Fifth Colloquium on Mathematics and Computer Science DMTCS proc. **AI**, 2008, 519–530

The size of random fragmentation intervals

Rafik Aguech†

Faculté des Sciences de Monastir, Département de Mathématiques, 5019 Monastir, Tunisia.

Two processes of random fragmentation of an interval are investigated. For each of them, there is a splitting probability at each step of the fragmentation process whose overall effect is to stabilize the global number of splitting events. More precisely, we consider two models. In the first model, the fragmentation stops which a probability p witch can not depend on the fragment size. The number of stable fragments with sizes less than a given $t \geq 0$, denoted by $K(t)$, is introduced and studied.
In the second one the probability to split a fragment of size x is $p(x) = 1 - e^{-x}$. For this model we utilize the contraction method to show that the distribution of a suitably normalized version of the number of stable fragments converges in law. It's shown that the limit is the fixed-point solution (in the Wasserstein space) to a distributional equation. An explicit solution to the fixed-point equation is easily verified to be Gaussian.

Keywords: Fragmentation models, fixed point, contraction method, Mellin transform.

1 Introduction

Fragmentation is a widely studied phenomena [12] with applications ranging from conventional fracture of solids [6] and collision induced fragmentation in atomic nuclei/ aggregates [1] to seemingly unrelated fields such as disordered systems [4] and geology. Fragmentation processes are also relevant to spin glasses, Boolean networks and genetic population.

We investigate two classes of stochastic fragmentation processes. In section 2 we consider a random fragmentation process introduced by Krapivsky, Ben-Naim and Grosse [5] where fragmentation stops stochastically, with a probability q of further fragmentation that not depend on the mass x of the fragment. In section 3, we consider the random fragmentation process investigated by Janson and Neininger [16], where splitting probability is, by nature, fragments length dependent. A particle having some mass x is broken, with probability $p(x) = 1 - e^{-x}$, into two pieces. The mass is distributed among the pieces at random in such a way that the proportions of the mass shared among different daughters are specified by some given probability distribution (the dislocation law). The process of fragmentation is repeated recursively for all pieces. For this model we derive a closed form expression of the mean and variance of the total number of stable fragments $N(x)$. Finally, using the contraction method in continuous times established by Janson Neininger [16], we prove a central limit Theorem of $N(x)$.

†Research supported by the Ministry of higher education, scientific research and technology (research units: 99/UR/15-10 and UR04DN04)

2 Homogeneous random fragmentation process

Let $(\xi_1, \xi_2, \cdots, \xi_m)$ be a given random vector. We assume that each ξ_j has a distribution that is absolutely continuous on $(0, 1)$ and

$$\sum_{j=1}^{m} \xi_j \overset{a.s}{=} 1,$$

i.e., that $(\xi_1, \cdots, \xi_m)$ belongs to the simplex.

We start with an interval of length 1. At step one, there is a probability $p \in (0, 1)$ of splitting the interval and so, with probability $q := 1 - p$ the initial unit fragment remains unchanged for ever. In the case of splitting, we obtain m ($m \geq 2$) fragments with random sizes $(\xi_1, \cdots, \xi_m)$. On each first generation sub-fragment, the splitting process is then iterated, independently, a property which we refer to in the sequel as the renewal structure of the process. Similar processes have been investigated by Krapivsky, Ben-Naim and Grosse [5]. The term homogeneous refers to the fact that in such models, the splitting probability is independent of fragment sizes at each step. This model can be understood from the discrete time binary Galton-Watson branching process.

Let N be the total number of stable fragments.

2.1 The mean

By the recursive nature of the process, we can write, almost surely

$$N = \sum_{j=1}^{m} 1\!\!\mathrm{I}_{\{j^{th}\ stable\}} + \sum_{j=1}^{m} 1\!\!\mathrm{I}_{\{j^{th}\ unstable\}} N^{(j)}, \tag{1}$$

where $(N^{(j)})_{\{1 \leq j \leq m\}}$ are independent copies of N and independent of $(\xi_1, \xi_2, \cdots, \xi_m)$.
Hence, $\mathbf{E}(N)$ the mean of N is given by the following proposition

Proposition 1 .

$$\mathbf{E}(N) = \begin{cases} \dfrac{mq}{1 - mp}, & \text{if } p < \dfrac{1}{m} \\ \\ +\infty, & \text{if } p \geq \dfrac{1}{m}. \end{cases}$$

The average total number of fragments diverges as the probability p *approches the critical point* $p_c = 1/m$, *reflecting the critical nature of the corresponding branching process.*

Remark: Let $\varphi_N(t) = \mathbf{E}(e^{tN})$, $t \in]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$.

