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A k-queue layout of a graph G consists of a linear order σ of V(G), and a partition of E(G) into k sets, each of which contains no two edges that are nested in σ. This paper studies queue layouts of graph products and powers.

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1 Introduction

Let G be a graph. (All graphs considered are finite, simple and undirected.) The vertex and edge sets of G are denoted by V(G) and E(G), respectively. The minimum and maximum degree of G are denoted by δ(G) and ∆(G), respectively. The density of G is η(G) := |E(G)|/|V(G)|.

A vertex ordering of G is a bijection σ : V(G) → {1, 2, . . . , |V(G)|}. In a vertex ordering σ of G, let Lσ(e) and Rσ(e) denote the endpoints of each edge e ∈ E(G) such that σ(Lσ(e)) < σ(Rσ(e)). Where the vertex ordering σ is clear from the context, we will abbreviate Lσ(e) and Rσ(e) by Le and Re, respectively. For edges e and f of G with no endpoint in common, there are the following three possible relations with respect to σ, as illustrated in Figure 1

(a) e and f nest if σ(Le) < σ(Lf) < σ(Re) < σ(Rf),
(b) e and f cross if σ(Le) < σ(Lf) < σ(Re) < σ(Rf),
(c) e and f are disjoint if σ(Le) < σ(Re) < σ(Lf) < σ(Rf).

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A queue in $\sigma$ is a set of edges $Q \subseteq E(G)$ such that no two edges in $Q$ are nested. Observe that when traversing $\sigma$ from left to right, the left and right endpoints of the edges in a queue are reached in first-in-first-out order—hence the name ‘queue’. Observe that $Q \subseteq E(G)$ is a queue if and only if for all edges $e, f \in Q$,

$$\sigma(L_e) \leq \sigma(L_f) \quad \text{and} \quad \sigma(R_e) \leq \sigma(R_f),$$

or

$$\sigma(L_f) \leq \sigma(L_e) \quad \text{and} \quad \sigma(R_f) \leq \sigma(R_e).$$

A $k$-queue layout of $G$ is a pair

$$(\sigma, \{Q_1, Q_2, \ldots, Q_k\})$$

where $\sigma$ is a vertex ordering of $G$, and $\{Q_1, Q_2, \ldots, Q_k\}$ is a partition of $E(G)$, such that each $Q_i$ is a queue in $\sigma$. The queue-number of a graph $G$, denoted by $\eta(G)$, is the minimum $k$ such that there is a $k$-queue layout of $G$.

Queue layouts were introduced by Heath et al. [15, 19]. Applications of queue layouts include sorting permutations [12, 20, 22, 24, 27], parallel process scheduling [3], matrix computations [23], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in the literature [7, 9, 10, 13, 25, 26]. Queue layouts of directed graphs [5, 11, 17, 18] and posets [16] have also been investigated.

Table 1 describes the best known upper bounds on the queue-number of various classes of graphs. Planar graphs are an interesting class of graphs for which it is not known whether the queue-number is bounded (see [6, 23]).

This paper studies queue layouts of graph products and graph powers. To prove optimality we use the following lower bound by Heath and Rosenberg [19]. See Pemmaraju [23] and Dujmović and Wood [9] for slightly more exact lower bounds.

**Lemma 1 ([19])** Every graph $G$ has queue-number $\eta(G) > \eta(G)/2$.

This paper is organised as follows. In Section 2 we introduce the concepts of strict queue layout and strict queue-number. Many of the upper bounds on the queue-number that are presented in later sections will be expressed as functions of the strict queue-number. In Section 3 we prove bounds on the queue-number of the power of a graph in terms of the queue-number of the underlying graph. In Section 4 we define the graph products that will be studied in later sections. In Section 5 we study the queue-number of the cartesian product of graphs. Finally in Section 6 we study the queue-number of the direct and strong products of graphs.

