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Asynchronous Cellular Automata and Brownian Motion

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This paper deals with some very simple interacting particle systems, *elementary cellular automata*, in the fully asynchronous dynamics: at each time step, a cell is randomly picked, and updated. When the initial configuration is simple, we describe the asymptotic behavior of the random walks performed by the borders of the black/white regions. Following a classification introduced by Fatès *et al.*, we show that four kinds of asymptotic behavior arise, two of them being related to Brownian motion.

Keywords: cellular automata, asynchronism, random processes, coalescent random walks

Contents

1 Introduction

1.1 Elementary Cellular Automata

Cellular automata are dynamical systems widely used the two last decades in order to modelize phenomena arising in game theory, economy, theoretical physics, biology, or theoretical computer science (complexity, computation). It consists of a (finite or countable) set of cells, the state of each cell at time k being a function of the state of its neighbours at time $k-1$. The set of possible states is finite, and, as we see, time is discrete. Cellular automata were introduced by von Neumann [vN66] in order to emulate *self-replication* in biology.

This paper deals more specifically with *elementary cellular automata* (ECA), introduced by Wolfram [Wol84], that is two-state automata (0/1 or white/black) with a finite and cyclic set of cells. Let us recall a few definitions.

Definition 1 *A (deterministic)* elementary cellular automaton *(ECA) is a triplet* $(n, x(0), \delta)$*, in which* n *stands for the number of cells,* $x(0) \in \{0,1\}^n$ *denotes the* initial configuration *and* $\delta : \{0,1\}^3 \to \{0,1\}$ *is the* local transition function*, or* local rule.

The first studies focused on the synchronous dynamic of $(n, x(0), \delta)$, i.e. the evolution of the configuration under iterations of the function A^δ on $x(0)$:

$$\begin{aligned} A^\delta : \qquad \{0,1\}^n &\to \{0,1\}^n \\ (x_1, \dots, x_n) &\mapsto (x'_1, \dots, x'_n) \end{aligned}$$

in which, for $i \in \{1, 2, \dots, n\}$, $x'_i = \delta(x_{i-1}, x_i, x_{i+1})$, that is, the n cells are updated simultaneously. It must be understood with the convention $x_{n+1} = x_1, x_0 = x_n$, so that the set of configuration is cyclic.

Thus, $(x(k); k = 0, 1, \dots)$ is a sequence of words of length n on the alphabet $\{0,1\}$. Alternatively, we shall consider configurations as doubly infinite periodic sequences $(x_n)_{n\in\mathbb{Z}}$, with period n. We will focus here only on *double-quiescent* ECA, i.e. ECA for which $\delta(0,0,0) = 0$ and $\delta(1,1,1) = 1$. This terminology has been introduced in [FMST05].

We are intersested here in the *asynchronous* dynamic : when the n cells are not updated simultaneously, but randomly picked and sequentially updated.

Definition 2 *The* fully asynchronous dynamic *of the automaton* δ *is the random process on* $\{0,1\}^n$ *defined by :*

$$\begin{aligned} &X_0 = x(0), \\ &X_k = A^\delta_{i_k} X_{k-1}, \textit{ for each } k \geq 1, \end{aligned}$$

where $(i_k)_{k\geq 1}$ *is a sequence of i.i.d. random variables, uniform in* $\{1,\ldots,n\}$ *and* A_j^δ *is the function defined by*

$$\begin{aligned} A_j^\delta : \quad & \{0,1\}^n \to \{0,1\}^n \\ & (x_1,\ldots,x_n) \mapsto (x'_1,\ldots,x'_n) \end{aligned}$$

in which $x'_j = \delta(x_{j-1}, x_j, x_{j+1})$*, while, if* $i \neq j$*,* $x'_i = x_i$.

Influence of asynchronism in ECA's has been studied for instance in [IB84, SdR99], with motivations in physics, and in biology. It turns out that asynchronism actually changes drastically the asymptotic behavior of cellular automata (see Figure 1.2 below for a simulation).

1.2 Worst expected convergence time

In the asynchronous case, for the 64 double-quiescent ECA's, the question of worst expected convergence time has been exhaustively investigated by Fatès *et al.* [FMST05], with surprising results, that we recall below. A local transition function δ is given by its eight transitions. A transition is said to be *active* if it changes the cell it is applied to. Of course δ is completely determined by its active transitions. Active transitions are labelled with a letter, as follows (a notation that proves to be quite handy when classifying ECA's).

A	B	C	D	E	F	G	H
000	001	100	101	010	011	110	111
1	1	1	1	0	0	0	0

For instance, the only cells possibly changed by the automaton $\delta = \mathrm{DG}$ are precisely the white cells surrounded by two black cells, and the black cells with a black cell on the left side and a white cell on the right side. Double-quiescent ECA are those for which neither A nor H appear. The automaton *Identity* is denoted $\varnothing$. For an automaton δ, $\mathfrak{F}_\delta$ denotes the set of fixed points of δ (of course, when δ is double-quiescent, $\{0^n, 1^n\} \subset \mathfrak{F}_\delta$).

Definition 3 *Given a fully asynchronous automaton* $(n, x(0), \delta)$*,* $T_n = T_n(\delta, x(0))$ *denotes the random variable*

$$T_n = \inf\{k \geq 0; X_k \in \mathfrak{F}_\delta\},$$

in which we use the convention $\inf\{\emptyset\} = +\infty$*. The* Worst Expected Convergence Time WECT_δ *is the real number*

$$\mathrm{WECT}_\delta = \max_{x(0)\in\{0,1\}^n} \mathbb{E}[T_n(\delta, x(0))].$$

Fatès *et al.* [FMST05] classify the 64 double-quiescent ECA's in five families, according to the asymptotic behavior of WECT_δ, when n is large. Let $\Theta(g_n)$ denote the set of sequences $f = (f_n)_{n\geq 1}$ that satisfy $c_1 \leq f_n/g_n \leq c_2$, for suitably chosen constants $c_1, c_2 \in (0, +\infty)$ that depend on f but not on n.