To obtain moment of order k, it suffices to derive k times φ_N.
Using equation (1), one can note that, the function $\varphi_N(t)$ is the solution of the equation

$$\varphi_N(t) = \Big(qe^t + p\varphi_N(t)\Big)^m, \quad |t| < \varepsilon.$$

2.2 Fragment length density

For $t \in [0,1]$, let $K(t)$ be the total number of stable fragments of length less than t, and $M(t) = \mathbf{E}(K(t))$. For $y \geq 0$ let $K(y,t)$ be the total number of stable fragments of length less than t given by a fragment of length y and let $M(y,t) = \mathbf{E}(K(y,t))$. The recursive nature of the process can be used to obtain

$$\begin{aligned} K(t) &= \sum_{j=1}^{m} 1\!\!1_{\{j^{th}\ stable\}} 1\!\!1_{\{\xi_j \leq t\}} + \sum_{j=1}^{m} 1\!\!1_{\{j^{th}\ unstable\}} K(\xi_j, t) \\ &= \sum_{j=1}^{m} 1\!\!1_{\{j^{th}\ stable\}} 1\!\!1_{\{\xi_j \leq t\}} + \sum_{j=1}^{m} 1\!\!1_{\{j^{th}\ unstable\}} 1\!\!1_{\{\xi_j \leq t\}} K(\xi_j, t) \\ &\quad + \sum_{j=1}^{m} 1\!\!1_{\{j^{th}\ unstable\}} 1\!\!1_{\{\xi_j > t\}} K(\xi_j, t). \end{aligned}$$

Moreover, for every $s,\ t \in [0,1]$, we have

$$M(s,t) = \begin{cases} M(1) = \mathbf{E}(N), & \text{if } s \leq t \\ \\ M(1, \dfrac{t}{s}) = M(\dfrac{t}{s}), & \text{if } s \geq t. \end{cases}$$

Consequently,

$$M(t) = \frac{\mathbf{E}(N)}{m} \sum_{j=1}^{m} F_j(t) + p \sum_{j=1}^{m} \int_t^1 M(\frac{t}{s}) f_j(s) ds, \tag{2}$$

where F_j (respectively f_j) is the distribution function (respectively the density) of the random ξ_j. With the help of the Mellin transform, Equation (2) can be solved, and we get:

$$\widehat{M}(s) = \int_0^{+\infty} t^{s-1} M(t) dt.$$

It is clear that $M(t)$ is well defined and piecewise-continuous verifying

$$\forall t \in [0,1],\ M(t) \in [0, E(N)].$$

Then the Mellin transform of M exists in the half of complex plane

$$\mathcal{S} = \{s \in \mathbb{C},\ \mathcal{R}e(s) > 0\}.$$

Denoting, for each $j \leq m$, by $\widehat{f_j}$ the Mellin transform of f_j. The Mellin transform of M reads

$$\widehat{M}(s) = \frac{\mathbf{E}(N)\Big(1 - \frac{1}{m}\sum_{j=1}^{m} \hat{f}_j(s+1)\Big)}{s(1 - p\sum_{j=1}^{m} \hat{f}_j(s+1))},\ \forall s \in \mathcal{S}. \tag{3}$$

Using the Mellin inverse, we deduce that, for all $c > 0$ and $t \in [0,1]$

$$M(t) = \frac{1}{2i\pi} \int_{c-\infty}^{c+i\infty} \widehat{M}(s) t^{-s} ds.$$

2.3 Applications

2.3.1 Uniform m fragmentations

In this subsection each unstable fragment with size l splits into m fragments, using $m-1$ independent, uniformly distributed cut points in the interval $[0, l]$.

$$\begin{aligned} K(t) &= \sum_{j=1}^{m} 1\!\mathrm{I}_{\{j^{th}\, stable\}} 1\!\mathrm{I}_{\{\xi_j \le t\}} + \sum_{j=1}^{m} 1\!\mathrm{I}_{\{j^{th}\, unstable\}} K(\xi_j, t) \\ &= \sum_{j=1}^{m} 1\!\mathrm{I}_{\{j^{th}\, stable\}} 1\!\mathrm{I}_{\{\xi_j \le t\}} + \sum_{j=1}^{m} 1\!\mathrm{I}_{\{j^{th}\, unstable\}} 1\!\mathrm{I}_{\{\xi_j \le t\}} K(\xi_j, t) \\ &\quad + \sum_{j=1}^{m} 1\!\mathrm{I}_{\{j^{th}\, unstable\}} 1\!\mathrm{I}_{\{\xi_j > t\}} K(\xi_j, t). \end{aligned}$$