† Dujmović and Wood [8] gave a simple proof of this result.
2 Strict Queue Layouts

Let $\sigma$ be a vertex ordering of a graph $G$. We say an edge $e$ is inside a distinct edge $f$, and $e$ and $f$ overlap, if

$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f).$$

A set of edges $Q \subseteq E(G)$ is a strict queue in $\sigma$ if no edge in $Q$ is inside another edge in $Q$. Alternatively, $Q$ is a strict queue in $\sigma$ if

$$\sigma(L_e) < \sigma(L_f) \quad \text{and} \quad \sigma(R_e) < \sigma(R_f),$$

or

$$\sigma(L_f) < \sigma(L_e) \quad \text{and} \quad \sigma(R_f) < \sigma(R_e).$$

Note that Equation (2) is obtained from Equation (1) by replacing “$\leq$” by “$<$”.

Hence a strict queue is a set of edges, no two of which are nested or overlapping, as illustrated in Figure 2. Note that edges forming a ‘butterfly’ can be in a single strict queue.

![Fig. 2: Relationships between pairs of edges with a common endpoint in a vertex ordering.](image-url)

A strict $k$-queue layout of $G$ is a pair $(\sigma, \{Q_1, Q_2, \ldots, Q_k\})$ where $\sigma$ is a vertex ordering of $G$, and $\{Q_1, Q_2, \ldots, Q_k\}$ is a partition of $E(G)$, such that each $Q_i$ is a strict queue in $\sigma$. We sometimes write
queue(e) = i for each edge e ∈ Qi. The strict-queue-number of a graph G, denoted by sqn(G), is the minimum k such that there is a strict k-queue layout of G.

Heath and Rosenberg [19] proved that a fixed vertex ordering of a graph G admits a k-queue layout of G if and only if it has no (k + 1)-edge rainbow, where a rainbow is a set of pairwise nested edges, as illustrated in Figure 3(a). Consider the analogous problem for strict queues: assign the edges of a graph G to the minimum number of strict queues given a fixed vertex ordering σ of G. As illustrated in Figure 3(b), a weak rainbow in σ is a set of edges R such that for every pair of edges e, f ∈ R, e is inside f or f is inside e.

![Fig. 3: (a) rainbow, (b) weak rainbow](image)

**Lemma 2** A vertex ordering of a graph G admits a strict k-queue layout of G if and only if it has no (k + 1)-edge weak rainbow.

**Proof:** A strict k-queue layout has no (k + 1)-edge weak rainbow since each edge of a weak rainbow must be in a distinct strict queue. Conversely, suppose we have a vertex ordering with no (k + 1)-edge weak rainbow. For every edge e ∈ E(G), let queue(e) be one plus the maximum number of edges in a weak rainbow consisting of edges that are inside e. If e is inside f then queue(e) < queue(f). Hence we have a valid strict queue assignment. The number of strict queues is at most k. □

A linear forest is a graph in which every component is a path. The linear arboricity of a graph G, denoted by la(G), is the minimum integer k such that E(G) can be partitioned in k linear forests; see [1, 2, 30, 31]. We have the following lower bounds on sqn(G).

**Lemma 3** The strict queue-number of every graph G satisfies:

(a) sqn(G) ≥ la(G) > η(G),

(b) sqn(G) ≥ la(G) ≥ Δ(G)/2, and

(c) sqn(G) ≥ δ(G).

**Proof:** Say Q is a strict queue in a vertex ordering σ of G. Every 2-edge path (u, v, w) in Q has σ(u) < σ(v) < σ(w) (or σ(u) < σ(w) < σ(v)). Thus no vertex is incident to three edges in Q, and Q induces a linear forest. Hence la(G) ≤ sqn(G).

Since a linear forest in G has at most |V(G)| − 1 edges, la(G) ≥ |E(G)|/(|V(G)| − 1) > η(G). This proves (a). At most two edges incident to each vertex are a linear forest. Thus la(G) ≥ Δ(G)/2. This proves (b).
In every vertex ordering of $G$, every edge incident to the first vertex is in a distinct strict queue. Hence $sqn(G) \geq \delta(G)$. This proves (c).

Obviously a proper edge $(\Delta(G) + 1)$-colouring \cite{28} can be combined with a $sqn(G)$-queue layout to obtain a strict queue layout.

