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On subcritical multi-type branching process in random environment

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We investigate a multi-type Galton-Watson process in a random environment generated by a sequence of independent identically distributed random variables. Suppose that the associated random walk constructed by the logarithms of the Perron roots of the reproduction mean matrices has negative mean and assuming some additional conditions, we find the asymptotics of the survival probability at time n as $n \to \infty$.

Keywords: branching processes in random environment, survival probability, limit theorems, random walks

Introduction

Branching processes in random environment are natural generalization of simple Galton-Watson processes. A branching process in random environment was first considered by Smith and Wilkinson (11). The subsequent papers investigated single- and multi-type Galton-Watson processes in random environment (3)-(8). A lot of papers is devoted to the study of the survival probability of single-type branching processes in random environment (see, for example, (1)-(9)). The asymptotics of the survival probability of the subcritical branching processes in random environment generated by a sequence of independent identically distributed random variables was found in (7) for single-type processes. The paper (5) deals with the asymptotics of the survival probability for the multi-type branching processes in random environment whose associated random walks satisfy the Doney-Spitzer condition.The present paper studies a case of subcritical multi-type processes branching processes in random environment. In particular, we generalize some results from (7) and (5).

Let $Z(n) = (Z_1(n), ..., Z_p(n)),\ n = 0, 1, ...,$ be a p-type Galton-Watson branching process in a random environment. This process can be described as follows.

Let $\mathbf{N}_0 = \{0, 1, 2, ...\}$ and $\mathbf{N}_0^p$ be the set of all vectors $t = (t_1, ..., t_p)$ with non-negative integer coordinates. Denote by $(\Delta_1, \mathcal{B}(\Delta_1))$ a set of probability measures on $\mathbf{N}_0^p$ with $\sigma-$algebra $\mathcal{B}(\Delta_1)$ of Borel sets endowed with the metric of total variation, and by $(\Delta, \mathcal{B}(\Delta))$ the $p-$times product of the space $(\Delta_1, \mathcal{B}(\Delta_1))$ on itself. Let $\mathbf{F} = (\mathbf{F}^{(1)}, ..., \mathbf{F}^{(p)})$ be a random variable (random measure) taking values in $(\Delta, \mathcal{B}(\Delta))$. An infinite sequence $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, ...)$ of independent identically distributed copies of $\mathbf{F}$ is said to form a random environment and we will say that $\mathbf{F}$ generates Π.

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A sequence of random $p-$dimensional vectors $Z\left(0\right), Z\left(1\right), Z\left(2\right), ...$ with non-negative integer coordinates is called a $p-$type branching process in random environment Π, if $Z\left(0\right)$ is independent of Π and for all $n \geq 0, z = (z_1, ..., z_p) \in \mathbf{N}_0^p$ and $f_0, f_1, ... \in \Delta$

$$\begin{aligned} &\mathcal{L}(Z\left(n+1\right) \mid Z(n) = (z_1, ..., z_p), \Pi = (f_1, f_2, ...)) \\ &\quad = \mathcal{L}(Z\left(n+1\right) \mid Z(n) = (z_1, ..., z_p), \mathbf{F}_n = f_n) \\ &\quad = \mathcal{L}(\sum_{i=1}^{p} \xi_{n,1}^{(i)} + ... + \xi_{n,z_i}^{(i)}), \end{aligned} \tag{1}$$

where $f_n = (f_n^{(1)}, f_n^{(2)}, ..., f_n^{(p)}) \in \Delta$, $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, ..., \xi_{n,z_i}^{(i)}, i = 1, ..., p$, are independent $p-$dimensional random vectors, and for each $i = 1, ..., p$ the random vectors $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, ..., \xi_{n,z_i}^{(i)}$ are identically distributed according to the measure $f_n^{(i)}$. Relation (1) defines a branching Galton-Watson process $Z(n)$ in random environment which describes the evolution of a particle population $Z(n) = (Z_1(n), ..., Z_p(n)),\ n = 0, 1, ...$, where $Z_i(n), i = 1, ..., p$, is the number of type i particles in the $n-$th generation.

This population evolves as follows. If $\mathbf{F}_n = (f_n^{(1)}, ..., f_n^{(p)}) \in \Delta$ then each of the $Z_i(n)$ particles of type i existing at the time n, produces offspring in accordance with the $p-$dimensional probability measure $f_n^{(i)}$ independently of the reproduction of other particles. Thus, the i-th component of the vector $Z(n+1) = (Z_1(n+1), ..., Z_p(n+1))$ is equal to the number of type i particles among all direct descendants of the particles of the $n-$th generation.

