) is a queue. Hence we have a \( d \)-queue layout. (At this point we have in fact proved that the lexicographical order admits a \( d \)-queue layout.)

We now prove that \( Q_{d-1} \cup Q_d \) is a queue, and thus we obtain the claimed \((d-1)\)-queue layout. Suppose two edges \( e \in Q_{d-1} \) and \( f \in Q_d \) are nested. Say

\[
e = (x_1, x_2, \ldots, x_d)(x_1, x_2, \ldots, x_{d-1} + 1, x_d),
\]

and

\[
f = (y_1, y_2, \ldots, y_d)(y_1, y_2, \ldots, y_{d-1} + 1, y_d + 1).
\]

Then for some \( s \), both \((x_1, x_2, \ldots, x_d)\) and \((y_1, y_2, \ldots, y_d)\) are in \( V_s \), and both \((x_1, x_2, \ldots, x_{d-1} + 1, x_d)\) and \((y_1, y_2, \ldots, y_{d-1} + 1, y_d + 1)\) are in \( V_{s+1} \).

Case 1. \((x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)\): Let \( j \) be the first dimension for which \( x_j < y_j \). If \( j \leq d - 2 \) then

\[
(x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1} + 1, y_d + 1).
\]
which implies that \( e \) and \( f \) are not nested. Observe that \( j \neq d \) as \((x_1, x_2, \ldots, x_d)\) and \((y_1, y_2, \ldots, y_d)\) differ in at least two coordinates, since \( \sum_i x_i = \sum_i y_i \). Thus \( j = d - 1 \). That is,
\[
x_{d-1} \leq y_{d-1} - 1. \tag{5}
\]
Since \( e \) and \( f \) are nested, we have \((y_1, y_2, \ldots, y_{d-1}, y_d + 1) \prec (x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d)\), which implies that \( y_{d-1} \leq x_{d-1} + 1 \). By Equation (5), \( x_{d-1} = y_{d-1} - 1 \). Since \( x_{d-1} + x_d = y_{d-1} + y_d \), we have \( x_d = y_d + 1 \), which implies that
\[
(y_1, y_2, \ldots, y_{d-1}, y_d + 1) = (x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d).
\]
That is, the right-hand endpoints of \( e \) and \( f \) are the same vertex. Hence \( e \) and \( f \) are not nested.

**Case 2.** \((y_1, y_2, \ldots, y_d) \prec (x_1, x_2, \ldots, x_d)\): By the same argument employed above, the first coordinate for which \((y_1, y_2, \ldots, y_d)\) and \((x_1, x_2, \ldots, x_d)\) differ is \( d - 1 \). That is,
\[
y_{d-1} < x_{d-1}. \tag{6}
\]
Since \( e \) and \( f \) are nested, we have \((x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1}, y_d + 1)\). Thus \( x_{d-1} + 1 < y_{d-1} \), which contradicts Equation (5). Hence \( e \) and \( f \) are not nested.

Therefore \( Q_1, Q_2, \ldots, Q_{d-2}, Q_{d-1} \cup Q_d \) is the desired \((d - 1)\)-queue layout. \(\square\)

More generally we have the following.

**Theorem 4** The queue-number of a generalised \( d \)-dimensional grid \( G = P_{n_1}^k \boxtimes P_{n_2}^k \boxtimes \cdots \boxtimes P_{n_d}^k \) (where \( n_i \geq k + 1 \)) satisfies:
\[
\frac{dk}{4} \leq \frac{dk}{2} - \frac{k(k + 1)}{4} \sum_{i=1}^{d} \frac{1}{n_i} < \text{qn}(G) \leq \lceil (d - \frac{1}{2})k \rceil.
\]

**Proof:** By Lemma 6, \( \text{qn}(P_n^k) = \lceil \frac{k}{2} \rceil \) and \( \text{sqn}(P_n^k) \leq k \). Thus, the upper bound follows from Corollary 1. Thus the lower bound follows from Lemma 1 and Equation (4).

By Theorem 4 with \( k = n - 1 \) we have the following.

**Corollary 2** The queue-number of the \( d \)-dimensional Hamming graph \( G = K_n \boxtimes K_n \boxtimes \cdots \boxtimes K_n \) satisfies:
\[
\frac{d(n - 1)}{4} < \text{qn}(G) \leq \lceil (d - \frac{1}{2})(n - 1) \rceil.
\]

A generalised \( d \)-dimensional toroidal grid is a graph \( C_{n_1}^k \boxtimes C_{n_2}^k \boxtimes \cdots \boxtimes C_{n_d}^k \) for all \( k \geq 1 \) and \( n_i \geq 2k + 1 \).