**Lemma 4** Every graph $G$ has strict queue-number $sqn(G) \leq (\Delta(G) + 1) \cdot sqn(G)$.

### 3 Graph Powers

Let $G$ be a graph, and let $d \in \mathbb{Z}^+$. The $d$-th power of $G$, denoted by $G^d$, is the graph with vertex set $V(G^d) = V(G)$, where $vw \in E(G^d)$ if and only if the distance between $v$ and $w$ in $G$ is at most $d$. The following general result is similar to a theorem of Dujmović and Wood \cite{10}.

**Theorem 1** For every graph $G$ and $d \in \mathbb{Z}^+$,

$$sqn(G^d) \leq \left(\frac{2sqn(G)^{d+1}}{2sqn(G) - 1}\right) - sqn(G) - 1.$$

**Proof:** Let $\sigma$ be the vertex ordering in a strict $sqn(G)$-queue layout of $G$. Consider $\sigma$ to be a vertex ordering of $G^d$. For every pair of vertices $v, w \in V(G)$ with $\sigma(v) < \sigma(w)$ and at distance $\ell \leq d$, fix a path $P(vw)$ from $v$ to $w$ in $G$ with exactly $\ell$ edges. Suppose $P(vw) = (x_0, x_1, \ldots, x_\ell)$, where $v = x_0$ and $w = x_\ell$. For each $1 \leq i \leq \ell$, let $dir(x_{i-1}x_i)$ be ‘+’ if $\sigma(x_{i-1}) < \sigma(x_i)$, and ‘−’ otherwise. Let $f(vw)$ be the vector

$$f(vw) = \left[(\text{queue}(x_{i-1}x_i), \text{dir}(x_{i-1}x_i)) : 1 \leq i \leq \ell\right].$$

Consider two edges $vw, pq \in E(G^d)$ with $f(vw) = f(pq)$. Then $|P(vw)| = |P(pq)|$. Let $P(vw) = (x_0, x_1, \ldots, x_\ell)$ and $P(pq) = (y_0, y_1, \ldots, y_\ell)$. We have $dir(x_0x_1) = dir(y_0y_1)$ and queue$(x_0x_1) = queue(y_0y_1)$. Thus $x_0 \neq y_0$. Without loss of generality $\sigma(x_0) < \sigma(y_0)$. By Equation (2), $\sigma(x_1) < \sigma(y_1)$. In general, $\sigma(x_{i-1}) < \sigma(y_{i-1})$ implies $\sigma(x_i) < \sigma(y_i)$, since queue$(x_{i-1}x_i) = queue(y_{i-1}y_i)$ and $dir(x_{i-1}x_i) = dir(y_{i-1}y_i)$. By induction, $\sigma(x_i) < \sigma(y_i)$ for all $0 \leq i \leq \ell$. In particular, $\sigma(w) < \sigma(q)$. Thus $vw$ and $pq$ can be in the same strict queue. If we partition the edges of $G^d$ by the value of $f$ we obtain a strict queue layout of $G^d$. The number of queues is

$$\sum_{\ell=1}^{d} (2sqn(G))^\ell = \left(\frac{2sqn(G)^{d+1}}{2sqn(G) - 1}\right) - sqn(G) - 1.$$

Observe that for the edges of $G$ we have counted $2sqn(G)$ queues. Of course we need only $sqn(G)$ queues. Thus the total number of queues is as claimed.

### 3.1 Powers of Paths and Cycles

In a vertex ordering $\sigma$ of a graph $G$, the width of an edge $e$ is $\sigma(R_e) - \sigma(L_e)$. The bandwidth of $G$ is the maximum width of an edge of $G$. The bandwidth of $G$, denoted by $bw(G)$, is the minimum bandwidth of a vertex ordering of $G$. Alternatively, $bw(G) = \min\{k : G \subseteq P_n^k\}$ for every $n$-vertex graph $G$. 
Heath and Rosenberg [19] observed that edges whose widths differ by at most one are not nested. Thus \( qn(G) \leq \lceil bw(G)/2 \rceil \), as mentioned in Table 1. In a vertex ordering, edges with the same width are not nested or overlapping, and thus form a strict queue. The next lemma follows.

**Lemma 5** Every graph \( G \) has strict queue-number \( sqn(G) \leq bw(G) \). \( \square \)

We have the following results that give more precise bounds on the queue-number and strict-queue-number of powers of paths and cycles than Theorem 1.

