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On expected number of maximal points in polytopes

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We answer an old question: what are possible growth rates of the expected number of vector-maximal points in a uniform sample from a polytope.

Keywords: -

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1 introduction

1.1 setup

We consider the following setup

- Fix a closed convex polyhedral cone K with nonempty interior in d-dimensional Euclidean space $V = \mathbb{R}^d$;
- The cone K defines a partial ordering on V: for $x, y \in V$,

$$x >_K y \quad \text{iff} \quad x - y \in K$$

  (we say that x *dominates* y or that y *is dominated by* x).

- a point $x \in X$ is said to be *K-maximal*, or simply *maximal* if there are no further point $x' \in X$ dominating x.

  (We will be using notation

$$max_K(X).$$

  for the set of maximal points in a set $X \subset V$.)

- Assume further that a convex (compact) polyhedron P is given, and that

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[†] Suppoorted in part by ARO

- X is a uniform size n sample from P.

The following question arises in various contexts:

Question 1 *Given P and K, find the expected size*

$$M_n = \mathbb{E}|max_K(X)|$$

of K-maximal elements in X, as a function of the sample size $n = |X|$.

I will not try to survey here all situations where computing M_n can be useful, just mention some keywords – *multicriterial optimization, geometric algorithms, convex hulls* – and refer the reader to [6, 8, 4, 5, **?**]

1.1.1 Convention

We will use $f \approx g$ as a synonym for

$$\frac{f}{g} \to c, 0 < c < \infty.$$

1.2 what was known so far

The Question 1 was addressed by many authors having different applications in mind; consequently, they arrived at partial answers. Two of the possible setting studied most occupy in some sense opposite corners of the space of all problems:

- If P is the unit square, and K is the positive quarter plane (in $d = 2$), the number of maximal point is the same as the number of records in an *iid* sample, i.e. *harmonic number*

  $$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

  More generally, in higher dimensions, if P is the unit cube, and K is the positive orthant ("Pareto cone"), the problem still is essentially combinatorial, and the expected number of maximal elements is the *incomplete polyzeta*: thus, in dimension d, the expected number of maximal elements is

  $$M_n = \frac{1}{n} \sum_{1 \leq i_1 \leq \cdot \leq \cdots \leq i_{d-1} \leq n} \frac{1}{i_1 i_2 \cdots i_{d-1}}.$$

  (this result was first established, it seems, in [2], and reproduced by many authors).

- If the polyhedron P is *in general position* with respect to the cone K (meaning: all flats spanned by facets of K and P intersect transversally), then

  $$M_n \approx n^{f/d}$$

  where f is the dimension of $max_K(P)$ [3].

The situations described above are in some sense the most degenerate and most generic ones, respectively. In the former case, all faces of P are parallel to some faces of K. Generically, a small perturbation would lead to a K, P being in general position. Intermediate situations are relevant, however: for example, if P is given as a set of solution of a system of linear inequalities $P = \{Ax \leq b\}$, the sparsity of the matrix A would lead to a problem intermediate between the those above.

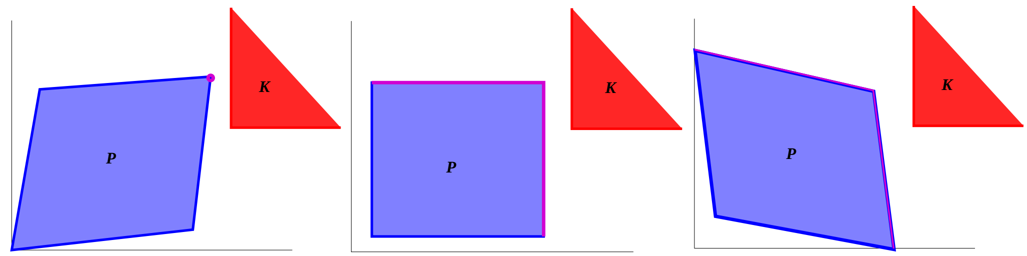

Figure 1: Generic deformations of the polyhedron P: in a smooth 1-dimensional family, typical representative has polynomial growth ($M_n \approx n^0$ on the left; $M_n \approx n^{1/2}$ on the right), and has logarithmic terms for exceptional values of parameter ($M_n \approx \log n$ for the middle display).