The main results

Let J^p be the set of all column vectors $s = (s_1, ..., s_p)^T, 0 \leq s_i \leq 1, i = 1, ..., p$. Here and later on the superscript T stands for transposition. For $s = (s_1, ..., s_p)^T$ and $t = (t_1, ..., t_p) \in \mathbf{N}_0^p$ set $s^t = \prod_{i=1}^{p} s_i^{t_i}$. Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with $\mathbf{F} = (\mathbf{F}^{(1)}, ..., \mathbf{F}^{(p)})$ generating Π a random $p-$dimensional column vector $F(s) = (F^{(1)}\left(s\right), ..., F^{(p)}\left(s\right))^T$, $s \in J^p$, whose components are $p-$dimensional (random) generating functions $F^{(i)}(s)$ corresponding to $\mathbf{F}^{(i)}, 1 \leq i \leq p$:

$$F^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}^{(i)}(t) s^t, s \in J^p. \tag{2}$$

In a similar way we associate with the component $\mathbf{F}_n = (\mathbf{F}_n^{(1)}, ..., \mathbf{F}_n^{(p)}), n \geq 0$, of the random environment $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, ...)$ a random vector $F_n(s) = (F_n^{(1)}\left(s\right), ..., F_n^{(p)}\left(s\right))^T$, $s \in J^p$, the components of which are multidimensional (random) generating functions $F_n^{(i)}(s)$, corresponding to $\mathbf{F}_n^{(i)}, 1 \leq i \leq p$,

$$F_n^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}_n^{(i)}(t) s^t. \tag{3}$$

Let $e_j, j = 1, ..., p$, be the $p-$dimensional row vector whose $j-$th component is equal to 1 and the others are zeros, $\overline{0} = (0, ..., 0)$ be the $p-$dimensional row vector all whose components are zeros, and let $\overline{1} = (1, ..., 1)^T$ be the $p-$dimensional column vector all whose components are equal to 1. For $x = (x_1, ..., x_p)$

and $y = (y_1, ..., y_p)^T$ we set $|x| = \sum_{i=1}^p |x_i|, (x, y) = \sum_{i=1}^p x_i y_i$. Let $A = (A(i,j))_{i,j=1}^p$ be an arbitrary positive $p \times p$ matrix. Denote by $\rho(A)$ the Perron root of A and by $u(A) = (u_1(A), ..., u_p(A))^T$ and $v(A) = (v_1(A), ..., v_p(A))$ the right and left eigenvectors of A corresponding to the eigenvalue $\rho(A)$ and such that

$$|v(A)| = 1, \ (v(A), u(A)) = 1.$$

For vector-valued generating functions $F(s)$ and $F_n(s)$ we introduce the mean matrices

$$M = M(\mathbf{F}) = (M(i,j))_{i,j=1}^p = \left(\frac{\partial F^{(i)}(\bar{1})}{\partial s_j} \right)_{i,j=1}^p$$

and

$$M_n = M_n(\mathbf{F}_n) = (M_n(i,j))_{i,j=1}^p = \left(\frac{\partial F_n^{(i)}(\bar{1})}{\partial s_j} \right)_{i,j=1}^p .$$

Let $\mathcal{C}_\alpha, 0 < \alpha < 1$, be the class of all matrices $A = (A(i,j))_{i,j=1}^p$ such that

$$\alpha \leq \frac{A(i_1, j_1)}{A(i_2, j_2)} \leq \alpha^{-1}, \ 1 \leq i_1, i_2, j_1, j_2 \leq p.$$

One of our basic hypotheses is the following condition.

Assumption A0. There exist a number $0 < \alpha < 1$ and a row vector $v = (v_1, ..., v_p), v_i > 0, i = 1, ..., p, |v| = 1$, such that

$$M \in \mathcal{C}_\alpha,$$

and

$$v(M) = v. \tag{4}$$

Thus, all mean matrices $M_n, n \geq 0$, have a common left eigenvector v, i.e.,

$$vM_n = \rho(M_n)v.$$

Set $\rho = \rho(M), \rho_n = \rho(M_n), n \geq 0$. It is not difficult to see that in our settings $X := \ln \rho$, $X_i := \ln \rho_{i-1}, i \geq 1$, are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk $S = (S_0, S_1, ...)$, where

$$S_n = X_1 + \cdots + X_n, \ n \geq 1, \ S_0 = 0.$$

Assumption A1. Suppose that

$$\mathbf{E}(\rho \log \rho) < 0. \tag{5}$$

Note that, by means of Jensen inequality, Assumption A1 implies

$$\mathbf{E}\rho < 1, \ -\infty \leq \mathbf{E} \log \rho < 0. \tag{6}$$

Observe, that the class of multi-type branching processes in random environment satisfying Assumptions A0 and A1 is an analog of the strongly subcritical case of single-type branching processes in random environment considered in (7).