**Lemma 6** The \( k \)-th power of a path \( P_n \) \((n \geq k + 1)\) has queue-number \( qn(P_n^k) = \lceil k/2 \rceil \) and strict queue-number \( sqn(P_n^k) = k \).

**Proof:** The bandwidth of a graph \( G \) can be thought of as the minimum integer \( k \) such that \( G \subseteq P_n^k \). Thus the upper bound is nothing more than the result \( qn(G) \leq \lceil bw(G)/2 \rceil \) of Heath and Rosenberg [19]. The lower bound follows since \( P_n^k \) contains a \((k + 1)\)-clique, which contains \( \lceil k/2 \rceil \) pairwise nested edges in any vertex ordering, all of which must be assigned to distinct queues.

The natural vertex-ordering of \( P_n^k \) has no \((k + 1)\)-edge weak rainbow. Thus \( sqn(P_n^k) \leq k \) by Lemma 2. The lower bound follows since \( P_n^k \) contains a \((k + 1)\)-clique, which contains a \( k\)-edge weak rainbow in any vertex ordering. \( \square \)

A graph is unicyclic if every connected component has at most one cycle. Heath and Rosenberg [19] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle has a 1-queue layout. More generally,

**Lemma 7** The \( k \)-th power of a cycle \( C_n \) \((n \geq 2k)\) has queue-number \( \frac{k}{2} < qn(C_n^k) \leq k \), and strict queue-number \( sqn(C_n^k) = 2k \).

**Proof:** Observe that \( \delta(C_n^k) = \Delta(C_n^k) = 2k \) and \( \eta(C_n^k) = k \). Thus the claimed lower bounds follow from Lemmata 1 and 3. For the upper bounds, say \( C_n = (v_1, v_2, \ldots, v_n) \). By considering the vertex ordering \((v_1, v_n; v_2, v_{n-1}; \ldots; v_i, v_{n-i+1}; \ldots; v_{\lceil n/2 \rceil}, v_{\lceil n/2 \rceil})\), we see that \( C_n^k \subseteq P_n^{2k} \). The result follows from Lemma 6. \( \square \)

### 4 Graph Products

Let \( G_1 \) and \( G_2 \) be graphs. Below we define a number of graph products whose vertex set is \( V(G_1) \times V(G_2) = \{(a, v) : a \in V(G_1), v \in V(G_2)\} \).

We classify a potential edge \((a, v)(b, w)\) as follows:

- **\( G_1\)-edge:** \( ab \in E(G_1) \) and \( v = w \).
- **\( G_2\)-edge:** \( a = b \) and \( vw \in E(G_2) \).
- **direct edge:** \( ab \in E(G_1) \) and \( vw \in E(G_2) \).
The cartesian product $G_1 \square G_2$ consists of the $G_1$-edges and the $G_2$-edges. The direct product $G_1 \times G_2$ consists of the direct edges. The strong product $G_1 \boxtimes G_2$ consists of the $G_1$-edges, the $G_2$-edges, and the direct edges. That is, $G_1 \boxtimes G_2 = (G_1 \square G_2) \cup (G_1 \times G_2)$. Note that other names abound for these graph products. Our notation is taken from the survey by Klavžar [21]. Assuming isomorphic graphs are equal, each of the above three products are associative, and for instance, $G_1 \square G_2 \square \cdots \square G_d$ is well-defined. Figure 4 illustrates these three types of graphs products.

![Graph Products](image)

**Fig. 4:** Examples of graph products: (a) cartesian, (b) direct, (c) strong.

The following lemma is well-known and easily proved.

**Lemma 8** For all graphs $G_1$ and $G_2$, the density satisfies

(a) $\eta(G_1 \square G_2) = \eta(G_1) + \eta(G_2)$,

(b) $\eta(G_1 \times G_2) = 2\eta(G_1) \cdot \eta(G_2)$,

(c) $\eta(G_1 \boxtimes G_2) = 2\eta(G_1) \cdot \eta(G_2) + \eta(G_1) + \eta(G_2)$.

5 The Cartesian Product

We have the following bounds on the queue-number of a cartesian product. In a vertex ordering $\sigma$ of a graph product, we abbreviate $\sigma((v, a))$ by $\sigma(v, a)$.