**Theorem 5** The queue-number of a generalised toroidal grid \( G = C_{n_1}^k \boxtimes C_{n_2}^k \boxtimes \cdots \boxtimes C_{n_d}^k \) (where \( n_i \geq 2k + 1 \)) satisfies:
\[
\frac{kd}{2} < \text{qn}(G) \leq (2d - 1)k.
\]

**Proof:** Since \( \eta(G) = kd \), we have that \( \text{qn}(G) > \frac{kd}{2} \) by Lemma 1. Thus \( \text{qn}(G) \geq \lceil \frac{d}{2} \rceil + 1 \). By Lemma 7, \( \text{qn}(C_{n_i}^k) \leq k \) and \( \text{sqn}(C_{n_i}^k) \leq 2k \). By Corollary 1, \( \text{qn}(G) \leq 2k(d - 1) + k = (2d - 1)k \). \(\square\)
6 Direct and Strong Products

We have the following bounds on the queue-number of direct and strong products.

**Theorem 6** For all graphs $G$ and $H$,

$$\text{qn}(G \times H) \leq 2\text{sqn}(G) \cdot \text{qn}(H) .$$

Furthermore, if $\text{sqn}(G) \leq c \cdot \eta(G)$ and $\text{qn}(H) \leq c \cdot \eta(H)$, then

$$\text{qn}(G \times H) > \frac{1}{c} \text{sqn}(G) \cdot \text{qn}(H) .$$

**Proof:** First we prove the upper bound. Let $k := \text{sqn}(G)$, and let $(\sigma, \{Q_1, Q_2, \ldots, Q_k\})$ be a strict $k$-queue layout of $G$. Let $\ell := \text{qn}(H)$, and let $(\pi, \{P_1, P_2, \ldots, P_\ell\})$ be an $\ell$-queue layout of $H$. For $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let

$$E'_{i,j} := \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(a) < \pi(b)\}$$

$$E''_{i,j} := \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(b) < \pi(a)\}$$

Then $\{E'_{i,j}, E''_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ is a partition of $E(G \times H)$ into $2k\ell$ sets. Let $\phi$ be the vertex ordering of $G \times H$ in which $\phi(v, a) < \phi(w, b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$. We claim that each set $E'_{i,j}$ and $E''_{i,j}$ is a queue in $\phi$.

Suppose that two edges $(v, a)(w, b), (x, c)(y, d) \in E'_{i,j}$ are nested. Without loss of generality, $\phi(v, a) < \phi(x, c) < \phi(y, d) < \phi(w, b)$. If $v \neq x$ and $y \neq w$, then $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ are nested in $\sigma$. If $v = x$ and $y = w$, then $\sigma(v) < \sigma(x) < \sigma(y) = \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in $\sigma$. If $v = x$ and $y \neq w$, then $\sigma(v) = \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in $\sigma$. Each of these outcomes contradict the assumption that $Q_i$ is a strict queue in $\sigma$. Otherwise $v = x$ and $y = w$, in which case $\pi(a) < \pi(c) < \pi(d) < \pi(b)$, and $ab$ and $cd$ are nested in $\pi$. This contradicts the assumption that $P_j$ is a queue in $\pi$. Thus each $E'_{i,j}$ is queue in $\phi$. By symmetry, each $E''_{i,j}$ is also a queue in $\phi$.

Now we prove the lower bound. Lemmata 1 and 8(b) imply that

$$\text{qn}(G \times H) > \eta(G \times H)/2 = \eta(G) \cdot \eta(H) \geq \frac{1}{c} \text{sqn}(G) \cdot \frac{1}{c} \text{qn}(H) .$$

\[\Box\]

**Theorem 7** For all graphs $G$ and $H$,

$$\text{qn}(G \boxtimes H) \leq 2\text{sqn}(G) \cdot \text{qn}(H) + \text{sqn}(G) + \text{qn}(H) .$$

Furthermore, if $\text{sqn}(G) \leq c \cdot \eta(G)$ and $\text{qn}(H) \leq c \cdot \eta(H)$, then

$$\text{qn}(G \boxtimes H) > \frac{1}{c^2} \text{sqn}(G) \cdot \text{qn}(H) + \frac{1}{2c^2} (\text{sqn}(G) + \text{qn}(H)) .$$

**Proof:** To prove the upper bound, observe that the vertex ordering $\phi$ defined in Theorems 2 and 6 is the same. By Theorem 2, $\phi$ admits a $\text{sqn}(G) + \text{qn}(H)$-queue layout of $G \boxtimes H$. By Theorem 6, $\phi$ admits a
2 sqn(G) · qn(H)-queue layout of G × H. Since G ⊠ H = (G □ H) ∪ (G × H), φ admits the claimed queue layout of G ⊠ H.

For the lower bound, Lemmata[1] and[8](c) imply that

\[ qn(G ⊠ H) > \frac{1}{2} \eta(G ⊠ H) = \eta(G) \cdot \eta(H) + \frac{1}{2} (\eta(G) + \eta(H)) \geq \frac{1}{c} sqn(G) \cdot \frac{1}{c} qn(H). \]

\[ \square \]

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References


Queue Layouts of Graph Products and Powers