Let ξ be a generic spacing distributed like ξ_1. It is known that ξ has the density

$$f(s) = f_\xi(s) = (m-1)(1-s)^{m-2}.$$

Thus

$$\begin{aligned} M(t) &= \mathbf{E}(N)\mathbf{Prob}(\xi \le t) + m(m-1)p \int_t^1 M(t/s)(1-s)^{m-2} ds \\ &= E(N)\Big[1-(1-t)^{m-1}\Big] + m(m-1)pt \int_t^1 \frac{M(y)}{y^m}(y-t)^{m-2} dy. \end{aligned} \tag{4}$$

From (3) one can obtain that, for all $s \in \mathcal{S}$

$$\widehat{M}(s) = \frac{\mathbf{E}(N)\Big((s+1)\cdots(s+m-1)-(m-1)!\Big)}{s\Big((s+1)\cdots(s+m-1)-pm!\Big)}. \tag{5}$$

2.3.2 Uniform 3-fragmentation case

In the particular case, when $m=3$, we get an explicit expression for $M(t)$. In fact from equation (4), we get

$$\begin{aligned} M(t) &= 3q\mathbf{Prob}(\xi_1 \le t) + 6p\int_0^t M(s,t)(1-s)ds + 6pt\int_t^1 M(s,t)(1-s)ds \\ &= 3q\mathbf{Prob}(\xi_1 \le t) + 6p\mathbf{E}(N)\int_0^t (1-s)ds + 6p\int_t^1 M(t/s)(1-s)ds \\ &= 3qt(2-t) + 6pt(1-\frac{t}{2})\mathbf{E}(N) + 6pt\int_t^1 \frac{M(y)}{y^2}dy - 6pt^2\int_t^1 \frac{M(y)}{y^3}dy. \end{aligned} \tag{6}$$

Let $P(t)$ be the fragment length density of stable fragments of length t. It follows

$$M(t) = \int_0^t P(x)dx.$$

Due to the last equality and (6), it follows

$$P(t) = 6q - 6qt + 6p\int_t^1 \frac{P(s)}{s}ds - 6pt\int_t^1 \frac{P(s)}{s^2}ds. \tag{7}$$

Then one can observe that the total length is conserved,

$$\int_0^1 tP(t) = 1.$$

Furthermore by integrating (7) we get the value of $\mathbf{E}(N)$ already given by Proposition 1,

$$\int_0^1 P(t)dt = \mathbf{E}(N).$$

The Mellin transformation $\widehat{M}$ can be obtained, in this case from (5) choosing $m = 3$

$$\widehat{M}(s) = \frac{\mathbf{E}(N)(s+3)}{(s+1)(s+2) - 6p}.$$

Using the Mellin inverse, it leads to following Theorem

Theorem 1 *For all* $t \in [0,1]$,

$$M(t) = \begin{cases} \dfrac{\mathbf{E}(N)}{\sqrt{1+24p}} t^{\frac{3+\sqrt{1+24p}}{2}} \Big[\dfrac{3+\sqrt{1+24p}}{2} t^{-\sqrt{1+24p}} + \dfrac{\sqrt{1+24p}-3}{2}\Big], & \textit{if } p < \dfrac{1}{3} \\ +\infty, & \textit{if } p \geq \dfrac{1}{3}. \end{cases}$$

Remark: Denoting by $\lambda(s,t)$, for all $0 \leq s \leq t \leq 1$, the mean of the number of stable fragments with length between s and t. It can be given by

$$\lambda(s,t) := M(t) - M(s).$$

3 Exponential fragmentation probability

Throughout this section we assume that the first segment is of length x (large enough) and the probability $p(s)$ that a new fragment remains unstable depends on the fragment size s. This is relevant for impact fragmentation and DNA segmentation where fragments have an intrinsic size scale below which the fragmentation probability becomes negligible.