**Theorem 2** For all graphs $G$ and $H$,

$$\text{qn}(G \square H) \leq \text{sqn}(G) + \text{qn}(H).$$
Furthermore, if for some constant \( c \) we have \( \text{sqn}(G) \leq c \cdot \eta(G) \) and \( \text{qn}(H) \leq c \cdot \eta(H) \), then
\[
\text{qn}(G \square H) \geq \frac{1}{2c} \left( \text{sqn}(G) + \text{qn}(H) \right).
\]

**Proof:** First we prove the upper bound. Let \( \sigma \) be the vertex ordering in a strict \( \text{sqn}(G) \)-queue layout of \( G \). Let \( \pi \) be the vertex ordering in a \( \text{qn}(H) \)-queue layout of \( H \). Let \( \phi \) be the vertex ordering of \( G \square H \) in which \( \phi(v, a) < \phi(w, b) \) if and only if \( \sigma(v) < \sigma(w) \), or \( v = w \) and \( \pi(a) < \pi(b) \).

For all edges \( e \) of \( G \) and for all vertices \( a \) of \( H \), we have \( \phi(L_e, a) < \phi(R_e, a) \). Similarly, for all edges \( e \) of \( H \) and for all vertices \( v \) of \( G \), we have \( \phi(v, L_e) < \phi(v, R_e) \).

Consider two \( G \)-edges \((L_e, a)(R_e, a)\) and \((L_f, b)(R_f, b)\) of \( G \square H \), for which \( e \) and \( f \) are in the same strict queue of \( G \). By Equation (2), without loss of generality, \( \sigma(L_e) < \sigma(L_f) \) and \( \sigma(R_e) < \sigma(R_f) \). Thus \( \phi(L_e, a) < \phi(L_f, b) \) and \( \phi(R_e, a) < \phi(R_f, b) \). Hence for each strict queue in \( G \), the corresponding \( G \)-edges of \( G \square H \) form a strict queue in \( \phi \).

Consider two \( H \)-edges \((v, L_e)(v, R_e)\) and \((w, L_f)(w, R_f)\) of \( G \square H \), for which \( e \) and \( f \) are in the same queue of \( H \). By Equation (1), without loss of generality, \( \pi(L_e) \leq \pi(L_f) \) and \( \pi(R_e) \leq \pi(R_f) \). First suppose that \( \sigma(v) \leq \sigma(w) \). Then \( \phi(v, L_e) \leq \phi(w, L_f) \) and \( \phi(v, R_e) \leq \phi(w, R_f) \). Thus \( \phi(v, L_e)(v, R_e) \) and \( (w, L_f)(w, R_f) \) are not nested in \( \phi \). Now suppose that \( \sigma(w) < \sigma(v) \). Then \( \phi(w, L_f) < \phi(w, R_f) \). Thus \( \phi(v, L_e)(v, R_e) \) and \( (w, L_f)(w, R_f) \) are disjoint. Thus for each queue in \( H \), the corresponding \( H \)-edges of \( G \square H \) form a queue in \( \phi \). Therefore \( \phi \) admits a \( \text{(sln}(G) + \text{qn}(H)) \)-queue layout of \( G \square H \).

Now we prove the lower bound. By Lemmata [1] and [8(a)], \( \text{qn}(G \square H) > \eta(G \square H)/2 = (\eta(G) + \eta(H))/2 \). The result follows since \( \eta(G) \geq \frac{1}{c} \text{sln}(G) \) and \( \eta(H) \geq \frac{1}{c} \text{qn}(H) \).

Theorem 2 has the following immediate corollary.

**Corollary 1** For all graphs \( G_1, G_2, \ldots, G_d \),
\[
\text{qn}(G_1 \square G_2 \square \cdots \square G_d) \leq \text{qn}(G_1) + \sum_{i=2}^{d} \text{sln}(G_i).
\]

\[ \square \]

### 5.1 Grids

A \( d \)-dimensional grid is a graph \( P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d} \), for all \( n_i \geq 1 \). Heath and Rosenberg [19] determined the queue-number of every 2-dimensional grid.