Theorem 1 (Fatès, Morvan, Schabanel & Thierry [FMST05]) *For* $\delta \neq \varnothing$*, either* WECT_δ *is infinite or it belongs to one of these four classes :* $\Theta(n\log n), \Theta(n^2), \Theta(n^3), \Theta(n\,2^n)$*. The corresponding families of automata are called respectively* Divergent*,* Coupon Collector*,* Quadratic*,* Cubic*, and* Exponential*.*

Class	δ	#
Identity	$\varnothing$	1
Coupon	E	2
	DE	1
Quadratic	B	4
	FG	2
	BDE	4
	BCDE	2
	BE	4
	EF	4
	BCE	2
	EFG	2
	BCDEF	4
	BEFG	4

Class	δ	#
Cubic	BDEF	2
	BDEG	2
	BCDEFG	1
	BEF	4
	BEG	4
	BCEFG	2
Exponential	BCEF	4
Divergent	BF	2
	BG	2
	BCF	4
	BCFG	1

Fig. 1: A classification of the 64 ECA's, according to the asymptotic behavior of WECT_δ.

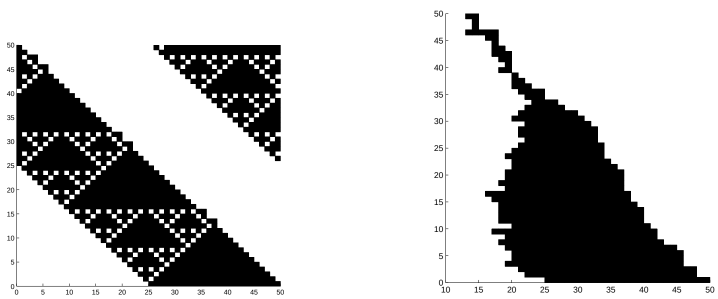

Fig. 2: Simulations of the synchronous and asynchronous dynamics for the rule BDFG, for $n = 50$ and $x(0) = 0^{25}1^{25}$.

This classification is remarkably similar to that introduced by Wolfram in a completely different context. For reasons of symmetry between black and white, or between left and right, the 64 cases reduce actually to 25. The main results of [FMST05] are summarized in Figure 1 (the third column gives the number of symmetries).

Without loss of generality, we assume in the sequel that n is even. Original motivation of this work was to refine the methods leading to Theorem 1. When the initial configuration contains only one black region, say $x(0) = 0^{n/2}1^{n/2}$, the whole sequence $(x(k))$ in the asynchronous dynamic contains only one black region (see Fig. 1.2), unless it has reached the fixed point 0^n. We assume from now on that $x(0) = 0^{n/2}1^{n/2}$. In a longer paper, we shall discuss the asymptotic behavior of the borders of black regions for an initial state with several black regions. We set $(R_0, L_0) = (0, n/2)$. For $k < T_n$, we define (R_k, L_k) by induction as the unique element of $\mathbb{Z}^2$ such that $x_{L_k}(k) = 0, x_{L_k+1}(k) = 1$, and $|L_k - L_{k-1}| \leq 1$, resp. $x_{R_k}(k) = 1, x_{L_k+1}(k) = 0$ and $|R_k - R_{k-1}| \leq 1$. This way, we can track if the black zone shifts, makes several revolutions, for instance.

Simulations suggest that there exists a continuous limit for the bi-dimensional process $(R_k, L_k)_{k\geq 0}$, after a suitable renormalization. Precisely, given some automaton, we exhibit some continuous process with values in $\mathbb{R}^2$ such that the following weak convergence holds (in a sense to be defined in the next Section):

$$n^{-1}\ \left(L_{\lfloor t\,\mathbb{E}[T_n]\rfloor}, R_{\lfloor t\,\mathbb{E}[T_n]\rfloor}\right)_{t\geq 0} \Rightarrow \left(X_t^{(1)}, X_t^{(2)}\right)_{t\geq 0}. \tag{1}$$

From this convergence of stochastic processes, we hope to deduce quantitative information on statistics of automata, e.g. on the r.v. $T_n/\mathbf{E}[T_n]$.

1.3 Convergence in $\mathcal{D}_p(I)$

If I is an interval $[0,T]$, with $0 \leq T \leq +\infty$, let $\mathcal{D}_p(I)$ be the set of *cadlag*[(i)] functions: $I \rightarrow \mathbb{R}^p$. We adress the convergence of random variables in $\mathcal{D}_p(I)$, endowed with the Skorohod topology. Recall that, when the limit is a continuous function, convergence in the Skorohod topology is equivalent to uniform convergence on compact sets, that is, convergence for the distance

$$d(f,g) = \sum_{k\geq 1} 2^{-k}\ \left(1 \wedge \sup_{t\leq k} \parallel f(t) - g(t) \parallel_{\mathbb{R}^p}\right).$$

Definition 4 (Convergence in $\mathcal{D}_p(I)$) *Let X (resp. $(X^{(n)})_{n\geq 0}$) be a random variable (resp. a sequence of random variables) with values in $\mathcal{D}_p(I)$. The sequence $X^{(n)}$ converges weakly to X, if for any function $\mathcal{L} : \mathcal{D}_p(I) \rightarrow \mathbb{R}$, bounded and continuous,*

$$\lim_n \mathbb{E}[\mathcal{L}(X^{(n)})] \ = \ \mathbb{E}[\mathcal{L}(X)].$$

We shall use the notation

$$X^{(n)} \Rightarrow X.$$

We use repeatedly the next two results:

Theorem 2 ([Bil68], Th. 5.1) *Let $h : \mathcal{D}_p(I) \rightarrow \mathcal{D}_p(I)$, and D_h be the set of discontinuity points of h. Assume that $X^{(n)} \Rightarrow X$ and that $\mathbb{P}(X \in \mathrm{D}_h) = 0$. Then*

$$h(X^{(n)}) \Rightarrow h(X).$$

Perhaps the most important result of convergence of stochastic processes is the convergence of renormalized random walks to the linear Brownian motion $(B_t)_{t\geq 0}$ [Bil68, Don51, RY99]:

Theorem 3 (Donsker [Don51]) *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$. Set $S_k = \sum_{i\leq k} X_i$. Then*

$$\left(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}}\right)_{t\geq 0} \Rightarrow (B_t)_{t\geq 0}.$$

(i) The terminology *cadlag* is usually applied to right-continuous functions that admit a left-limit at each point of $(0,T]$. It is an acronym for the french expression *continue à droite, limite à gauche*.

