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An extremal problem on potentially $K_{p,1,1}$-graphic sequences†

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A sequence $S$ is potentially $K_{p,1,1}$ graphical if it has a realization containing a $K_{p,1,1}$ as a subgraph, where $K_{p,1,1}$ is a complete 3-partite graph with partition sizes $p, 1, 1$. Let $\sigma(K_{p,1,1}, n)$ denote the smallest degree sum such that every $n$-term graphical sequence $S$ with $\sigma(S) \geq \sigma(K_{p,1,1}, n)$ is potentially $K_{p,1,1}$ graphical. In this paper, we prove that $\sigma(K_{p,1,1}, n) \geq 2\left\lfloor\left(\frac{(p+1)(n-1)+2}{2}\right)\right\rfloor$ for $n \geq p+2$. We conjecture that equality holds for $n \geq 2p+4$. We prove that this conjecture is true for $p = 3$.

AMS Subject Classifications: 05C07, 05C35

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1 Introduction

If $S = (d_1, d_2, \ldots, d_n)$ is a sequence of non-negative integers, then it is called graphical if there is a simple graph $G$ of order $n$, whose degree sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is precisely $S$. If $G$ is such a graph then $G$ is said to realize $S$ or be a realization of $S$. A graphical sequence $S$ is potentially $H$ graphical if there is a realization of $S$ containing $H$ as a subgraph, while $S$ is forcibly $H$ graphical if every realization of $S$ contains $H$ as a subgraph. Let $\sigma(S) = d(v_1) + d(v_2) + \ldots + d(v_n)$, and $\lfloor x \rfloor$ denote the largest integer less than or equal to $x$. We denote $G + H$ as the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Let $K_k$, and $C_k$ denote a complete graph on $k$ vertices, and a cycle on $k$ vertices, respectively. Let $K_{p,1,1}$ denote a complete 3-partite graph with partition sizes $p, 1, 1$.

Given a graph $H$, what is the maximum number of edges of a graph with $n$ vertices not containing $H$ as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [3] in 1938 and in general by Turán [12]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer $l$ such that every $n$-term graphical sequence $S$ with

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\( \sigma(S) \geq l \) is forcibly \( H \)-graphical. Here we consider the following variant: determine the minimum even integer \( l \) such that every \( n \)-term graphical sequence \( S \) with \( \sigma(S) \geq l \) is potentially \( H \)-graphical. We denote this minimum \( l \) by \( \sigma(H, n) \). Erdős, Jacobson and Lehel [4] showed that \( \sigma(K_k, n) \geq (k-2)(2n-k+1)+2 \) and conjectured that equality holds. They proved that if \( S \) does not contain zero terms, this conjecture is true for \( k = 3, n \geq 6 \). The conjecture is confirmed in [5], [7], [8], [9] and [10].

Gould, Jacobson and Lehel [5] also proved that \( \sigma(pK_2, n) = (p-1)(2n-2)+2 \) for \( p \geq 2 \); \( \sigma(C_4, n) = 2\left(\frac{3n-1}{2}\right) \) for \( n \geq 4 \). Luo [11] characterized the potentially \( K_{r,s} \)-graphic. Here we consider the following variant: determine the minimum even \( n \) for \( K_{r,s} \)-graphical. We denote \( H, n \) by \( S \), and conjectured that equality holds. They proved that if \( S \) does not contain zero terms, this conjecture is true for \( k = 3, n \geq 6 \). The conjecture is confirmed in [5], [7], [8], [9] and [10].

Lai [13] gave sufficient conditions for a graphic sequence being potentially \( K_{r,s} \)-graphic, and determined \( \sigma(K_{r,s}, n) \) for \( r = 3, 4 \). Lai [6] proved that \( \sigma(K_4-e, n) = 2\left(\frac{3n-1}{2}\right) \) for \( n \geq 7 \). In this paper, we prove that \( \sigma(K_{p,1,1}, n) \geq 2\left(\left(p+1\right)(n-1)+2\right)/2 \) for \( n \geq p + 2 \). We conjecture that equality holds for \( n \geq 2p + 4 \). We prove that this conjecture is true for \( p = 3 \).