**Lemma 9 ([19])** Every 2-dimensional grid has queue-number one.

A generalised \( d \)-dimensional grid is a graph \( G = P_{n_1}^k \square P_{n_2}^k \square \cdots \square P_{n_d}^k \), for all \( k \geq 1 \) and \( n_i \geq k + 1 \). Now \( P_{n_i}^k \) has \( kn - k(k + 1)/2 \) edges. Thus \( \eta(P_{n_i}^k) = k - \frac{k(k + 1)}{2n_i} \). By Lemma [8(a)],
\[
\eta(G) = \sum_{i=1}^{d} \left( k - \frac{k(k + 1)}{2n_i} \right) = dk - \frac{1}{2} k(k + 1) \sum_{i=1}^{d} \frac{1}{n_i}.
\]

Lemma [9] generalises as follows.

\[ \square \]
Theorem 3 For all $d \geq 2$, the queue-number of a $d$-dimensional grid $G = P_{n_1} \sqcap P_{n_2} \sqcap \cdots \sqcap P_{n_d}$ satisfies:
\[
\frac{d}{4} \leq \frac{1}{2} \left( d - \sum_{i=1}^{d} \frac{1}{n_i} \right) < \text{qn}(G) \leq d - 1.
\]

Proof: The lower bound follows from Lemma 1 and Equation 4 with $k = 1$.

For the upper bound, we have $\text{qn}(G) = 1$ by Lemma 9. Obviously $\text{sqn}(G) = 1$ for all $i \geq 3$. Thus $\text{qn}(G) \leq d - 1$ by Corollary 1.

We now give an alternative proof of the upper bound using a different construction. The graph $G$ can be thought of as having vertex set $(x_1, x_2, \ldots, x_d) : 1 \leq x_i \leq n_i, 1 \leq i \leq d)$, where two vertices $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are adjacent if and only if $|x_i - y_i| = 1$ for some $i$, and $x_j = y_j$ for all $j \neq i$. We say this edge is in the $i$-th dimension. For all $s \geq 0$, let $V_s$ be the set of vertices
\[
V_s = \{(x_1, x_2, \ldots, x_d) : \sum_{i=1}^{d} x_i = s\}.
\]

Order the vertices $(V_0, V_1, \ldots)$, where each $V_s$ is ordered lexicographically. If $vw$ is an edge then $v$ and $w$ differ in exactly one coordinate, and $v \in V_s$ and $w \in V_{s+1}$ for some $s$. Thus if two edges $vw$ and $pq$ are nested then $v, p \in V_s$ and $w, q \in V_{s+1}$ for some $s$. Let $Q_i$ be the set of edges in the $i$-th dimension. Consider two edges $e$ and $f$ in $Q_i$. Say
\[
eq (x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_d)\)
\]
and
\[
f = (y_1, y_2, \ldots, y_d)(y_1, y_2, \ldots, y_i, y_{i+1}, \ldots, y_d)\).
\]

Without loss of generality $(x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)$, which implies that
\[
(x_1, \ldots, x_{i-1}, x_i + j, x_{i+1}, \ldots, x_d) \prec (y_1, \ldots, y_i - 1, y_i + j, y_{i+1}, \ldots, y_d)\).
\]

Thus $e$ and $f$ are not nested, and $Q_i$ is a queue. Hence we have a $d$-queue layout. (At this point we have in fact proved that the lexicographical order admits a $d$-queue layout.)

We now prove that $Q_{d-1} \cup Q_d$ is a queue, and thus we obtain the claimed $(d-1)$-queue layout. Suppose two edges $e \in Q_{d-1}$ and $f \in Q_d$ are nested. Say
\[
eq (x_1, x_2, \ldots, x_d)(x_1, x_2, \ldots, x_{d-1} + 1, x_d)\),
\]
and
\[
f = (y_1, y_2, \ldots, y_d)(y_1, y_2, \ldots, y_{d-1}, y_d + 1)\).
\]

Then for some $s$, both $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are in $V_s$, and both $(x_1, x_2, \ldots, x_{d-1} + 1, x_d)$ and $(y_1, y_2, \ldots, y_{d-1}, y_d + 1)$ are in $V_{s+1}$.

