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On randomly colouring locally sparse graphs

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We consider the problem of generating a random \( q \)-colouring of a graph \( G = (V, E) \). We consider the simple Glauber Dynamics chain. We show that if for all \( v \in V \) the average degree of the subgraph \( H_v \) induced by the neighbours of \( v \in V \) is \( \ll \Delta \) where \( \Delta \) is the maximum degree and \( \Delta > c_1 \ln n \) then for sufficiently large \( c_1 \), this chain mixes rapidly provided \( q/\Delta > \alpha \), where \( \alpha \approx 1.763 \) is the root of \( \alpha = e^{1/\alpha} \). For this class of graphs, which includes planar graphs, triangle free graphs and random graphs \( G_{n,p} \) with \( p \ll 1 \), this beats the \( 11\Delta/6 \) bound of Vigoda [20] for general graphs.

Keywords: Counting Colourings, Sampling, Markov Chains

1 Introduction

Markov Chain Monte Carlo (MCMC) is an important tool in sampling from complex distributions. It has been successfully applied in several areas of Computer Science, most notably volume computation [3, 13, 16] and estimating the permanent of a non-negative matrix [12]. It was used by Jerrum [10] to generate a random \( q \)-colouring of a graph \( G \), provided \( q > 2\Delta \). This has led to the challenging problem of determining the smallest value of \( q \) for which it is possible to generate a (near)-uniform sample from the set \( Q \) of proper \( q \)-colourings of \( G \) in polynomial time. We cannot expect the chain to mix for \( q \leq \Delta + 1 \) and at present it is unknown as to whether or not it mixes rapidly for say \( q = \Delta + 2 \). Vigoda [20] improved Jerrum’s result by reducing the lower bound on \( q \) to \( 11\Delta/6 \). This is still the best lower bound on \( q \) for general graphs.

The lack of complete success on the general problem has led to the analysis of restricted classes of graphs. Suppose that we consider Glauber dynamics on the set \( Q \). Specifically we will consider the heat bath dynamics, which may be described as follows. We start from an arbitrary proper \( q \)-colouring \( X_0 \in Q \). At step \( t > 0 \) of the process, in state \( X_{t-1} \in Q \), we choose a vertex \( v_t \in V \) uniformly

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at random. Then we choose $j_t$ uniformly at random from the colours with which $v_t$ may be properly coloured, given $X_{t-1}(V \setminus v_t)$. We recolour $v_t$ with $j_t$ to give $X_t \in \mathcal{Q}$.

Dyer and Frieze [2] considered this process restricted to the class of graphs $G(c_1, c_2)$: the set of graphs with $n$ vertices, maximum degree $\Delta \geq c_1 \log n$ and girth $g \geq c_2 \log \Delta$. They showed using the idea of “burn-in” that for $c_1, c_2$ sufficiently large, Glauber Dynamics mixed in $O(n \log n)$ time, provided $q > \alpha \Delta$ where $\alpha \approx 1.763$ is the root of $\alpha = e^{1/\alpha}$. Molloy [17] improved this result by reducing the lower bound on $q$ to being more than $\beta \Delta$ where $\beta \approx 1.489$ is the root of $(1 - e^{-1/\beta})^2 + \beta e^{-1/\beta} = 1$. The girth assumption were then relaxed by Hayes [7] to $g \geq 5$ for $k/\Delta > \alpha$ and $g \geq 6$ for $k/\Delta > \beta$. Subsequently, Hayes and Vigoda [8] made considerable progress, using a non-Markovian coupling, and reduced the lower bound on $k/\Delta$ to $(1 + \epsilon)$ for all $\epsilon > 0$, which is nearly optimal. Their result requires girth $g \geq 9$. However, the large maximum degree restriction remained. This was replaced by $\Delta \geq \Delta_0$ in Dyer, Frieze, Hayes and Vigoda [5], with the same restrictions on girth as in [7]. Dyer, Flaxman, Frieze and Vigoda [4] show that for sparse random graphs, the number of colours required for rapid mixing is of order the average rather than maximum degree whp. Goldberg, Martin and Paterson [6] prove results on the related notion of strong spatial mixing.

In this paper we avoid girth restrictions and consider locally sparse graphs instead. We say that a graph $G = (V, E)$ is $\gamma$-locally sparse if for all $v \in V$, the average degree of the graph induced by the neighbourhood $N(v)$ is at most $\gamma$. Thus planar graphs are always 6-locally-sparse and triangle free graphs are 0-locally-sparse.