Set

$$\zeta := \sum_{t\in\mathbf{N}_0^p} \sum_{i=1}^{p} v_i \sum_{j,k=1}^{p} \mathbf{F}^{(i)}(t) t_j t_k, \ \eta = \zeta/(\rho^2 v^*), \tag{7}$$

$$\zeta_n := \sum_{t\in\mathbf{N}_0^p} \sum_{i=1}^{p} v_i \sum_{j,k=1}^{p} \mathbf{F}_n^{(i)}(t) t_j t_k, \ \eta_n = \zeta/(\rho_n^2 v^*),$$

where $v = (v_1, ..., v_p)$ is from (4), $v^* = \min(v_1, ..., v_p)$.

Introduce the random variables

$$Q^{(i)}(n) = \mathbf{P}(Z(n) \neq \vec{0}^T | Z(0) = e_i, \Pi), \ Q(n) = (Q^{(1)}(n), ..., Q^{(p)}(n))$$

and let

$$q_i(n) = \mathbf{P}(Z(n) \neq \vec{0}^T | Z(0) = e_i) = \mathbf{E}Q^{(i)}(n).$$

Note that under Assumptions A0 and A1 $Q^{(i)}(n) \to 0$ a.s. as $n \to \infty$ for all $1 \le i \le p$, since in view of (6)

$$(v, Q(n)) \le \min_{0\le k\le n-1} |vM_0 \cdots M_k| \le \exp\{\min_{0\le k\le n-1} S_k\} \to 0$$

as $n \to \infty$. Denote by $u(n) = (u_1(n), ..., u_p(n))^T := u(M_0 \cdots M_n), n \ge 0$, the right eigenvector of the product $M_0 \cdots M_n$, corresponding to the Perron root $\rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n$.

The main result of the paper is the following statement.

Theorem 1 *Suppose that Assumptions A0 and A1 are valid, and*

$$\mathbf{E}(\rho \log^+ \zeta) < \infty, \tag{8}$$

where ζ is from (7). Then, as $n \to \infty$,

$$q_i(n) \sim c_i(\mathbf{E}\rho)^n, \ c_i > 0, \ i = 1, ..., p. \tag{9}$$

In conclusion of this section we give an example where condition (8) is fulfilled.

Since

$$x \log^+(y/x^2) \le x \log^+ y + 2 \sup_{t>0} t \log(1/t) \le x \log^+ y + 2/e, \ x > 0, y > 0,$$

we see that if the measure $\mathbf{F}$ generating our random environment has a bounded support, i.e., there exists a $p-$dimensional cube

$$K = \{t = (t_1, ..., t_p) \in \mathbf{N}_0^p, \ 0 \le t_i \le b, b > 0, 1 \le i \le p\},$$

such that $\mathbf{P}(\mathbf{F}(K) = 1) = 1$, then condition (8) holds.

1 Auxiliary results

The following assertion for the asymptotics of $u(n)$ has been obtained in (5).

Theorem 2 *If Assumption A0 is valid, then there exist a random vector $u = (u_1, ..., u_p)^T$ and a function $g(n) \geq 0, g(n) \to 0, n \to \infty$, such that with probability 1*

$$|u_i(n) - u_i| \leq g(n),\ i = 1, ..., p.$$

In addition,

$$(v, u) = 1,\ \alpha \leq u_i \leq 1/v^*,$$

$v^* = \min(v_1, ..., v_p)$ *and* $v = (v_1, ..., v_p)$ *is from (4).*

The following theorem describes the behavior of $Q_i(n)$ as $n \to \infty$.

Theorem 3 *Assume Assumption A0 and let*

$$\mathbf{E} \log \rho < 0. \tag{10}$$

Then with probabilty 1, as $n \to \infty$,

$$\frac{Q_i(n)}{(v, Q(n))} \to u_i,\ i = 1, ..., p, \tag{11}$$

where $u = (u_1, ..., u_p)^T$ is from Theorem 2.