1.4 The results

In this paper, we study the case where the initial configuration $x(0)$ is composed by a single black region: $x(0) = 0^{n/2}1^{n/2}$. The space renormalization must be $1/n$, and the time renormalization has to be $\mathcal{O}\left(\mathbb{E}[T_n]^{-1}\right)$, as shown in equation (1). Roughly speaking, renormalization of a discrete process can lead to four different behaviors, ordered by increasing degree of randomness:

- the sequence converges to a non null, non-random, process,
- the sequence converges to a random process (e.g. to the standard linear Brownian motion),
- the sequence is tight (relatively compact) but different subsequences converge to different limit processes,
- the sequence is not tight (unbounded).

Our results are roughly summarized below:

quadratic	$\rightarrow$	non-random limit
cubic	$\rightarrow$	reflected (and-or) coalescent Brownian motions
exponential	$\rightarrow$	no limit (untight)
divergent	$\rightarrow$	reflected Brownian motions

so that three of the four previous cases occur when renormalizing ECA's as in (1).

2 Quadratic automata : non-random limit

2.1 The automaton FG

For $t \geq 0$, set

$$\psi(t) = (\psi_1(t), \psi_2(t)) = \left(\tfrac{1}{2} + t, 1 - t\right).$$

Due to Theorem 1, only the time-renormalization n^2 can, eventually, lead to a nontrivial limit process. Actually, a limit process exists, and this limit is non-random. Recall that (L_k, R_k) is the process of the borders of the black region,

$$\ell_n(t) = L_{\lfloor tn^2 \rfloor \wedge T_n}/n, \qquad r_n(t) = R_{\lfloor tn^2 \rfloor \wedge T_n}/n. \tag{2}$$

Theorem 4 *The following convergence holds in $\mathcal{D}_2(\mathbb{R}_+)$:*

$$(\ell_n, r_n) \Rightarrow \left(\psi\left(t \wedge \tfrac{1}{4}\right)\right)_{t \geq 0}.$$

Proof: First, consider the Markov chain $(\tilde{L}_k, \tilde{R}_k)_{k\geq 0}$ defined by $(\tilde{L}_0, \tilde{R}_0) = (n/2, 0)$, and

$$(\tilde{L}_{k+1}, \tilde{R}_{k+1}) = \begin{cases} (\tilde{L}_k, \tilde{R}_k) & \text{with probability } \frac{n-2}{n} \\ (\tilde{L}_k + 1, \tilde{R}_k) & \text{with probability } \frac{1}{n} \\ (\tilde{L}_k, \tilde{R}_k - 1) & \text{with probability } \frac{1}{n} \end{cases}$$

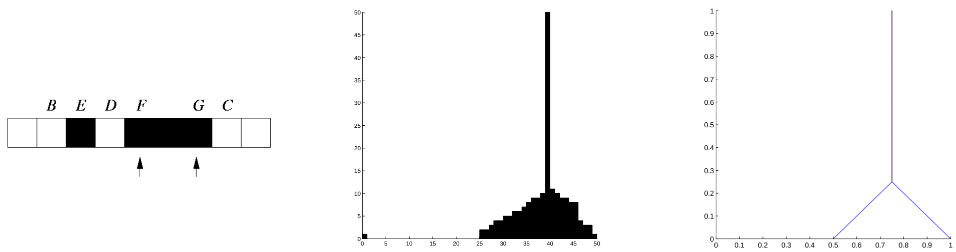

Fig. 3: Automaton FG, and its limit process ψ.

For $t \geq 0$, set

$$\tilde{\ell}_n(t) = \tilde{L}_{\lfloor tn^2 \rfloor}/n, \qquad \tilde{r}_n(t) = \tilde{R}_{\lfloor tn^2 \rfloor}/n.$$

We have

$$\left| \mathbb{E}[\tilde{\ell}_n(t)] - \psi_1(t) \right| = t - \frac{\lfloor tn^2 \rfloor}{n^2},$$

thus for x, T two positive constants, and for n large enough,

$$\mathbb{P}\left(\sup_{t \leq T} \left| \tilde{\ell}_n(t) - \psi_1(t) \right| \geq x \right) \leq \mathbb{P}\left(\sup_{t \leq T} \left| \tilde{\ell}_n(t) - \mathbb{E}[\tilde{\ell}_n(t)] \right| \geq \frac{x}{2} \right).$$

We need the following bound:

Lemma 4.1 (Kolmogorov's inequality, [Bil95], Th. 22.4) *Let $(Y_{k,n})_{k \geq 0}$ denote sequences of i.i.d. random variables such that $\mathbb{E}[Y_{1,n}] = 0$, $\mathbb{E}\left[{Y_{1,n}}^2\right] = c_n < \infty$. One notes $S_{k,n} = Y_{1,n} + \cdots + Y_{k,n}$. For any k and $x > 0$,*

$$\mathbb{P}\left(\max_{1 \leq l \leq k} |S_{l,n}| \geq x \right) \leq c_n k / x^2. \tag{3}$$