1.3 gap problem

It has been noticed [8] that there is a certain *gap* in the possible asymptotic behaviors of M_n as a function of n, at least in dimension 2: if the expected number grows faster than $\log(n)$, then it is asymptotically $\Omega(\sqrt{n})$. This, clearly, motivates the problem:

Question 2 *What are the possible asymptotics of M_n, at least for convex polyhedral K, P (and uniform sample from P)?*

In other words, what else, beyond observed so far behaviors $M_n \approx n^{f/d}$; $M_n \approx \log^{d-1}(n)$ can happen within our polyhedral setup?

In this work we answer both Questions, 1 and 2.

1.3.1 acknowledgment

I was told about the gap problem by Mordecai Golin during AofA'06; many thanks!

2 main result

Consider the set of points

$$\Delta = \{(m, c) \in \mathbb{Z}^2, 0 \leq c \leq m \leq d\}.$$

We will call a pair $(r, \mu), r > 0, \mu \in \mathbb{N}$ *admissible*, if the intersection of the ray

$$c = rm, m \geq 0$$

with the set Δ contains at least at least mu points (beyond the origin). In particular, in an admissible pair (r, μ), r is rational.

Theorem 1 *The expected number of maximal elements $M_n = \mathbb{E}(max_K(X))$ with respect to a convex closed polyhedral cone K, where X is the uniform sample from polyhedral $P \subset V$ of size n, satisfies*

$$M_n \approx n^{1-r} \log^{\mu-1}(n), \tag{1}$$

for some admissible pair (r, μ). For any admissible (r, μ), there exists a pair (P, K) satisfying (1).

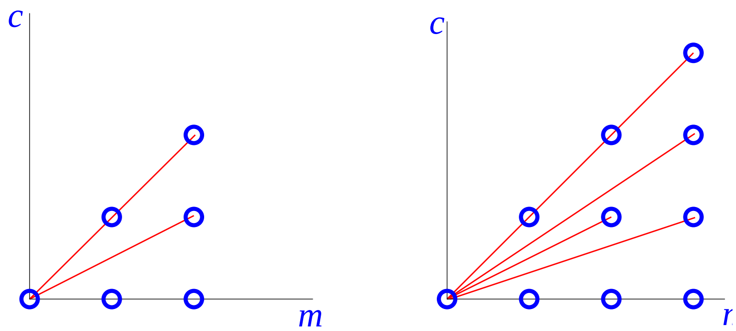

Figure 2: Asymptotics of M_n in dimensions 2,3.

2.1 examples

1. For $d = 2$, the Figure 2.1 shows all possible alternatives: the point $(m, c) = (1, 1)$ corresponds to $M_n \approx n^0$ (compare Fig. 1.2, left); the point $(2, 2)$ correspond to $M_n \approx \log n$ (Fig. 1.2, middle) and the point $(2, 1)$ corresponds to $M_n \approx n^{1/2}$. In particular, one can see the "gap".

2. For $d = 3$, asymptotic growth rates for M_n are
   - $M_n \approx n^{1-r}, r = 1/3, 1/2, 2/3;$
   - $M_n \approx \log^{\nu}(n), \nu = 0, 1, 2.$

3. For illustrative purposes, some examples in dimension 8: The three segments illustrate the following

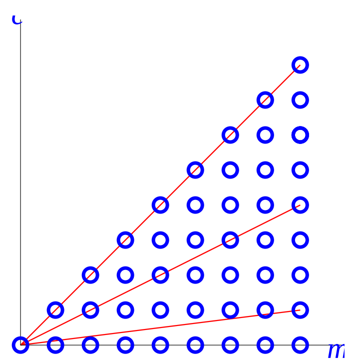

Figure 3: Asymptotics of M_n in dimension 8.

asymptotics (bottom-to-top):

- $M_n \approx n^{7/8};$
- $M_n \approx n^{3/4} \log^3(n);$

- $\log^7(n)$.

Remark: In generic situation (when all faces of P are transversal to all faces of K), all multiplicities μ are equal to 0.

2.2 specific polytopes and cones

The general result of the previous section depends, of course, on the results describing the asymptotics of M_n for specific instances of (P, K). To describe those, we will need to define *flags* and their *spectra*.