Note that relation (11) has been proved in (5) for the case when Assumption A0 and condition

$$\mathbf{P}(S_n > 0) \to a,\ \ n \to \infty, \tag{12}$$

where $0 < a < 1$, are valid (see Theorem 2 in (5)). The arguments presented in (5) are still valid if we replace condition (12) by (10).

We need the following notations. For $s \in J^p$ set

$$F_{k,n}(s) = F_k(F_{k+1}(...F_{n-1}(s))),\ 0 \leq k < n-1,\ F_{n,n}(s) = s,$$

$$F_{k,n}(s) = F_{k-1}(F_{k-2}(...F_n(s))),\ k > n \geq 0.$$

It is not difficult to see that $Q(n) = \vec{1} - F_{0,n}(\vec{0})$ and

$$\frac{1}{(v, \vec{1} - F_{0,n}(s))} = \frac{e^{-S_0}}{(v, \vec{1} - F_{0,n}(s))}$$

$$= \frac{e^{-S_n}}{(v, \vec{1} - F_{n,n}(s))} + \sum_{k=0}^{n-1} \left(\frac{e^{-S_k}}{(v, \vec{1} - F_{k,n}(s))} - \frac{e^{-S_{k+1}}}{(v, \vec{1} - F_{k+1,n}(s))} \right)$$

$$= \frac{e^{-S_n}}{(v, \vec{1} - s)} + \sum_{k=0}^{n-1} e^{-S_k} g_k(F_{k+1,n}(s)),\ s \in J^p, \tag{13}$$

where

$$g_k(s) := \frac{1}{(v, \vec{1} - F_k(s))} - \frac{1}{\rho_k(v, \vec{1} - s)},\ k \geq 0,\ s \in J^p. \tag{14}$$

The following bound for $g_n(s), n \geq 0$, is from Lemma 1 in (5)

Lemma 1 *Assume Assumption A0. Then*

$$0 \le g_n(s) \le \eta_n,\ n \ge 0,\ s \in J^p.$$

We need also the following statement.

Lemma 2 *If Assumption A0 and condition (10) are valid, then, as* $n \to \infty$*,*

$$\mathbf{E}Q^{(i)}(n) \sim \mathbf{E}(u_i(n)(v, Q(n))),\ i = 1, ..., p.$$

The proof of the lemma is elementary (it is based on Theorems 2, 3) and Fatou's lemma and we omit it.

2 Proof of the main result

We need some additional notation. Let Δ_2 be the set of all possible tuples $f(s) = (f^{(1)}(s), ..., f^{(p)}(s)), s \in J^p$, where $f^{(i)}(s), i = 1, ..., p$, are $p-$dimensional probability generating functions.

In fact we identify the set Δ_2 with the space $(\Delta, \mathcal{B}(\Delta))$. Since Assumption A1 yields $\mathbf{E}\rho < 1$ we can introduce the random vector

$$\bar{F}(s) = (\bar{F}^{(1)}(s), ..., \bar{F}^{(p)}(s))^T,\ s \in J^p,$$

where $\bar{F}^{(i)}(s),\ i = 1, ..., p$, are multidimensional (random) generating functions such that

$$\mathbf{E}\varphi(\bar{F}(s)) = \frac{\mathbf{E}(\rho\varphi(F(s))}{\mathbf{E}\rho} \tag{15}$$

for every non-negative measurable function φ on Δ_2. (Recall, that $F(s)$ was defined by (2)). In a similar way we introduce, for $n \ge 0$,

$$\bar{F}_n(s) = (\bar{F}_n^{(1)}(s), ..., \bar{F}_n^{(p)}(s))^T,\ s \in J^p,$$

by the relation

$$\mathbf{E}\varphi(\bar{F}_n(s)) = \frac{\mathbf{E}(\rho\varphi(F_n(s)))}{\mathbf{E}\rho},$$

where $F_n(s)$ was defined by (3). It is not difficult to see that $\bar{F}_0(s), ..., \bar{F}_{n-1}(s)$ are i.i.d. copies of $\bar{F}(s)$ and

$$\mathbf{E}\varphi(\bar{F}_0(s), ..., \bar{F}_{n-1}(s)) = \frac{\mathbf{E}(\exp(S_n)\varphi(F_0(s), ..., F_{n-1}(s)))}{(\mathbf{E}\rho)^n},$$

where φ is from (15).