Let us write

$$\tilde{L}_k - (n/2) = B_1 + \ldots + B_k, \tag{4}$$

in which the B_i's are i.i.d. random variables with $\mathbb{P}(B_i = 1) = 1 - \mathbb{P}(B_i = 0) = 1/n$. Applying Lemma 4.1 with $S_{k,n} = \tilde{L}_k - (n/2) - (k/n)$, $Y_{i,n} = B_i - (1/n)$, $c_n = \frac{n-1}{n^2}$ and $k = \lfloor Tn^2 \rfloor$, one obtains

$$\begin{aligned} \mathbb{P}\left(\sup_{t \leq T} \left| \tilde{\ell}_n(t) - \mathbb{E}[\tilde{\ell}_n(t)] \right| \geq \frac{x}{2} \right) &= \mathbb{P}\left(\max_{1 \leq \ell \leq \lfloor Tn^2 \rfloor} |S_{\ell,n}| \geq \frac{nx}{2} \right) \\ &\leq 4 \lfloor Tn^2 \rfloor n^{-3} x^{-2}. \end{aligned}$$

With $x = n^{-1/2+\delta}$, for some $\delta \in (0, 1/2)$, it leads to

$$\mathbb{P}\left(\sup_{t \leq T} \left| \tilde{\ell}_n(t) - \psi_1(t) \right| \geq x \right) \leq T\, n^{-2\delta}.$$

The same argument holds for the right border $\tilde{R}_k$. It follows that

$$\left(\tilde{\ell}_n,\, \tilde{r}_n\right) \Rightarrow \psi. \tag{5}$$

Now, the process $(\tilde{L}_k, \tilde{R}_k)_{k\geq 0}$ is designed to have the same distribution as $(L_k, R_k)_{k\geq 0}$, as long as $L_k \leq R_k - 1$. More precisely, if τ and $\mathcal{L}$ denote the operators defined on $\mathcal{D}_2(0, +\infty)$ by

$$\tau(f) = \inf\{t \geq 0\ ;\ f_1(t) \geq f_2(t) - 1\},$$

and

$$\mathcal{L}\,(f) = \left(f\,(t \wedge \tau(f))\right)_{t\geq 0},$$

then we have:

$$(\ell_n,\, r_n) \overset{\text{law}}{=} \mathcal{L}\left(\tilde{\ell}_n, \tilde{r}_n\right). \tag{6}$$

Theorem 2 allows us to conclude, since, in the relation (5), the limit point ψ is a point of continuity of $\mathcal{L}$, and since $\left(\psi\left(t \wedge \frac{1}{4}\right)\right)_{t\geq 0} = \mathcal{L}\psi$. □

2.2 Other quadratic automata.

Quadratic automata are roughly divided into two sub-families. FG belongs to the first one, with automata B, EF, EFG, BDE, BE, BCDE and BCE. The proof adapts easily to all of them, and they converge to non-random limits. The second family contains BCDEF and BEFG. Their behavior is slightly different. A border (say, the left-border) essentially drifts to the right (with small random perturbations), whereas the right-border performs a symmetric random walk. However, these random perturbations are of order $\mathcal{O}\left(n^{1/2}\right)$ and are erased by the space renormalization factor $1/n$, so that the limit is also deterministic. We get the following convergence:

Theorem 5 *For automata BCDEF and BEFG, the following convergence holds in $\mathcal{D}_2(\mathbb{R}_+)$:*

$$(\ell_n, r_n) \Rightarrow \left(\psi'\left(t \wedge \tfrac{1}{4}\right)\right)_{t\geq 0},$$

where $\psi'(t) = (\frac{1}{2}, 1 - t)$.

Proof: This proof and the proof of Theorem 4 are similar. One just has to replace the sequence (B_i) in (4) by a sequence of i.i.d. r.v., with

$$\mathbb{P}(B_i = 1) = \mathbb{P}(B_i = -1) = \tfrac{1}{2}(1 - \mathbb{P}(B_i = 0)) = 1/n.$$

□

3 Cubic automata : interactions between Brownian motions

3.1 The automaton BCEFG

The class of cubic automata provides a variety of interesting limit processes, related with the standard linear Brownian motion [RY99]. For sake of brevity, we focus on the automaton BCEFG: its limit process

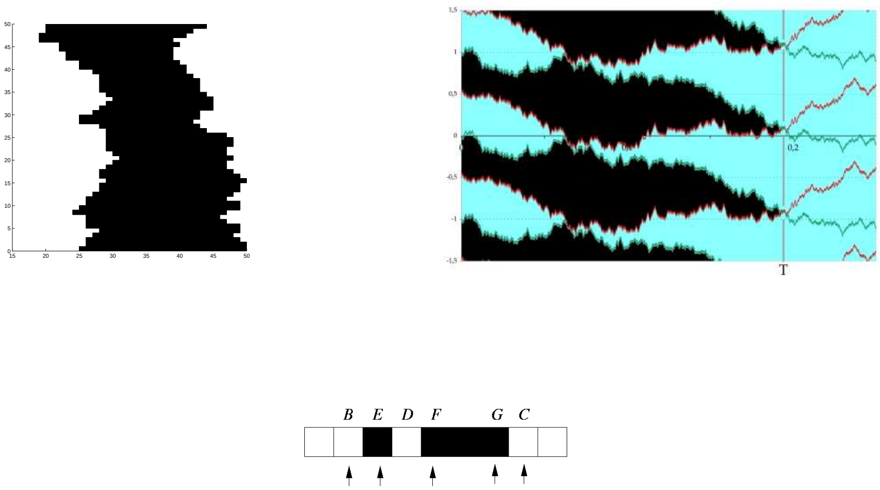

Fig. 4: The automaton BCEFG: a simulation for $n = 50$ and the limit process (here $T \sim 0.195\dots$).

can be described by reflection and coalescence of two independent standard linear Brownian motions W_1 and W_2: set $B_t^{(1)} = 0.5 + \sqrt{2}\,W_1(t)$ (resp. $B_t^{(2)} = \sqrt{2}\,W_2(t)$). For $t \geq 0$, set

$$\ell_n(t) = \frac{L_{\lfloor tn^3 \rfloor \wedge (T_n - 1)}}{n}, \qquad r_n(t) = \frac{R_{\lfloor tn^3 \rfloor \wedge (T_n - 1)}}{n}.$$