## 2 Main results.

**Theorem 1** \( \sigma(K_{p,1,1}, n) \geq 2\left(\left(p+1\right)(n-1)+2\right)/2 \), for \( n \geq p + 2 \).

**Proof:** If \( p = 1 \), by Erdős, Jacobson and Lehel [4], \( \sigma(K_{1,1,1}, n) \geq 2n \). Then \( \sigma(K_{1,1,1}, n) \) is true.

If \( p = 2 \), by Gould, Jacobson and Lehel [5], \( \sigma(K_{2,1,1}, n) = \sigma(K_4-e, n) \geq \sigma(C_4, n) = 2\left(\frac{3n-1}{2}\right) \).

Then \( \sigma(K_{2,1,1}, n) \) is true. Then we can suppose that \( p \geq 3 \).

We first consider odd \( p \). If \( n \) is odd, let \( n = 2m + 1 \), by Theorem 9.7 of [2], \( K_{2m} \) is the union of one 1-factor \( M \) and \( m - 1 \) spanning cycles \( C_1^n, C_2^n, \ldots, C_{m-1}^{n-1} \). Let

\[
H = C_1^n \cup C_2^n \cup \ldots \cup C_{m-1}^{n-1} + K_1
\]

(1)

Then \( H \) is a realization of \( ((n-1), p^{n-1}) \), where the symbol \( x^y \) stands for \( y \) consecutive terms \( x \). Since \( K_{p,1,1} \) contains two vertices of degree \( p+1 \) while \( ((n-1), p^{n-1}) \) only contains one integer \( n-1 \) greater than degree \( p \), \( ((n-1), p^{n-1}) \) is not potentially \( K_{p,1,1} \)-graphic. Thus

\[
\sigma(K_{p,1,1}, n) \geq (n-1) + p(n-1) + 2 = 2\left(\left(p+1\right)(n-1)+2\right)/2 \).
\]

(2)

Next, if \( n \) is even, let \( n = 2m + 2 \), by Theorem 9.6 of [2], \( K_{2m+1} \) is the union of \( m \) spanning cycles \( C_1^n, C_2^n, \ldots, C_{m}^{n-1} \). Let

\[
H = C_1^n \cup C_2^n \cup \ldots \cup C_{m-1}^{n-1} + K_1
\]

(3)

Then \( H \) is a realization of \( ((n-1), p^{n-1}) \), and we are done as before. This completes the discussion for odd \( p \).

Now we consider even \( p \). If \( n \) is odd, let \( n = 2m + 1 \), by Theorem 9.7 of [2], \( K_{2m} \) is the union of one 1-factor \( M \) and \( m - 1 \) spanning cycles \( C_1^n, C_2^n, \ldots, C_{m-1}^{n-1} \). Let

\[
H = M \cup C_1^n \cup C_2^n \cup \ldots \cup C_{m-1}^{n-1} + K_1
\]

(4)

Then \( H \) is a realization of \( ((n-1), p^{n-1}) \), and we are done as before.
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Next, if $n$ is even, let $n = 2m + 2$, by Theorem 9.6 of [2], $K_{2m+1}$ is the union of $m$ spanning cycles $C_1, C_2, \ldots, C_m$. Let

$$C_1 = x_1x_2 \ldots x_{2m+1}x_1$$

$$H = (C_1^1 \cup C_2^1 \cup \ldots \cup C_m^1 + K_1) - \{x_1x_2, x_3x_4, \ldots, x_{2m-1}x_{2m}, x_2x_{m+1} \}$$

Then $H$ is a realization of $((n-1)\{p\}^2, (p-1)\{1\})$. It is easy to see that $((n-1)\{p\}^2, (p-1)\{1\})$ is not potentially $K_{p,1,1}$ graphic. Thus

$$\sigma(K_{p,1,1}, n) \geq (n-1) + p(n-2) + p - 1 + 2$$

$$= 2\left(\left(\frac{p+1}{n-1} + \frac{1}{2}\right)\right).$$

This completes the discussion for even $p$, and so finishes the proof of Theorem 1.

\[\square\]

**Theorem 2** For $n = 5$ and $n \geq 7$, $\sigma(K_{3,1,1}, n) = 4n - 2$.

For $n = 6$, if $S$ is a 6-term graphical sequence with $\sigma(S) \geq 22$, then either there is a realization of $S$ containing $K_{3,1,1}$ or $S = (4^6)$. (Thus $\sigma(K_{3,1,1}, 6) = 26$.)