If $p(s) = 1\!\!\mathrm{I}_{s \geq 1},\ m = 2$ and ξ is uniform on $[0,1]$, this model has been studied by Sibuya and Itoh [15]. Recently Janson and Neininger [16] studied the general case where $p(s) = 1\!\!\mathrm{I}_{s \geq 1},\ m \geq 2$ and the support of the distribution of $(\xi_1, \xi_2, \cdots, \xi_m)$ on the standard simplex has an interior point.

In our model, we assume that, $m = 2,\ \xi_1$ is uniform on $[0,1]$ and

$$p(s) = 1 - e^{-s}.$$

It is natural to consider the fragmentation process as a tree $\mathcal{T}(\infty)$, with the root representing the original object (first fragment of size x), its children representing the pieces of the first fragmentation, and so on. We let the fragmentation go on for ever, although we ignore what happens to stable pieces. Let us mark each node with the mass of the corresponding object. A node is unstable if the corresponding fragment is unstable. We define the fragmentation tree $\mathcal{T}(x)$ to be the subtree of $\mathcal{T}(\infty)$ consisting of all unstable nodes.

As mentioned in [16], our model can help to approximate the binary search tree by a fragmentation tree. In fact, for binary search trees, we have n random (uniformly distributed) points in an interval, split the interval by the first of these points, and continue recursively splitting each subinterval that contains at least one of the points. If we scale the initial interval to have length n, then the probability that a subinterval of length x contains at least one point is $\approx 1 - e^{-x}$.

Let $N(x)$ be the total number of stable fragments when we start with an interval of length x $\big(N(x)$ is also the number of nodes in $\mathcal{T}(x)\big)$ and

$$m(x) = \mathbf{E}(N(x)),\ V(x) = \mathbf{Var}(N(x)).$$

By the recursive construction of the fragmentation process, we have, almost surely

$$N(x) = 1\!\mathrm{I}_{\{x\ stable\}} + 1\!\mathrm{I}_{\{x\ unstable\}} \sum_{j=1}^{2} N^{(j)}(\xi_j x), \tag{8}$$

where $(N^{(j)}(.))_{j\geq 1}$ are copies of $N(.)$, independent of each other and of (ξ_1, ξ_2).

3.1 The mean

By equation (8), the mean $m(x)$ satisfies

$$m(x) = e^{-x} + \frac{2(1-e^{-x})}{x} \int_0^x m(t)dt. \tag{9}$$

By the nature of the process, we have $\lim_{x\to 0} m(x) = 1$. The solution of the equation (9) can be written as

$$\begin{aligned} m(x) &= \varphi(x)\Big[\int_0^x \Big\{[e^{-t} - \frac{2(1-e^{-t})}{t}]\exp(-\int_0^t \frac{2(1-e^{-s})}{s}ds)\Big\}dt\Big] \\ &\quad + e^{-x} - \frac{2(1-e^{-x})}{x} + \varphi(x), \end{aligned}$$

where

$$\varphi(x) = \frac{2(1-e^{-x})}{x} \exp\Big(\int_0^x \frac{2(1-e^{-t})}{t}dt\Big).$$

Consequently

Theorem 2 *As $x \to +\infty$:*

$$m(x) = 2\gamma x + O(1), \tag{10}$$

with

$$\begin{aligned}\gamma &= \exp\Big[\int_0^1 \frac{2(1-e^{-t})}{t}dt\Big]\exp\Big[-2\int_1^{+\infty}\frac{e^{-t}}{t}dt\Big] \\ &+ \int_0^{+\infty}\Big\{\frac{1}{t^2}\Big[e^{-t}-\frac{2(1-e^{-t})}{t}\Big]\exp\Big[-2\int_t^{+\infty}\frac{e^{-s}}{s}ds\Big]\Big\}dt.\end{aligned}$$

Remark: If each unstable interval is divided at $(b-1)$ randomly chosen points, that is, $(b-1)$ independent points from the uniform $[0,x]$ distribution, in this case the mean function $m(x)$ satisfy the following differential equation

$$\frac{d^{b-1}}{dx^{b-1}}\Big(x^{b-1}m(x)\Big) - b!(1-e^{-x})m(x) = b!e^{-x}.$$

3.2 The variance

Let $S(x) = 1\!\!\mathrm{I}_{\{x\ stable\}}$ and $V(x) = \mathbf{Var}(N(x))$. In view of (8)