We have

Theorem 6 *Set*

$$(B_t^+, B_t^-) = \left(B_t^{(1)} \vee B_t^{(2)}, B_t^{(1)} \wedge B_t^{(2)}\right),$$

and

$$T = \inf\{t \geq 0; |B_t^{(1)} - B_t^{(2)}| \geq 1\} = \inf\{t \geq 0; B_t^+ - B_t^- \geq 1\}.$$

Then

$$(\ell_n(t),\ r_n(t))_{t \geq 0} \Rightarrow (B_{t\wedge T}^+, B_{t\wedge T}^-)_{t \geq 0}.$$

Proof: First, we study a simpler Markov , $(\tilde{L}_k^{(n)}, \tilde{R}_k^{(n)})_{k\geq 0} = (\tilde{L}_k, \tilde{R}_k)_{k \geq 0}$, with values in $\mathbb{Z}^2$, starting at $(\frac{n}{2}, 0)$. Its transition probabilities $p_{(x,y),(z,t)}$ are defined as follows:

- if $y = x - 1$,

$$p_{(x,y),(x+1,y)} = p_{(x,y),(x,y-1)} = p_{(x,y),(y,x)} = \tfrac{1}{n}, \qquad p_{(x,y),(x,y)} = \tfrac{n-3}{n},$$

- if $y = x - n + 1$,

$$p_{(x,y),(x-1,y)} = p_{(x,y),(x,y+1)} = p_{(x,y),(x+1,y-1)} = \tfrac{1}{n}, \qquad p_{(x,y),(x,y)} = \tfrac{n-3}{n},$$

- else,

$$p_{(x,y),(x-1,y)} = p_{(x,y),(x,y-1)} = p_{(x,y),(x+1,y)} = p_{(x,y),(x,y+1)} = \tfrac{1}{n}, \qquad p_{(x,y),(x,y)} = \tfrac{n-4}{n}.$$

We take p symmetric, that is: $p_{(y,x),(t,z)} = p_{(x,y),(z,t)}$. The transitions of $(\tilde{L}_k, \tilde{R}_k)_{k\geq 0}$ are designed with the purpose that the Markov chain

$$\left(\tilde{L}_k^+, \tilde{R}_k^-\right) = \left(\tilde{L}_k \vee \tilde{R}_k,\ \tilde{L}_k \wedge \tilde{R}_k\right)$$

has the same distribution as $(L_k, R_k)_{k\geq 0}$, as long as $L_k - n \leq R_k - 1$. These processes, when suitably renormalized, converges to Brownian-like stochastic processes. More precisely, for $t \geq 0$, set

$$\left(\tilde{\ell}_n, \tilde{r}_n, \tilde{\ell}_n^+, \tilde{r}_n^-\right)(t) = n^{-1}\left(\tilde{L}_{\lfloor tn^3 \rfloor}, \tilde{R}_{\lfloor tn^3 \rfloor}, \tilde{L}^+_{\lfloor tn^3 \rfloor}, \tilde{R}^-_{\lfloor tn^3 \rfloor}\right).$$

Lemma 6.1

$$(\tilde{\ell}_n, \tilde{r}_n) \Rightarrow \left(B_t^{(1)},\ B_t^{(2)}\right)_{t\geq 0}. \tag{7}$$

Proof of the Lemma: This Lemma is a consequence of the following Proposition, which is a particular case of ([EK86], Chap.7, Th 4.1).

Proposition 1 *Let $\tilde{\ell}_n, \tilde{r}_n, a_n, b_n, c_n$ be some random elements in $\mathcal{D}_1(\mathbb{R}_+)$, and let $(\mathcal{F}_t^n)_{t\geq 0}$ be the filtration defined by $\mathcal{F}_t^n = \sigma\left(\tilde{\ell}_n(s), \tilde{r}_n(s); s \leq t\right)$. Suppose that*

1. *For each n, $\tilde{\ell}_n$ and $\tilde{r}_n$ are $\mathcal{F}_t^n$-martingales.*
2. *For each n, $\tilde{\ell}_n^2 - a_n$, $\tilde{r}_n^2 - b_n$ and $\tilde{\ell}_n\tilde{r}_n - c_n$ are $\mathcal{F}_t^n$-martingales.*

Assume furthermore that for each constant $\mathcal{T} > 0$, the following convergences hold in probability:

$$\sup_{t\leq\mathcal{T}} |a_n(t) - 2t| \to 0, \tag{8}$$

$$\sup_{t\leq\mathcal{T}} |b_n(t) - 2t| \to 0, \tag{9}$$

$$\sup_{t\leq\mathcal{T}} |c_n(t)| \to 0. \tag{10}$$

Then

$$\left(\tilde{\ell}_n(t), \tilde{r}_n(t)\right)_{t\geq 0} \Rightarrow \left(\sqrt{2}B_t^1, \sqrt{2}B_t^2\right)_{t\geq 0},$$

where B_t^1, B_t^2 are two independent Brownian motions.

We apply Proposition 1 with

$$a_n(t) = b_n(t) = 2\frac{\lfloor tn^3 \rfloor}{n^3}$$
$$c_n(t) = \frac{1}{n^3} \sum_{\ell=0}^{\lfloor tn^3 \rfloor - 1} \mathbf{1}_{|\tilde{L}_\ell - \tilde{R}_\ell| = 1} - \mathbf{1}_{|\tilde{L}_\ell - \tilde{R}_\ell| = n-1}.$$

Simple calculations show that

1. $(\tilde{\ell}_n(t))_{t \geq 0}$ and $(\tilde{r}_n(t))_{t\geq 0}$ are $\mathcal{F}_t^n$-martingales,
2. $\tilde{\ell}_n^2 - a_n$ and $\tilde{r}_n^2 - b_n$ are $\mathcal{F}_t^n$-martingales,
3. $\tilde{\ell}_n \tilde{r}_n - c_n$ is a $\mathcal{F}_t^n$-martingale.