Consider the (ranked) *face poset* of the polytope P: its elements are faces of P of all dimensions ordered by inclusion (thus the maximal face is of dimension d, the polytope P itself, then the facets of P of dimension $d-1$ and so on. We will denote the set of faces of dimension l as $\mathcal{P}_l$.

Flags are the chains of *adjoining* proper faces of P:

$$\mathcal{F} = (f_1, f_2, \ldots, f_l) : f_1 \subset f_2 \subset \ldots \subset f_l, f_i \in \mathcal{P}_{d_i}; d_l < d.$$

A flag if *full* if it has length d, i.e. it includes facets of all dimensions between 0 and $d-1$.

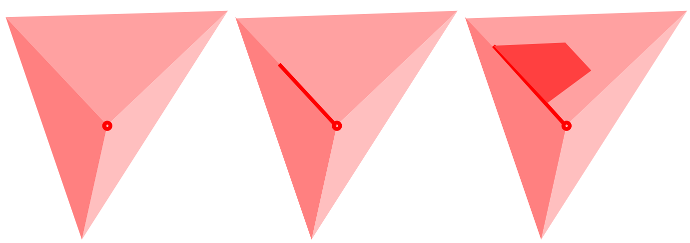

Figure 4: A flag.

We associate to each face f of P its *deficiency* $\delta(f)$: it is just the dimension of the intersection

$$(K + x) \cap P$$

for a (relative) interior point $x \in f$. One can easily check that the deficiency is well-defined, i.e. does not depend on particular point in the relative interior of a face.

Deficiencies in a flag do not decrease:

$$\delta(f_1) \leq \delta(f_2), f_1 \subset f_2.$$

Definition: The spectrum of a flag $F = (f_1 \subset f_2 \subset \ldots \subset f_l)$ is the multiset

$$\sigma(\sigma) = \left\{ \frac{c(f_1)}{m(f_1)}, \frac{c(f_2)}{m(f_2)}, \ldots, \frac{c(f_l)}{m(f_l)} \right\}$$

(where elements are counted with multiplicities), where

$$c(f_i) = codim(f_i) = d - \dim(f_i)$$

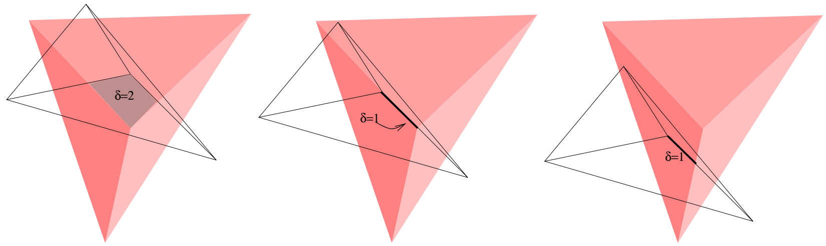

Figure 5: Deficiencies of some of the facets for polytope on Figure 2.2; the deficiencies of the facets in the flag on Figure 2.2 are equal to $(0, 1, 2)$, and its spectrum is $\{1^3\}$ (1 with multiplicity 3).

is the codimension of a face, and

$$m(f_i) = d - \delta(f_i)$$

is its *codeficiency*.

Definition: For a flag $\mathcal{F}$ define its leading exponent to be

$$r(\mathcal{F}) = \min\{r : r \in \sigma(\mathcal{F})\},$$

the smallest number in the flag spectrum. The multiplicity $\mu(\mathcal{F})$ is the multiplicity of $r(\mathcal{F})$ in $\sigma(\mathcal{F})$.

The spectra of flags are crucial for us due to our second main result,

Theorem 2 *The asymptotics of M_n, as $n \to \infty$ is given by*

$$M_n \approx n^{1-r_*} \log^{\mu-1}(n), \tag{2}$$

where $r_* = \min_{\mathcal{F}} r(\mathcal{F})$ is the smallest of the leading exponents of flags, and μ is the largest multiplicity of r_* among all the flags.

Remark: It is immediate that extending a flag only increases its spectra, and thus can lead only to faster growing terms in (2). Hence to analyze the leading asymptotics of M_n it is enough to consider only full flags.