$$\begin{aligned}V(x) &= (e^{-x}-e^{-2x}) + 2\mathbf{Var}\Big(S(x)N(\xi x)\Big) - 4e^{-x}(1-e^{-x})\mathbf{E}\Big[m(\xi x)\Big] \\ &\quad +2(1-e^{-x})\mathbf{E}\Big[m(\xi x)m((1-\xi x))\Big] - 2(1-e^{-x})^2\Big(\mathbf{E}[m(\xi x)]\Big)^2 \\ &= (e^{-x}-e^{-2x}) + 2(1-e^{-x})\mathbf{E}\Big[N^2(\xi x)\Big] - 2(1-e^{-x})^2\Big(\mathbf{E}\Big[m(\xi x)\Big]\Big)^2 \\ &\quad +2(1-e^{-x})\mathbf{E}\Big[m(\xi x)m((1-\xi)x)\Big] - 4e^{-x}(1-e^{-x})\mathbf{E}\Big[m(\xi x)\Big] - 2(1-e^{-x})^2\Big(\mathbf{E}[m(\xi x)]\Big)^2\end{aligned}$$

Finally, $V(x)$, satisfies

$$V(x) = \frac{2(1-e^{-x})}{x}\int_0^x V(t)dt + h(x), \tag{11}$$

where

$$\begin{aligned}h(x) &= 2(1-e^{-x})\mathbf{E}\Big[m(\xi x)m((1-\xi)x)\Big] \\ &+ (e^{-x}-e^{-2x}) + \frac{(2e^{-x}-1)e^{-x}(m(x)-e^{-x})^2}{2(1-e^{-x})} - 2e^{-x}(m(x)-e^{-x}).\end{aligned}$$

Let $H(t) = h(t) - \dfrac{2(1-e^{-t})}{t}$. Then, the solution of equation (11) can be given by

$$\begin{aligned}V(x) &= \frac{2(1-e^{-x})}{x}\exp\Big(\int_0^x \frac{2(1-e^{-t})}{t}dt\Big) + H(x) \\ &+ \frac{2(1-e^{-x})}{x}\exp\Big(\int_0^x \frac{2(1-e^{-t})}{t}dt\Big)\Big[\int_0^x H(t)\exp\Big(-\int_0^t \frac{2(1-e^{-s})}{s}ds\Big)dt\Big].\end{aligned}$$

Consequently, we have the following Theorem, which describes an asymptotic expression of $V(x)$, for large values of x

Theorem 3 *Asymptotically, as x large enough, we have*

$$\mathbf{Var}(N(x)) = 2\lambda x - \frac{2}{x} + O(e^{-x/2}), \tag{12}$$

where

$$\lambda = \exp\Big(\int_0^1 (\frac{1-e^{-t}}{t})dt - \int_1^{+\infty} \frac{e^{-t}}{t}dt\Big) + \int_0^{+\infty} \frac{H(t)}{t^2} \exp\Big(-2\int_t^{+\infty} \frac{e^{-s}}{s}ds\Big)dt.$$

3.3 Limit law

Let

$$N_*(x) = \frac{N(x)-m(x)}{\sqrt{V(x)}}.$$

Based on some heuristics in the structure of the problem, a solution is guessed for the limit distribution of $N_*(x)$. The guess is then verified by showing convergence of the distribution function to that of the guessed limit in some metric space. The contraction method was introduced by Rï£¡sler [14]. Rachev and Rï£¡schendorf [9] added several useful extensions. General contraction theorems and multivariate extensions were added by Rï£¡sler [11], and Neininger [8]. Rï£¡sler and Rï£¡schendorf [13] provide a valuable survey. Recently general contraction theorems and multivariate extensions were added, with continuous parameter, by Janson and Neininger [16].

By the recursive decomposition of the process $N(x)$, the mean $m(x)$ satisfies

$$m(x) = e^{-x} + (1-e^{-x})\sum_{j=1}^{2} \mathbf{E}\Big(m(\xi_j x)\Big). \tag{13}$$