Proof of Theorem 1. In view of Lemma 2 to prove Theorem 1 it is sufficient to show that

$$\mathbf{E}(u_i(n)(v, \vec{1} - F_{0,n}(\vec{0}))) \sim c_i(\mathbf{E}\rho)^n,\ c_i > 0,\ i = 1, ..., p.$$

Fix $i = 1, ..., p$ and denote by

$$u(n, 0) = (u_1(n, 0), ..., u_p(n, 0))^T := u(M_{n-1}M_{n-2} \cdots M_0),\ n \ge 1,$$

the right eigenvector of the product $M_{n-1}M_{n-2}\cdots M_0$, corresponding to the Perron root $\rho(M_{n-1}M_{n-2}\cdots M_0)=\rho_{n-1}\cdots\rho_0$. Using relation

$$\mathbf{E}(u_i(n)(v,\vec{1}-F_{0,n}(\vec{0})))=\mathbf{E}(u_i(n,0)(v,\vec{1}-F_{n,0}(\vec{0})))$$

and setting

$$\varphi(F_0(s),...,F_{n-1}(s))=\exp(-S_n)(v,\vec{1}-F_{n,0}(s))$$

in (15) we have

$$\mathbf{E}(u_i(n)(v,\vec{1}-F_{0,n}(\vec{0})))=(\mathbf{E}\rho)^n\mathbf{E}(\bar{u}_i(n,0)\exp(-\bar{S}_n)(v,\vec{1}-\bar{F}_{n,0}(\vec{0}))),$$

where $\bar{u}_i(n,0),\bar{S}_n,\bar{F}_{n,0}$, etc. are the analogues of $u_i(n,0),S_n,F_{n,0}$, defined in the terms of the $\bar{F}_i, i\geq 0$. For instance, $\bar{S}_n=\sum_{i=0}^{n-1}\log\bar{\rho}_i$. Replacing F_k by $\bar{F}_k$ in (13) we have

$$\exp(-\bar{S}_n)(v,\vec{1}-\bar{F}_{n,0}(\vec{0}))=(1+\sum_{k=1}^{n}\bar{\zeta}_{k-1}\exp(\bar{S}_k))^{-1},$$

where $\bar{\zeta}_k=\bar{g}_k(\bar{F}_{k,0}(\vec{0})), k\geq 0$. Hence

$$\mathbf{E}(u_i(n)(v,\vec{1}-F_{0,n}(\vec{0})))=(\mathbf{E}\rho)^n\mathbf{E}(\bar{u}_i(n,0)(1+\sum_{k=1}^{n}\bar{\zeta}_{k-1}\exp(\bar{S}_k))^{-1}).$$

Observe that $\lim_{n\to\infty}\bar{u}_i(n,0)=\lim_{n\to\infty}\ u_i(n)=u_i$ a.s. by Theorem 2. Thus, to prove (9) it suffices to show that with probability 1

$$\sum_{k=1}^{\infty}\bar{\zeta}_{k-1}\exp(\bar{S}_k)<\infty. \tag{16}$$

By Lemma 1

$$\bar{\zeta}_k\leq\bar{\eta}_k:=\sum_{t\in\mathbf{N}_0^p}\sum_{i=1}^{p}v_i\sum_{j,l=1}^{p}\bar{\mathbf{F}}_k^{(i)}(t)t_jt_l/(\bar{\rho}_k^2v^*),\ k\geq 0.$$

Condition (8) implies, that $\bar{S}_n, n\geq 0$, is a random walk with negative drift

$$\mathbf{E}\log\bar{\rho}=\frac{\mathbf{E}(\rho\log\rho)}{\mathbf{E}\rho}<0.$$

Since by the strong law of large numbers $\lim_{n\to\infty}\bar{S}_n/n=\mathbf{E}\log\bar{\rho}$ a.s., to prove (16) it suffices to show that

$$\lim_{n\to\infty}\sup\frac{\log^+\bar{\eta}_n}{n}=0\ \ \textit{a.s.} \tag{17}$$

Using (15), (8), and the estimate $\sup_{x>0}(x\log 1/x)=e^{-1}$, it is not difficult to check that

$$\begin{aligned}\mathbf{E}\log^+\bar{\eta}_n&=\mathbf{E}\log^+\frac{\bar{\zeta}_n}{\bar{\rho}_n^2}\leq\frac{1}{\mathbf{E}\rho}+\mathbf{E}\log^+\bar{\zeta}_n\\&=\frac{1}{\mathbf{E}\rho}+\frac{\mathbf{E}(\rho\log^+\bar{\zeta})}{\mathbf{E}\rho}<\infty.\end{aligned} \tag{18}$$

Relation (17) follows from (18). This entails the desired result.

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