**Theorem 1.1** Suppose that $q \geq (\alpha + \epsilon) \Delta$ where $\epsilon$ is a small positive constant. Let $G$ be an $n$-vertex $\gamma$-locally sparse graph with $\gamma \leq \epsilon^2 \Delta/10$ and $\Delta \geq c_1 \log n$. If $c_1 = c_1(\epsilon)$ is sufficiently large then the Glauber dynamics converges to within variation distance $e^{-1}$ from uniform over $\mathcal{Q}$ in at most $O(n \ln n)$.

Notice that if $G = G_{n,p}$ and $\frac{c_1 \log n}{n} \leq p \leq \epsilon^2 / 11$ then whp $G$ satisfies the conditions of the theorem. Note also that the chromatic number of a triangle-free graph is $O(\Delta/\log \Delta)$ – see Johansson [14] or Molloy and Reed [18] or Alon, Krivelevich and Sudakov [1] or Vu [21].

Our proof uses coupling and relies on a recent idea from Hayes and Vigoda [9] that utilises the fact that we can couple against the steady state distribution of the chain. Note that the theorem generalises Theorem 4 of [9].

In what follows we will assume that $n$ is sufficiently large and $\epsilon$ is sufficiently small to satisfy our inequalities.

## 2 Preliminaries

We will consider two copies of Glauber Dynamics, $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$. Here $X_0$ is an arbitrary colouring and $Y_0$ is chosen from the uniform (stationary) distribution over $\mathcal{Q}$. At time $t$, the Hamming distance between $X_t, Y_t$ is defined by

$$H(X_t, Y_t) = \sum_{v \in V} 1_{X_t(v) \neq Y_t(v)}.$$  

We will couple the two processes as in Jerrum [10]. Here $v_t$ is the same in both processes and then the choice of colours is maximally coupled. For vertex $w$ let

$$A(X_t, w) = \{c \in [q] : c \notin X_t(N(w))\}$$
be the set of colours available to colour \( w \) in \( X_t \) if \( v_t = w \).  
Let \( a(X_t, w) = |A(X_t, w)| \) and define the terms \( A(Y_t, w) \), \( a(Y_t, w) \) analogously.

It is shown in [9] that

\[
E(H(X_{t+1}, Y_{t+1}) - H(X_t, Y_t)) \leq -\frac{1}{n} H(X_t, Y_t) + \frac{1}{n} \sum_{w \in V} \frac{|\{u \in N(w) : X_t(u) \neq Y_t(u)\}|}{\max\{a(X_t, w), a(Y_t, w)\}}. \tag{1}
\]

We will show that for \( w \in V \) and \( \delta = \epsilon/10 \),

\[
\Pr(a(Y_t, w) \leq \Delta/(1 - \delta)) \leq n^{-4}. \tag{2}
\]

Assuming that \( a(Y_t, w) \geq \Delta/(1 - \delta) \) in (1) we get

\[
E(H(X_{t+1}, Y_{t+1}) - H(X_t, Y_t)) \leq -\frac{1}{n} H(X_t, Y_t) + \frac{1}{n} \frac{H(X_t, Y_t)\Delta}{\Delta/(1 - \delta)} \leq -\frac{\delta}{n} H(X_t, Y_t).
\]

So conditional on an event of probability \( 1 - O(n^{-3}) \), we have

\[
E(H(X_{t+1}, Y_{t+1}) | X_t, Y_t) \leq \left(1 - \frac{\delta}{n}\right) H(X_t, Y_t).
\]

Thus if \( T = n(1 + \ln n)\delta^{-1} \) then conditional on an event of probability \( 1 - O(n^{-2} \log n) \), we have

\[
E(H(X_T, Y_T)) \leq e^{-1}
\]

and so unconditionally

\[
E(H(X_T, Y_T)) \leq e^{-1} + o(1).
\]

Hence the mixing time of the Glauber Dynamics is \( O(n \ln n) \) as claimed.

3 Bounding the number of available colours

Fix \( v \in W \) and let \( H_v \) be the subgraph of \( G \) induced by \( N(v) \). Let \( B(v) \) be the vertices of \( N(v) \) that have degree at least \( \gamma \delta^{-1} \) in \( H_v \). Note that \( \gamma \delta^{-1} \leq \epsilon \Delta \) and

\[
|B(v)| \leq \delta|N(v)|, \tag{3}
\]

since \( G \) is \( \gamma \)-locally-sparse.