**Proof:** For $n \geq 5, \sigma(K_{3,1,1}, n) \geq 2\left(\left(3 + 1\right)(n-1) + 2\right) = 4n - 2$. We need to show that if $S$ is an $n$-term graphical sequence with $\sigma(S) \geq 4n - 2$, then there is a realization of $S$ containing a $K_{3,1,1}$ (unless $S = (4^6)$). Let $d_1 \geq d_2 \geq \cdots \geq d_n$, and let $G$ be a realization of $S$.

**Case** $n = 5$: If a graph with size $q \geq 9$, then clearly it contains a $K_{3,1,1}$, so that $\sigma(K_{3,1,1}, 5) \leq 4n - 2$.

**Case** $n = 6$: If $\sigma(S) = 22$, we first consider $d_6 \leq 2$. Let $S'$ be the degree sequence of $G - v_6$, so $\sigma(S') \geq 22 - 2 \times 2 = 18$. Then $S'$ has a realization containing a $K_{3,1,1}$. Therefore $S$ has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 3$. It is easy to see that $S$ is one of $(5^3, 4^2, 3^2)$ or $(4^3, 3^2)$. Obviously, all of them are potentially $K_{3,1,1}$-graphic. Next, if $\sigma(S) = 24$, we first consider $d_6 \leq 3$. Let $S'$ be the degree sequence of $G - v_6$, so $\sigma(S') \geq 24 - 3 \times 2 = 18$. Then $S'$ has a realization containing a $K_{3,1,1}$. Therefore $S$ has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 4$. It is easy to see that $S = (4^6)$. Obviously, $(4^6)$ is graphical and $(4^6)$ is not potentially $K_{3,1,1}$ graphic. Finally, suppose that $\sigma(S) \geq 26$. We first consider $d_6 \leq 4$. Let $S'$ be the degree sequence of $G - v_6$, so $\sigma(S') \geq 26 - 2 \times 4 = 18$. Then $S'$ has a realization containing a $K_{3,1,1}$. Therefore $S$ has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 5$. It is easy to see that $S = (5^6)$. Obviously, $(5^6)$ is potentially $K_{3,1,1}$-graphic.

**Case** $n = 7$: First we assume that $\sigma(S) = 26$. Suppose $d_7 \leq 2$ and let $S'$ be the degree sequence of $G - v_7$, so $\sigma(S') \geq 26 - 2 \times 2 = 22$. Then $S'$ has a realization containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore $S$ has a realization containing a $K_{3,1,1}$ or $S = (5^1, 4^5, 1^1)$. Obviously, $(5^1, 4^5, 1^1)$ is potentially $K_{3,1,1}$-graphic. In either event, $S$ has a realization containing a $K_{3,1,1}$. Now we assume that $d_7 \geq 3$. It is easy to see that $S$ is one of $(6^1, 5^1, 3^3), (6^1, 4^2, 3^4), (5^2, 4^3, 3^1), (5^1, 4^3, 3^2)$. Obviously, all of them are potentially $K_{3,1,1}$-graphic. Next, if $\sigma(S) = 28$, Suppose $d_7 \leq 3$. Let $S'$ be the degree sequence of $G - v_7$, so $\sigma(S') \geq 28 - 3 \times 2 = 22$. Then
This finishes the inductive step, and thus Theorem 2 is established.

\( S' \) has a realization containing a \( K_{3,1,1} \) or \( S' = (4^6) \). Therefore \( S \) has a realization containing a \( K_{3,1,1} \) or \( S = (5^2, 4^4, 2^1) \). Obviously, \( (5^2, 4^2, 2^1) \) is potentially \( K_{3,1,1} \)-graphic. In either event, \( S \) has a realization containing a \( K_{3,1,1} \). Now we assume that \( d_T \geq 4 \), then \( S = (4^7) \). Clearly, \( (4^7) \) has a realization containing a \( K_{3,1,1} \). Finally, suppose that \( \sigma(S) \geq 30 \). If \( d_T \leq 4 \). Let \( S' \) be the degree sequence of \( G - v_T \), so \( \sigma(S') \geq 30 - 2 \times 4 = 22 \). Then \( S' \) has a realization containing a \( K_{3,1,1} \) or \( S' = (4^6) \). Therefore \( S \) has a realization containing a \( K_{3,1,1} \) or \( S = (5^3, 4^3, 3^1) \). Clearly, \( (5^3, 4^3, 3^1) \) has a realization containing a \( K_{3,1,1} \). Now we consider \( d_T \geq 5 \). It is easy to see that \( \sigma(S) \geq 5 \times 7 = 35 \). Obviously \( \sigma(S) \geq 36 \).