The Theorem will be proved once it is etablished that for each $\mathcal{T} > 0$,

$$\sup_{t \leq \mathcal{T}} \left| 2\frac{\lfloor tn^3 \rfloor}{n^3} - 2t \right| \to 0, \text{ in probability,} \tag{11}$$
$$\sup_{t \leq \mathcal{T}} \left| \frac{1}{n^3} \sum_{\ell=0}^{\lfloor tn^3 \rfloor - 1} \mathbf{1}_{|\tilde{L}_\ell - \tilde{R}_\ell| = 1} - \mathbf{1}_{|\tilde{L}_\ell - \tilde{R}_\ell| = n-1} \right| \to 0, \text{ in probability.} \tag{12}$$

Only (12) is nontrivial. We will denote by L_k^p the *local time* in p at time k of the random walk $(|\tilde{L}_\ell - \tilde{R}_\ell|)_{k\geq 0}$; that is

$$L_k^p = \sum_{\ell=0}^{k} \mathbf{1}_{|\tilde{L}_\ell - \tilde{R}_\ell| = p}.$$

It is proved in Appendix that there exists C such that for each p,

$$\mathbb{E}[L_k^p] \leq n + Cn^{3/4}k^{1/4}.$$

Hence, by the Markov inequality,

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \leq \mathcal{T}} \frac{1}{n^3} \left| L^0_{\lfloor tn^3 \rfloor} - L^{n-1}_{\lfloor tn^3 \rfloor} \right| > \varepsilon \right) &\leq n^{-3}\varepsilon^{-1} \mathbb{E}\left[\sup_{t \leq \mathcal{T}} \left| L^0_{\lfloor tn^3 \rfloor} - L^{n-1}_{\lfloor tn^3 \rfloor} \right| \right] \\
&\leq n^{-3}\varepsilon^{-1} \mathbb{E}\left[\sup_{t \leq \mathcal{T}} |L^0_{\lfloor tn^3 \rfloor}| + |L^{n-1}_{\lfloor tn^3 \rfloor}| \right] \\
&= n^{-3}\varepsilon^{-1} \left(\mathbb{E}[|L^0_{\lfloor \mathcal{T}n^3 \rfloor}|] + \mathbb{E}[|L^{n-1}_{\lfloor \mathcal{T}n^3 \rfloor}|] \right) \\
&\leq 2C\,\mathcal{T}^{1/4} n^{-3/2} \varepsilon^{-1},
\end{aligned}$$

which converges to zero when $\mathcal{T}$ is fixed. □

Now, since the operator Λ defined on $\mathcal{D}_2(0, +\infty)$ by

$$\Lambda\,(f) = (f_1(t) \vee f_2(t),\ f_1(t) \wedge f_2(t))_{t \geq 0}$$

is continuous, it follows that

$$(\tilde{\ell}_n^+, \tilde{r}_n^-) \Rightarrow \left(B_t^+, B_t^-\right)_{t\geq 0}.$$

The stochastic process $\left(B_t^+, B_t^-\right)_{t\geq 0}$ is often called a planar Brownian motion reflected at a line (here the first bisectrix). Finally, using the operators τ and $\mathcal{L}$ defined at Section 2.1, we have again:

$$(\ell_n, r_n) \stackrel{\text{law}}{=} \mathcal{L}\left(\tilde{\ell}_n^+, \tilde{r}_n^-\right). \tag{13}$$

Again, Theorem 2 allows us to conclude, since, due to properties of sample paths of the standard Brownian motion (cf. [RY99], Chap.2, Th.2.2), the limit point $(B_t^+, B_t^-)_{t\geq 0}$ is almost surely a point of continuity of $\mathcal{L}$. □

3.2 Automata BDEF, BEF, BCDEFG, BCEFG : Brownian motion

Up to symmetries, there are 6 different cubic automata. The same arguments show that four of them admit a continuous limit with a n^3-time-renormalization: automata BDEF, BEF, BCDEFG, BCEFG. All these limits involve the standard Brownian motion: resp. reflected and stopped, reflected, coalescent, coalescent and reflected BM. The proofs differ only by the choice of the operator Λ.

3.3 Automata BDEG, BEG : no convergence

The n^3-time-renormalization is not suitable for these two automata, that behave as quadratic automata. It is primarily due to the fact that

$$\mathbb{E}[n^{-1}L_{\lfloor tn^3 \rfloor}] = 1/2 + tn,$$

that does not converge.

4 Exponential automaton : no convergence

4.1 The automaton BDFG

Fig. 5: The automaton BDFG.

BDFG is, up to symmetries, the only *exponential automaton*. Simulations suggest that its behavior is quite different from those already encountered. The right border essentially drifts to the left (with small random perturbations), while the left border, that would be a symmetric random walk, is pushed to the left by the right border. Actually, the size of the black region $Z_k^{(n)} = |R_k - L_k|$ performs a *biased* random walk on $\{1, \dots, n\}$, reflected at 1, absorbed at n. According to [FMST05],

$$\mathbb{E}[T_n] = \frac{1}{9}\, n\, 2^n + \mathcal{O}(n^2).$$

As opposed to the previous cases, it turns out that the process

$$z_n = \left(n^{-1} Z^{(n)}_{\lfloor t\, n\, 2^n \rfloor}\right)_{t \geq 0}$$

is not weakly convergent. Actually, the sequence (z_n) is not *tight* (ii). This is a consequence of the next Proposition, a slight modification of ([Ald78], Cor. 1), very powerful in this case:

Proposition 2 *Assume that the sequence (z_n) converges in $\mathcal{D}(\mathbb{R})$. Let (τ_n, δ_n) be a sequence such that*

(i) *for all n, τ_n is a stopping time w.r.t the process $(z_n)_{t\geq 0}$ (with its natural filtration) and τ_n takes its values in a finite set,*

(ii) *(δ_n) is a sequence of real numbers converging to zero.*

Then

$$z_n(\tau_n + \delta_n) - z_n(\tau_n) \xrightarrow{\mathrm{P}} 0,\ n \to \infty. \tag{14}$$

Now, set

$$\begin{aligned} t_n &= n^{-1} 2^{-n} T_n, \\ \tau_n &= 2 \wedge \inf\{u > 0; z_n(u) \geq 1/2\}, \\ \delta_n &= \tfrac{1}{n}. \end{aligned}$$

The r.v. τ_n is a stopping time w.r.t. z_n, and it takes its values in the finite set $\{n^{-1}2^{-n}k \,:\, 1 \leq k \leq 2n2^n\}$. We show that these sequences (τ_n) and (δ_n) violate the condition (14), as would do any subsequence. Incidentally, the fact that any subsequence violates (14) precludes tightness for the sequence (z_n).