Remark: In fact, the entire asymptotic expansion of M_n is governed by the spectra of flags. Essentially, for each pair $(c(\mathcal{F}), m(\mathcal{F}))$ such that $r = c/m$ occurs in the flag spectrum with multiplicity μ contributes to the asymptotic expansion of M_n an asymptotic series

$$\sum_{0 \le l; 0 \le \nu < \mu} a_{l,\nu} n^{1-(c+l)/m} \log^{\nu}(n).$$

We will not go into details, postponing a detailed exposition of this (and all the remaining) topics to a separate paper.

3 techniques

The techniques are a melange of Fubini theorem, an elementary version of *resolution of singularities* and some fairly standard results from the theory of generalized functions. We will not even attempt to present any details of the proofs (which, while not complicated, would require quite a bit of supporting machinery), but rather will sketch the major steps and outline main constructions.

3.1 Fubini

We will denote by λ the Lebesgue measure on V, and by $|P| = \lambda(P)$ the volume of the polytope P.

For a point $x \in P$, denote by

$$u(x) = |P|^{-1}\lambda(K + x \cap P)$$

the probability that a random point from P dominates x.

Lemma 1 *The function u is piece-wise polynomial in V: there exists a polyhedral subdivision of P (*spline subdivision*) such that the restriction of u to each of its polyhedra is polynomial.*

We will use the following formula which is a more or less straightforward corollary of Fubini and a formula for M_n implied by conditioning on the positions of a point in X and finding the probability that this point is maximal:

Lemma 2

$$M_n = n \int (1-t)^{n-1} dB(t), \tag{3}$$

where

$$B(t) = |P|^{-1}\lambda\{x \in P : u(x) \le t\}, \tag{4}$$

is the probability that M evaluated at a random point in P is $\le t$.

The advantage of using (3 is the decoupling of geometry: we can concentrate now on the properties of the function $B(t)$. Indeed, Karamata-type Tauberian theorems would translate the asymptotics of B near zero into the the asymptotics of $M_n, n \to \infty$.

To analyze B near 0, we need to analyze u near the points where u vanishes. This can be done locally.

3.2 resolution of singularities

The domains where u is polynomial can adjoin the faces of P in rather intricate fashion. Hence a resolution of P would be helpful. In particular, it would be convenient to arrive at the domain with controlled singularities, specifically, with normal intersections (where each face of codimension k is adjoined to at most k faces of codimension 1).

To do so we resolve the singularities of P (i.e. its faces of codimension 2 and higher), in such a way that all facets (independently of their dimensions) would lift to faces of dimension $(d-1)$ in the resolved polytope, the flags of length 2 (pairs of incident facets) would lift to faces of dimension $(d-2)$ and so on, with full flags corresponding to the vertices in the resolution.

The easiest way to visualize such a resolution is as follows:

For each face f^l of dimension $l < d$ of the polytope P consider the ϵ^{l+1}-tube $T(f)$ around this face (i.e. the set of all points in P at the distance at most ϵ^{l+1} from the affine subspace (of dimension l) spanned by f^l. For small enough ϵ this tubes intersect transversally, and only if the corresponding faces adjoin.

Let P_ϵ be the complement to the union of these tubes,

$$P_\epsilon = P - \cup_{l<d} \cup_{f^l \subset P} T(f).$$

It is immediate that the facets of P_ϵ are in one-to-one correspondence with flags of P.

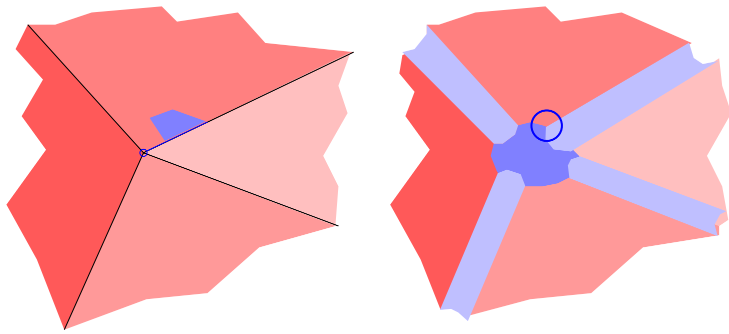

Figure 6: Resolution of vertices: non-simple vertex becomes (topologically) simple after the resolution. Vertices of the resolved polytope correspond to the flags of the original polytope P: one such pair, flag $\leftrightarrow$ vertex, is shown.