Case 1. $(x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)$: Let $j$ be the first dimension for which $x_j < y_j$. If $j \leq d - 2$ then
\[
(x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1}, y_d + 1)\),
\]
which implies that $e$ and $f$ are not nested. Observe that $j \neq d$ as $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ differ in at least two coordinates, since $\sum_i x_i = \sum_i y_i$. Thus $j = d - 1$. That is,

$$x_{d-1} \leq y_{d-1} - 1 .$$

(5)

Since $e$ and $f$ are nested, we have $(y_1, y_2, \ldots, y_{d-1}, y_d + 1) \prec (x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d)$, which implies that $y_{d-1} \leq x_{d-1} + 1$. By Equation (5), $x_{d-1} = y_{d-1} - 1$. Since $x_{d-1} + x_d = y_{d-1} + y_d$, we have $x_d = y_d + 1$, which implies that

$$(y_1, y_2, \ldots, y_{d-1}, y_d + 1) = (x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) .$$

That is, the right-hand endpoints of $e$ and $f$ are the same vertex. Hence $e$ and $f$ are not nested.

**Case 2.** $(y_1, y_2, \ldots, y_d) \prec (x_1, x_2, \ldots, x_d)$: By the same argument employed above, the first coordinate for which $(y_1, y_2, \ldots, y_d)$ and $(x_1, x_2, \ldots, x_d)$ differ is $d - 1$. That is,

$$y_{d-1} < x_{d-1} .$$

(6)

Since $e$ and $f$ are nested, we have $(x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1}, y_d + 1)$. Thus $x_{d-1} + 1 < y_{d-1}$, which contradicts Equation (5). Hence $e$ and $f$ are not nested.

Therefore $Q_1, Q_2, \ldots, Q_{d-2}, Q_{d-1} \cup Q_d$ is the desired $(d-1)$-queue layout. □

More generally we have the following.

**Theorem 4** The queue-number of a generalised $d$-dimensional grid $G = P_{n_1}^k \square P_{n_2}^k \square \cdots \square P_{n_d}^k$ (where $n_i \geq k + 1$) satisfies:

$$\frac{dk}{4} \leq \frac{d}{2} - \frac{k(k+1)}{4} \sum_{i=1}^d \frac{1}{n_i} < qn(G) \leq \lceil (d - \frac{1}{2})k \rceil .$$

**Proof:** By Lemma 6, $qn(P_{n_1}^k) = \lceil \frac{k}{2} \rceil$ and $sqn(P_{n_1}^k) \leq k$. Thus, the upper bound follows from Corollary 1. Thus the lower bound follows from Lemma 1 and Equation (4). □

By Theorem 4 with $k = n - 1$ we have the following.

**Corollary 2** The queue-number of the $d$-dimensional Hamming graph $G = K_n \square K_n \square \cdots \square K_n$ satisfies:

$$\frac{d(n-1)}{4} < qn(G) \leq \lceil (d - \frac{1}{2})(n-1) \rceil .$$

A generalised $d$-dimensional toroidal grid is a graph $C_{n_1}^k \square C_{n_2}^k \square \cdots \square C_{n_d}^k$ for all $k \geq 1$ and $n_i \geq 2k + 1$.

**Theorem 5** The queue-number of a generalised toroidal grid $G = C_{n_1}^k \square C_{n_2}^k \square \cdots \square C_{n_d}^k$ (where $n_i \geq 2k + 1$) satisfies:

$$\frac{kd}{2} < qn(G) \leq (2d - 1)k .$$

**Proof:** Since $\eta(G) = kd$, we have that $qn(G) > \frac{kd}{2}$ by Lemma 1. Thus $qn(G) \geq \lceil \frac{k}{2} \rceil + 1$. By Lemma 7, $qn(C_{n_1}^k) \leq k$ and $sqn(C_{n_1}^k) \leq 2k$. By Corollary 1, $qn(G) \leq 2k(d-1) + k = (2d-1)k$ □
Queue Layouts of Graph Products and Powers

6 Direct and Strong Products

We have the following bounds on the queue-number of direct and strong products.