Let

\[
N^*(v) = N(v) \setminus B(v) = \{w_1, w_2, \ldots, w_d\}.
\]

Now let us fix the colours \( \kappa(v) \) used at

\[
v \in W_v = V \setminus N^*(v).
\]

Let us use the term allowable for colorings of \( N^*(v) \) which respect this conditioning. Let \( \Omega \) be the set of allowable colourings of \( N^*(v) \).
Let $a^*(\sigma, v)$ be the number of colours not used on $N^*(v)$. Note that (3) implies
\[ a(\sigma, v) \geq a^*(\sigma, v) - \delta|N(v)|. \] (4)

Now consider the following process $P_{\sigma}$ for producing an allowable colouring of $H_v$. Here $\sigma \in \Omega$. We let $\sigma_0 = \sigma$ and for $j = 1, 2, \ldots, d$ let $\sigma_j$ be obtained from $\sigma_{j-1}$ as follows: Keep $\sigma_j(w_k) = \sigma_{j-1}(w_k)$ for $k \neq j$ and choose $\sigma_j(w_j)$ randomly from what is available to it.

Let $Z_\sigma$ be the number of colours not appearing on a vertex in $N^*(v)$ if we start with $\sigma_0 = \sigma$.

**Lemma 3.1** If $\sigma$ is chosen uniformly from $\Omega$ then for any $c > 0$,
\[ \Pr(a^*(\sigma, v) \geq c) = \Pr(Z_\sigma \geq c). \]

**Proof** We first prove that
\[ \text{If } \sigma_0 \text{ is chosen uniformly from } \Omega \text{ then } \sigma_d \text{ is also uniform over } \Omega. \] (5)

We do this by induction on $j$, with base case $j = 0$.
\[
\Pr(\sigma_j = \sigma) = \sum_{\sigma' \in \Omega} \Pr(\sigma_j = \sigma | \sigma_{j-1} = \sigma') \Pr(\sigma_{j-1} = \sigma')
= \frac{1}{|\Omega|} \sum_{\sigma' \sim \sigma} \Pr(\sigma_j = \sigma | \sigma_{j-1} = \sigma')
\]
Here $\sigma' \sim \sigma$ if $\sigma, \sigma'$ differ only at $w_j$.
\[
= \frac{1}{|\Omega|} \sum_{\sigma' \sim \sigma} \frac{1}{|\{\sigma': \sigma' \sim \sigma\}|}
= \frac{1}{|\Omega|}.
\]

Now $a^*(\sigma_d, v) = Z_{\sigma_0}$ and so
\[
\Pr(a^*(\sigma_d, v) \geq c) = \Pr(Z_{\sigma_0} \geq c)
\]
and the lemma follows from (5).

For $w \in N^*(v)$ let
\[ L(w) = [q] \setminus \{\kappa(u) : u \in N(w) \setminus N^*(v)\} \]
be the colours not specifically barred from $w$ by the current conditioning. Then let
\[ L^*(w_j) = [q] \setminus \{\sigma_{j-1}(u) : u \neq w_j\} \text{ for } j = 1, 2, \ldots, d \]
be the colours available to $w_j$ when it is re-coloured by $\sigma_j$. 
We will first estimate the (conditional) expectation of $Z_\sigma$ for arbitrary $\sigma$. Suppose that $x \in [q]$. Let $\theta_{x,j} = 1_{x \in L(w_j)}$ and let $\theta_{x,j}^* = 1_{x \in L^*(w_j)}$. Then we have

$$\Pr(x \notin \sigma_d(N^*(v))) = \prod_{j=1}^{d} \Pr(\sigma_d(w_j) \neq x \mid \sigma_d(w_i) \neq x, 1 \leq i < j)$$

$$= \prod_{j=1}^{d} E\left((1 - \frac{1}{|L^*(w_j)|})^{\theta_{x,j}^*}\right)$$

$$\geq \prod_{j=1}^{d} \left(1 - \frac{1}{|L(w_j)| - \gamma \delta - 1}\right)^{\theta_{x,j}}$$

since $|L^*(w_j)| \geq |L(w_j)| - \gamma \delta$ and $L^*(w_j) \subseteq L(w_j)$ implying $\theta_{x,j}^* \leq \theta_{x,j}$.