It is convenient to generate the sequence (Z_k) with the help of a sequence of i.i.d. r.v. $(Y_0, Y_1, \dots)$ such that $Y_i = -1$, (resp. 0, 1) with probabilities $\frac{2}{n}$ (resp. $\frac{n-3}{n}$, $\frac{1}{n}$), as follows:

$$Z_{k+1} = Z_k + Y_k\, \mathbf{1}_{0<Z_k<n} + \mathbf{1}_{Z_k=0 \text{ and } Y_k=1}$$

For any $0 < \varepsilon < 1/3$, we see that

$$\begin{aligned} \mathbb{P}(|z_n(\tau_n + \delta_n) - z_n(\tau_n)| > \varepsilon) &= \mathbb{P}(|Z_{n2^n(\tau_n+\delta_n)} - Z_{n2^n\tau_n}| > n\varepsilon) \\ &\geq \mathbb{P}(|Z_{n2^n(\tau_n+\delta_n)} - Z_{n2^n\tau_n}| > n\varepsilon; \tau_n \leq 1) \\ &\geq \mathbb{P}(Y_{n2^n\tau_n+1} + \dots Y_{n2^n\tau_n+2^n} \leq -n\varepsilon; \tau_n \leq 1). \end{aligned}$$

(ii) Meaning that its closure is not even compact, cf. [Bil68] for definitions.

The two events on the right hand side are independent. More precisely,

$$\mathbb{P}(|z_n(\tau_n+\delta_n)-z_n(\tau_n)|>\varepsilon)\geq\mathbb{P}(Y_1+\ldots Y_{2^n}\leq -n\varepsilon)\mathbb{P}(\tau_n\leq 1)$$

in which $\lim_n \mathbb{P}(Y_1+\cdots+Y_{2^n}\leq n\varepsilon)=1$ by the Bienaymé-Tchebychev inequality. It is more involved to prove that $\liminf_n \mathbb{P}(\tau_n\leq 1)>0$, so we only give a sketch of the proof. Let us consider the number N and the positions of excursions of Z_n that reach n but not $2n$, and that occur before T_n: N has a geometric distribution with parameter $(2^n+2)^{-1}$, so that $\mathbb{E}[N]=2^n+1$, and so that, with a probability exponentially close to 1, $N\geq 2$. Given that $N\geq 2$ and $T_n=\ell$, the first excursion of Z_n that reaches n takes place before all the other excursions of the same kind (since $N\geq 2$, there exists at least another one of the kind), and approximately before half the other excursions. Thus the conditional expectation of the first return of z_n to 0 after τ_n, given $N\geq 2$ and $T_n=\ell$, is not larger than $n^{-1}2^{-n}\,\ell/2$, or than $t_n/2$. Markov inequality entails that the conditional probability that $\tau_n\leq 3t_n/4$ is larger than 1/3. As a consequence, $\mathbb{P}(\tau_n\leq 1)$ is larger than $\mathbb{P}(t_n\leq 4/3)/3\sim(1-e^{-12})/3$, and (14) does not hold.

5 Divergent automata

5.1 The automata BCFG, BF and CF: reflected Brownian motions

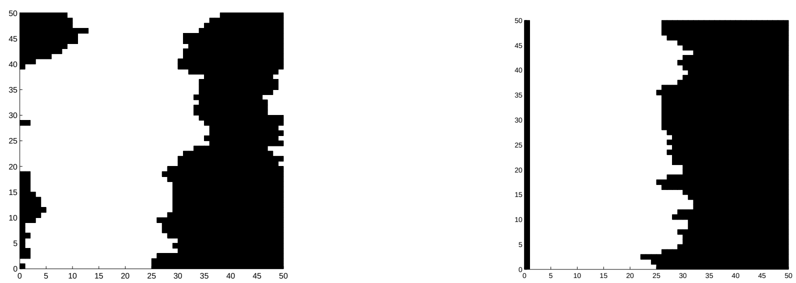

Fig. 6: Simulations for divergent automata BCFG and BF.

The limit processes of these three divergent automata are related to reflected Brownian motions. The main difference with Section 3 is that coalescence does not occur. In order to state our results for the automaton BCFG, we shall use the same tools and notations as in Section 3. Set

$$(L_t,R_t)=\left(B_t^{(2)},B_t^{(1)}\right)\;+\;(-1,1)\,\frac{\lfloor B_t^{(2)}-B_t^{(1)}\rfloor}{2}\;+\;(0,1),$$

if $\lfloor B_t^{(2)}-B_t^{(1)}\rfloor$ is even,

$$(L_t,R_t)=\left(B_t^{(1)},B_t^{(2)}\right)\;+\;(1,-1)\,\frac{\lfloor B_t^{(2)}-B_t^{(1)}\rfloor}{2}\;+\;(0.5,-0.5),$$

if $\lfloor B_t^{(2)}-B_t^{(1)}\rfloor$ is odd. One can see (L,R) as two self-reflected Brownian motions on the circle.

Set

$$(\ell_n(t), r_n(t)) = \left(\frac{1}{n}L_{\lfloor tn^3 \rfloor}, \frac{1}{n}R_{\lfloor tn^3 \rfloor}\right),$$

with these notations one gets the following result.