**Theorem 6** For all graphs $G$ and $H,$

$$q_n(G \times H) \leq 2q_n(G) \cdot q_n(H).$$

Furthermore, if $q_n(G) \leq c \cdot \eta(G)$ and $q_n(H) \leq c \cdot \eta(H),$ then

$$q_n(G \times H) > \frac{1}{c} q_n(G) \cdot q_n(H).$$

**Proof:** First we prove the upper bound. Let $k := q_n(G),$ and let $(\sigma, \{Q_1, Q_2, \ldots, Q_k\})$ be a strict $k$-queue layout of $G.$ Let $\ell := q_n(H),$ and let $(\pi, \{P_1, P_2, \ldots, P_\ell\})$ be an $\ell$-queue layout of $H.$ For $1 \leq i \leq k$ and $1 \leq j \leq \ell,$ let

$$E'_{i,j} := \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(a) < \pi(b)\}$$

$$E''_{i,j} := \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(b) < \pi(a)\}$$

Then $\{E'_{i,j}, E''_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ is a partition of $E(G \times H)$ into $2k\ell$ sets. Let $\phi$ be the vertex ordering of $G \times H$ in which $\phi(v, a) < \phi(w, b)$ if and only if $\sigma(v) < \sigma(w),$ or $v = w$ and $\pi(a) < \pi(b).$

We claim that each set $E'_{i,j}$ and $E''_{i,j}$ is a queue in $\phi.$

Suppose that two edges $(v, a)(w, b), (x, c)(y, d) \in E'_{i,j}$ are nested. Without loss of generality, $\phi(v, a) < \phi(x, c) < \phi(y, d) < \phi(w, b).$ If $v \neq x$ and $y \neq w,$ then $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w),$ and the edges $vw, xy \in Q_i$ are nested in $\sigma.$ If $v = x$ and $y = w,$ then $\sigma(v) = \sigma(x) = \sigma(y) = \sigma(w),$ and the edges $vw, xy \in Q_i$ overlap in $\sigma.$ If $v = x$ and $y \neq w,$ then $\sigma(v) = \sigma(x) < \sigma(y) < \sigma(w),$ and the edges $vw, xy \in Q_i$ overlap in $\sigma.$ Each of these outcomes contradict the assumption that $Q_i$ is a strict queue in $\sigma.$ Otherwise $v = x$ and $y = w,$ in which case $\pi(a) < \pi(c) < \pi(d) < \pi(b),$ and $ab$ and $cd$ are nested in $\pi.$ This contradicts the assumption that $P_\ell$ is a queue in $\pi.$ Thus each $E'_{i,j}$ is queue in $\phi.$ By symmetry, each $E''_{i,j}$ is also a queue in $\phi.$

Now we prove the lower bound. Lemmata [1] and [3(b)] imply that

$$q_n(G \times H) > \eta(G \times H)/2 = \eta(G) \cdot \eta(H) \geq \frac{1}{c} q_n(G) \cdot \frac{1}{c} q_n(H).$$

\[\square\]

**Theorem 7** For all graphs $G$ and $H,$

$$q_n(G \boxtimes H) \leq 2q_n(G) \cdot q_n(H) + q_n(G) + q_n(H).$$

Furthermore, if $q_n(G) \leq c \cdot \eta(G)$ and $q_n(H) \leq c \cdot \eta(H),$ then

$$q_n(G \boxtimes H) > \frac{1}{c^2} q_n(G) \cdot q_n(H) + \frac{1}{c^2} (q_n(G) + q_n(H)).$$

**Proof:** To prove the upper bound, observe that the vertex ordering $\phi$ defined in Theorems [2] and [6] is the same. By Theorem [2] $\phi$ admits a $q_n(G) + q_n(H)$-queue layout of $G \boxtimes H.$ By Theorem [6] $\phi$ admits a
\[ \text{sqn}(G) \cdot \text{qn}(H) \text{-queue layout of } G \times H. \] Since \[ G \boxtimes H = (G \square H) \cup (G \times H), \] \( \phi \) admits the claimed queue layout of \( G \boxtimes H. \)

For the lower bound, Lemmata 1 and 8(c) imply that

\[ \text{qn}(G \boxtimes H) > \frac{1}{2} \eta(G \boxtimes H) = \eta(G) \cdot \eta(H) + \frac{1}{2} (\eta(G) + \eta(H)) \geq \frac{1}{c} \text{sqn}(G) \cdot \frac{1}{c} \text{qn}(H). \]

\[ \square \]

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References


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