Proposition 1 *The (topological) polyhedron P_ϵ resolves P: there exists a (natural) mapping $\pi : P_\epsilon \to P$ which is a diffeomorphism on the interior of P_ϵ taking each codimension 1 facet of P_ϵ to the corresponding facet of P. (More generally, the faces of P_ϵ of codimension k correspond to flags in P of length k.) Further, the polyhedron P_ϵ is a manifold with corners: locally, it is diffeomorphic to an open ball at the origin in $\mathbb{R}^d$ intersected with $s \leq d$ halfspaces $x_1 \geq 0, \ldots, x_s \geq 0$.*

Also, the resolution moves the boundaries between the spline domains away from the vertices of P_ϵ:

Proposition 2 *The boundaries between polyhedra of the spline subdivision lift to subvarieties (with corners) of P_ϵ which do not contain the vertices of P_ϵ.*

3.3 Gelfand-Leray forms and elementary Laplace integrals

Using the proposition 1 we can localize the integral (4). Indeed, we can represent

$$B(t) = |P|^{-1} \int_P \mathbf{1}(u(x) < t) d\lambda \tag{5}$$

as

$$|P|^{-1} \int_{P_\epsilon} \mathbf{1}(\pi^* u(x) < t) d\rho \tag{6}$$

$$= \quad |P|^{-1} \sum_{\mathcal{F}} \int_{P_\epsilon} \phi_{\mathcal{F}}(y) \mathbf{1}(\pi^* u < t) d\rho. \tag{7}$$

Here $\rho = \pi^* \lambda$ is the pull-back of the Lebesgue measure (possible as π is a diffeomorphism onto the interior of P), and

$$\sum_{\mathcal{F}} \phi_{\mathcal{F}} = 1, \phi_{\mathcal{F}} \geq 0$$

is the partition of unity such that $\phi_{\mathcal{F}} = 0$ outside of the small vicinity of the stratum of P_ϵ corresponding to flag $\mathcal{F}$.

The advantage of the decomposition (7) stems from the following

Lemma 3 *Let y be an interior point of a s-dimensional facet of P_ϵ corresponding to a flag $\mathcal{F} = (f_1 \subset f_2 \subset \ldots \subset f_l)$. In particular, one can choose a coordinate system centered at y such that near the origin, the polyhedron P_ϵ is given by $\{y_1 \geq 0, \ldots, y_l \geq 0\}$. Then*

- *the density of ρ behaves as*
$$\frac{d\rho}{d\lambda}(y) = U_\rho(y) \prod_1^s y_i^{c_i - 1},$$
and the lift of the function u behave
$$\pi^* u(y) = U_u(y) \prod_1^s y_i^{m_i}.$$

Here $c_i = c(f_i)$ and $m_i = m(f_i)$, and U_ρ, U_u are nonvanishing functions, U_ρ smooth, and U_u continuous and smooth in vicinities of the vertices of P_ϵ.

Now we could apply directly the results of [1], expressing the asymptotics of the “elementary Laplace integral”
$$\int_{y \geq 0} U(y) e^{-t \prod_i y_i^{m_i}} y_1^{c_1 - 1} \cdot \ldots \cdot y_s^{c_s - 1} dy_1 \ldots dy_s,$$
or, more directly, can analyze the poles of the Mellin transform
$$\int_y (\pi^* u(y))^z y_1^{c_1 - 1} \cdot \ldots \cdot y_s^{c_s - 1} dy_1 \ldots dy_s$$
and apply the results of [10] relating them to the Laplace integrals. One can then see that the leading terms of the asymptotics come from vicinities of the vertices of P_ϵ, leading to Theorem 2.

4 concluding remarks

- The main results of this note give an algorithm of computing the asymptotics of M_n, in the polyhedral setup. The method requires, on its face, to enumerate and to analyze all the flags of a polyhedron, the number of which grows superexponentially with the dimension. There are obvious shortcuts, and a more efficient way to find the growth rates of M_n might be quite feasible.
- What happens if the cone K is not polyhedral but rather semi-algebraic? A lot of the elements of the proofs survive; but some auxiliary results (especially Lemma 3) would need some rethinking.

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