Theorem 7 *For the automaton BCFG,*

$$(\ell_n,\, r_n) \Rightarrow (L, R).$$

For the automaton BF, $(\ell_n,\, r_n) \Rightarrow (W, 1)$, in which W denotes a standard linear Brownian motion starting at 0.5, reflected at 0 and 1, while for the automaton CF, only the renormalized width $z_n = r_n - \ell_n$ of the black region converges to W, while $(\ell_n,\, r_n)$ is untight: more precisely, one can see that

$$(\ell_n(t)/n, r_n(t)/n)_{t\geq 0} \;\Rightarrow\; (t, 0.5 + t)_{t\geq 0}.$$

5.2 The automaton BCF

This automaton behaves a lot like the exponential automaton BDFG of Section 4, with the difference that its width is reflected at 0 but also at $n-1$. The hitting time of the barrier $n-1$ has again an expectation $n\ 2^n$, but then the whole process starts again. For the same reasons as in Section 4, the sequence of processes z_n is not tight.

5.3 The automaton BG

Starting from $x(0) = 0^{n/2}1^{n/2}$, the automaton BG cannot reach a fixed point. However, the dynamic is similar to that of quadratic automata, and the limit is indeed deterministic, when the renormalization is that of Section 2 : here

$$(\ell_n(t), r_n(t)) = \left(\frac{1}{n}L_{\lfloor tn^2 \rfloor}, \frac{1}{n}R_{\lfloor tn^2 \rfloor}\right).$$

Theorem 8 *For automaton BG, the following convergence holds in $\mathcal{D}_2(\mathbb{R}_+)$:*

$$(\ell_n, r_n) \Rightarrow \left(\frac{1}{2} - t, 1 - t\right)_{t\geq 0}.$$

Appendix

Lemma 8.1 *Let $(Z_\ell)_{\ell\geq 0}$ be a random walk on $\mathbb{Z}$, $\mathbb{P}(Z_{\ell+1} = Z_\ell + 1) = \mathbb{P}(Z_{\ell+1} = Z_\ell - 1) = 1/n$, $\mathbb{P}(Z_{\ell+1} = Z_\ell) = \frac{n-2}{n}$, starting from z_0. There exists a constant C such that pour each p*

$$\mathbb{E}[L_k^p] \leq n + Cn^{3/4}k^{1/4}.$$

Proof: Let $(\tilde{Z}_\ell)_{\ell>0}$ be a r.w. on $\mathbb{Z}$, $\mathbb{P}(\tilde{Z}_{\ell+1} = \tilde{Z}_\ell + 1) = \mathbb{P}(\tilde{Z}_{\ell+1} = \tilde{Z}_\ell - 1) = 1/2$, starting from z_0. If $\ell \geq n$,

$$\mathbb{P}(Z_\ell = p) = \sum_{j=0}^{\ell} \mathbb{P}(B_{\ell,2/n} = j)\mathbb{P}(\tilde{Z}_j = p),$$

where $B_{\ell,2/n}$ is a binomial r.v., with parameters $(\ell, 2/n)$.

$$\begin{aligned}
\mathbb{P}(\tilde{Z}_\ell = p) &\leq \sum_{|j-\frac{2\ell}{n}|\leq\left(\frac{2\ell}{n}\right)^{1/4}} \mathbb{P}(B_{\ell,2/n} = j)\mathbb{P}(\tilde{Z}_j = p) + \sum_{|j-\frac{2\ell}{n}|>\left(\frac{2\ell}{n}\right)^{1/4}} \mathbb{P}(B_{\ell,2/n} = j)\mathbb{P}(\tilde{Z}_j = p) \\
&\leq \sum_{|j-\frac{2\ell}{n}|\leq\left(\frac{2\ell}{n}\right)^{1/4}} \mathbb{P}(B_{\ell,2/n} = j)\mathbb{P}(\tilde{Z}_j = p) + \mathbb{P}(|B_{\ell,2/n} - \tfrac{2\ell}{n}| > \left(\tfrac{2\ell}{n}\right)^{1/4}) \\
&\leq \sum_{|j-\frac{2\ell}{n}|\leq\left(\frac{2\ell}{n}\right)^{1/4}} \mathbb{P}(B_{\ell,2/n} = j)\mathbb{P}(\tilde{Z}_j = p) + 2\exp(-\sqrt{2}n).
\end{aligned}$$

Here we have used that $\mathbb{P}(|B_{r,q} - rq| > h) \leq 2\exp(-\frac{2h^2}{r})$ (see for example [Bol85],Chap.I,Cor.4). Hence,

$$\begin{aligned}
\mathbb{P}(Z_\ell = 0) &\leq \max_{|j-\frac{2\ell}{n}|\leq(\frac{2\ell}{n})^{1/4}} \mathbb{P}(B_{\ell,2/n} = j) \times 2\left(\tfrac{2\ell}{n}\right)^{1/4} \max_{|j-\frac{2\ell}{n}|\leq(\frac{2\ell}{n})^{1/4}} \mathbb{P}(\tilde{Z}_j = p) + 2\exp(-\sqrt{2}n). \\
&\leq C_1\left(\tfrac{n}{\ell}\right)^{1/2}\left(\tfrac{\ell}{n}\right)^{1/4}\left(\tfrac{n}{\ell}\right)^{1/2} + 2\exp(-\sqrt{2}n) \leq C_2\left(\frac{n}{\ell}\right)^{3/4}. \qquad (15)
\end{aligned}$$

This last inequality is the consequence of two well-known facts (see [Fel70],Chap.VI):

1. The central term in the binomial distribution $B_{r,q}$ is bounded above by $\frac{C}{\sqrt{rq(1-q)}}$.
2. $\mathbb{P}(\tilde{S}_j = p)$ is bounded above by $\frac{C}{\sqrt{j}}$, C being independent of z_0 and p.

Now, for each $k > n$,

$$\begin{aligned}
\mathbb{E}[L_k^0] = \sum_{\ell=0}^{k} \mathbb{P}(Z_\ell = 0) &\leq n + \sum_{\ell=n+1}^{k} \mathbb{P}(Z_\ell = 0) \\
&\leq n + C_2 \sum_{\ell=n+1} \left(\frac{n}{\ell}\right)^{3/4} \leq n + C_3 n^{3/4} k^{1/4}.
\end{aligned}$$

□

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