We start from the recursive decomposition (8), adapted in the form (using (13))

$$\begin{aligned} \frac{N(x)-m(x)}{\sqrt{V(x)}} \quad &= \quad \frac{1\!\!1_{\{x\, stable\}} - e^{-x}}{\sqrt{V(x)}} \\ &+ 1\!\!1_{\{x\, instable\}} \frac{N^{(1)}(\xi_1 x) - m(\xi_1 x)}{\sqrt{V(\xi_1 x)}} \sqrt{\frac{V(\xi_1 x)}{V(x)}} \\ &+ 1\!\!1_{\{x\, instable\}} \frac{N^{(2)}(\xi_2 x) - m(\xi_2 x)}{\sqrt{V(\xi_2 x)}} \sqrt{\frac{V(\xi_2 x)}{V(x)}} \\ &+ \frac{\mathbf{E}\Big[N^{(1)}(\xi_1 x) + N^{(2)}(\xi_2 x)\Big]}{\sqrt{V(x)}} \Big[1\!\!1_{\{x\, instable\}} - (1-e^{-x})\Big], \end{aligned}$$

where ξ is a random variable with law $\mathcal{U}_{[0,1]}$. The last equation can be written as

$$N_*(x) = \sum_{j=1}^{2} N_*^{(j)}(\xi_j x)\sqrt{\frac{V(\xi_j x)}{V(x)}} + D(x), \tag{14}$$

where

$$\begin{aligned} D(x) &= 1\!\!\mathrm{I}_{\{x\, stable\}} \sum_{j=1}^{2} N_*^{(j)}(\xi_j x)\sqrt{\frac{V(\xi_j x)}{V(x)}} + \frac{1\!\!\mathrm{I}_{\{x\, stable\}} - e^{-x}}{\sqrt{V(x)}} \\ &+ \frac{\mathbf{E}\Big[N^{(1)}(\xi x) + N^{(2)}((1-\xi)x)\Big]}{\sqrt{V(x)}}\Big[1\!\!\mathrm{I}_{\{x\, unstable\}} - (1-e^{-x})\Big]. \end{aligned}$$

In order to apply Theorem 5.1 [16], we introduce the map T on the space $\mathcal{M}$ of probability mesures on $\mathbb{R}$ by

$$\begin{aligned} T \quad &: \quad \mathcal{M} \longrightarrow \mathcal{M} \\ T(\mu) \quad &\stackrel{\mathcal{D}}{=} \quad \sum_{j=1}^{2} \sqrt{\xi_j} X^{(j)}, \end{aligned} \tag{15}$$

where $X^{(j)} \stackrel{\mathcal{D}}{=} \mu$ for $j = 1, 2$. We need the following result

Proposition 2 *As $x \longrightarrow +\infty$, we have*

(i) $D(x) \xrightarrow{L^3} 0.$

(ii) $\sqrt{\dfrac{V(\xi_j x)}{V(x)}} \xrightarrow{L^3} \sqrt{\xi_j},\ j = 1, 2.$

(iii) $\displaystyle\sum_{j=1}^{2} \mathbf{E}\Big(\sqrt{\xi_j}^3\Big) = \frac{4}{5}.$

To prove this Proposition we need the following Lemma

Lemma 1 *For all $x > 0$, $\mathbf{E}\Big([N(x)]^3\Big) < \infty$. Furthermore $\sup\limits_{0\le y\le x} \mathbf{E}\Big([N(y)]^3\Big) < \infty$.*

Proof:

For $j = 2, 3$, let $m_j(x) = \mathbf{E}([N(x)]^j)$ and

$$L(x) = 6(1-e^{-x})\mathbf{E}\Big[m_2(\xi x)m((1-\xi)x)\Big] - \frac{2(1-e^{-x})}{x}.$$

By equation (8),

$$\begin{aligned} m_3(x) &= L(x) + \frac{(2(1-e^{-x})}{x}\exp\Big(\int_0^x \frac{2(1-e^{-t})}{t}dt\Big) \\ &+ \frac{2(1-e^{-x})}{x}\int_0^x L(t)\exp\Big(\int_t^x \frac{2(1-e^{-y})}{y}dy\Big)dt. \end{aligned}$$

This implies that

$$m_3(x) < \infty.$$

The final statement follows because $0 \le N(y) \le N(x)$ when $0 \le y \le x$. $\square$

Proof of Proposition :

(i) For x large enough we have, almost surely $\dfrac{V(\xi_j x)}{V(x)} \le 1$. Then $\forall j \in \{1,2\}$, by the dominated convergence theorem

$$\begin{aligned}\mathbf{E}\Big(1\!\!1_{\{x\,stable\}} N_*^{(j)}(\xi_j x)\sqrt{\frac{V(\xi_j x)}{V(x)}}\Big)^3 \quad &\le \mathbf{E}\Big(1\!\!1_{\{x\,stable\}} N_*^{(j)}(\xi_j x)\Big)^3 \\ &\le \mathbf{E}(1\!\!1_{\{x\,stable\}})\mathbf{E}\Big(N_*^{(j)}(\xi_j x)\Big)^3 \\ &\le e^{-x} \sup_{0\le y\le x} \mathbf{E}(N(y)^3).\end{aligned}$$