Clearly, \( S \) has a realization containing a \( K_{3,1,1} \).

We proceed by induction on \( n \). Take \( n \geq 8 \) and make the inductive assumption that for \( 7 \leq t < n \), whenever \( S_1 \) is a \( t \)-term graphical sequence such that

\[
\sigma(S_1) \geq 4t - 2
\]

then \( S_1 \) has a realization containing a \( K_{3,1,1} \). Let \( S \) be an \( n \)-term graphical sequence with \( \sigma(S) \geq 4n - 2 \). If \( d_n \leq 2 \), let \( S' \) be the degree sequence of \( G - v_n \). Then \( \sigma(S') \geq 4n - 2 - 2 \times 2 = 4(n - 1) - 2 \). By induction, \( S' \) has a realization containing a \( K_{3,1,1} \). Therefore \( S \) has a realization containing a \( K_{3,1,1} \). Hence, we may assume that \( d_n \geq 3 \). By Proposition 2 and Theorem 4 of [5] (or Theorem 3.3 of [7]) \( S \) has a realization containing a \( K_4 \). By Lemma 1 of [5], there is a realization \( G \) of \( S \) with \( v_1, v_2, v_3, v_4 \), the four vertices of highest degree containing a \( K_4 \). If \( d(v_2) = 3 \), then \( 4n - 2 \leq \sigma(S) \leq n - 1 + 3(n - 1) = 4n - 4 \). This is a contradiction. Hence, we may assume that \( d(v_2) \geq 4 \). Let \( v_1 \) be adjacent to \( v_2, v_3, v_4, y_1 \). If \( y_1 \) is adjacent to one of \( v_2, v_3, v_4 \), then \( G \) contains a \( K_{3,1,1} \). Hence, we may assume that \( y_1 \) is not adjacent to \( v_2, v_3, v_4 \). Let \( v_3 \) be adjacent to \( v_1, v_3, v_4 \). If \( y_2 \) is adjacent to one of \( v_1, v_3, v_4 \), then \( G \) contains a \( K_{3,1,1} \). Hence, we may assume that \( y_2 \) is not adjacent to \( v_1, v_3, v_4 \). Since \( d(y_1) \geq d_n \geq 3 \), there is a new vertex \( y_3 \), such that \( y_1 y_3 \in E(G) \).

**Case 1:** Suppose \( y_3 v_3 \in E(G) \). If \( y_3 v_4 \in E(G) \), then \( G \) contains a \( K_{3,1,1} \). Hence, we may assume that \( y_3 v_4 \notin E(G) \). Then the edge interchange that removes the edges \( y_1 y_3, v_3 v_4 \) and \( v_2 y_2 \) and inserts the edges \( y_1 v_2, y_3 v_1 \) and \( y_2 v_3 \) produces a realization \( G' \) of \( S \) containing a \( K_{3,1,1} \).

**Case 2:** Suppose \( y_3 v_3 \notin E(G) \). Then the edge interchange that removes the edges \( y_1 y_3, v_3 v_4 \) and \( v_2 y_2 \) and inserts the edges \( y_1 v_2, y_3 v_3 \) and \( y_2 v_4 \) produces a realization \( G' \) of \( S \) containing a \( K_{3,1,1} \).

This finishes the inductive step, and thus Theorem 2 is established.
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We make the following conjecture:

**Conjecture 1** $\sigma(K_{p,1,1}, n) = 2\left[\left(\left(p + 1\right)(n - 1) + 2\right)/2\right]$, for $n \geq 2p + 4$.

This conjecture is true for $p = 1$, by Theorem 3.5 of [4], for $p = 2$, by Theorem 1 of [6], and for $p = 3$, by the above Theorem 2.

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References