Using Lemma 1, there exists a constant c such that

$$\Big\|1\!\!1_{\{x\,stable\}} \sum_{j=1}^{2} N_*^{(j)}(\xi_j x)\sqrt{\frac{V(\xi_j x)}{V(x)}}\Big\|_3 \le c e^{-x/3}.$$

(ii) Is trivially using Theorem 3.
(iii) The random variables ξ_1 and ξ_2 have the same law the (0,1) uniform distribution, then

$$\sum_{j=1}^{2} \mathbf{E}\Big(\sqrt{\xi_j}^{3}\Big) = 2\int_0^1 t^{3/2} dt = 4/5. \qquad \square$$

Since the conditions of Theorem 5.1 [16] are satisfied one can derive the following result.

Proposition 3 *$N_*(x)$ converges in distribution to a limit N_*, where $\mathcal{L}(N_*)$ is the unique fixed point of T given in (15) subject to*

$$\mathbf{E}\Big(N_*^3\Big) < \infty,\ \mathbf{E}(N_*) = 0 \textit{ and } \mathbf{Var}(N_*) = 1.$$

It's not difficult to prove that the law $\mathcal{N}(0,1)$ is a fixed point of the map T, leading to our last Theorem.

Theorem 4 *As $x \longrightarrow +\infty$, we have a version of the central limit Theorem*

$$N_*(x) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Acknowledgments.

I thank the University of Versailles Saint Quentin for supporting a research visit. Thanks also are due to anonymous referees for comments that greatly helped us improving the presentation of our results.

References

[1] Campi X. , Krivine H. , Sator N. and Plagnol E. (2000). *Eur. Phys. J. D.*, **11, 233**.

[2] Devroye L. (1987). Branching processes in the analysis of the height of trees. *Acta Informatica*, **24**, 277–298.

[3] Devroye, L. (1986). A note on the height of binary search trees. *Journal of the ACM*, **33**, 489–498.

[4] Derrida B. and Flyvbjerg H. (1987). *J. Phys. A* , **20, 5273** .

[5] Krapivsky, P.L., Grosse, I., Ben-Naim, E. (2004). Stable distributions in stochastic fragmentation. *J.Phys. A: Math. Gen.*, **37**, 2863-2880.

[6] Lawn B.R. and Wilshaw T.R. (1975). Fracture of Brittle Solids. *Cambridge University Press, Cambridge.*

[7] Neininger, R., Rï£¡schendorf, L. (2004). A general limit theorem for recursive algorithms and combinatorial structures. *Ann. Appl. Probab*, **14**, 378–418 .

[8] Neininger, R. (2001). On a multivariate contraction method for random recursive structures with applications to Quicksort. *Random Structures Algorithms*, **19**, 498–524.

[9] Rachev, S. and Rüschendorf, L. (1995). Probability metrics and recursive algorithms. *Advances in Applied Probability*, **27**, 770–799.

[10] Rachev, S.T. (1991). Probability Metrics and the Stability of Stochastic Models. *Wiely, New York*.

[11] Rï£¡sler, U. (2001). On the analysis of stochastic divide and conquer algorithms. *Algorithmica*, **29**, 238–261.

[12] RednerFor S. (1990). Statistical Models for the Fracture of Disordered Media, *ed. H.J. Herrmann and S. Roux (Elsevier Science, New York).*

[13] Rösler, U. and Rüschendorf, L. (2001). The contraction method for recursive algorithms. *Algorithmica*, **29**, 3–33.

[14] Rï£¡sler, U. (1991). A limit theorem for “Quicksort”. *RAIRO Inform. Théor. Appl.*, **25**, 85–100.

[15] Sibuya M., Itoh Y. (1987). Random sequential bisection and its associated binary tree.
Ann. Inst. Satist. Math., **39**, 69–84.

[16] Svante J., Neininger, R.(2007). The size of random fragmentation trees.
Probab. Theory Relat. Fields, **to appear**.

[17] Zolotarev, V.M. (1977). Ideal metrics in the problem of approximating the distributions of sums of independent random variables (Russian). *Teor. Verojatnost. i Primenen*, **22**, 449–465.