Then, following [2],

$$E(Z_\sigma) \geq \sum_{x \in [q]} \prod_{j=1}^{d} \left(1 - \frac{1}{|L(w_j)| - \gamma \delta - 1}\right)^{\theta_{x,j}}$$

$$\geq q \left(\prod_{x \in [q]} \prod_{j=1}^{d} \left(1 - \frac{1}{|L(w_j)| - \gamma \delta - 1}\right)^{\theta_{x,j}}\right)^{1/q}$$

$$= q \left(\prod_{j=1}^{d} \left(1 - \frac{1}{|L(w_j)| - \gamma \delta - 1}\right)^{|L(w_j)|}\right)^{1/q}$$

$$\geq q \exp\left\{-\frac{1}{q} \sum_{j=1}^{d} \frac{|L(w_j)|}{|L(w_j)| - 1 - \gamma \delta - 1}\right\}, \quad \text{using } 1 - x \geq e^{-x/(1-x)} \text{ for } 0 < x < 1,$$

$$\geq q \exp\left\{-\frac{\Delta}{q} \cdot \frac{|L(w_j)|}{q - \Delta - 1 - \gamma \delta - 1}\right\}$$

$$\geq \left(1 + \frac{\epsilon}{2}\right) \Delta. \quad (6)$$

(If $f(x) = xe^{-x/2}$ then $f(\alpha) = 1$ and $f'(\alpha) \sim .891$.)

We will now prove that for all $\sigma \in \Omega$, $Z_\sigma$ is concentrated around its mean via the use of the Azuma-Hoeffding martingale inequality. To this end, let $x_1, x_2, \ldots, x_d$ be the colours assigned to $w_1, w_2, \ldots, w_d$. Thus we can write $Z_\sigma = Z_\sigma(x_1, x_2, \ldots, x_d)$. Now let

$$Z_{\sigma,i} = Z_{\sigma,i}(x_1, x_2, \ldots, x_i) = E(Z \mid x_1, x_2, \ldots, x_i).$$

We will show next that for all feasible colours $x_1, x_2, \ldots, x_i, x'_i$ that

$$|Z_{\sigma,i}(x_1, x_2, \ldots, x_{i-1}, x_i) - Z_{\sigma,i}(x_1, x_2, \ldots, x_{i-1}, x'_i)| \leq 2. \quad (7)$$
The aforementioned inequality will then imply that for any \( t \geq 0 \),
\[
\Pr(Z_{\sigma} - E(Z_{\sigma}) \leq -t) \leq e^{-t^2/(2d)}
\]
and then taking \( t = \epsilon \Delta / 4 \) and using (6) we get
\[
\Pr \left( Z_{\sigma} \leq \left( 1 + \frac{\epsilon}{4} \right) \Delta \right) \leq e^{-\epsilon^2 \Delta / 32}.
\]
This together with Lemma 3.1 and (4) implies (2).

To prove (7), fix \( i, x_1, x_2, \ldots, x_i, x_i^* \). In one instance of \( \mathcal{P}_{\sigma} \) we start by colouring \( w_1, w_2, \ldots, w_i \) with \( x_1, x_2, \ldots, x_i \) to produce colouring \( \tau \). In another instance we start by colouring \( w_1, w_2, \ldots, w_i \) with \( x_1, x_2, \ldots, x_i^* \) to produce colouring \( \tau^* \).

We couple these two constructions in order to minimise the expected difference in the number of vertices \( U \) with a different colour. A paths of disagreement argument gives that
\[
E(U) \leq 1 + \sum_{j=i+1}^{d} \left( \frac{\gamma \delta^{-1}}{|L(w_j)| - \gamma \delta^{-1}} \right)^{j-i} \leq 2
\]
and (7) follows.

**Explanation of (8):** We claim that if \( c_j, c_j^* \) is the colour of \( v_j \) in \( \sigma_d, \sigma_d^* \) respectively, then
\[
\Pr(c_j \neq c_j^*) \leq \left( \frac{\gamma \delta^{-1}}{|L(w_j)| - \gamma \delta^{-1}} \right)^{j-i}.
\]
This is because if \( c_j \neq c_j^* \) then there is a path of disagreements \( v_{i_1}, v_{i_2}, \ldots, v_{i_s} \) where \( i = i_1 < i_2 < \cdots < i_s = j \) such that \( c_{i_r} \neq c_{i_r}^* \) for \( 1 \leq r \leq s \). There are at most \((\lambda \delta^{-1})^{j-i}\) such paths and each has probability at most \((|L(w_j)| - \gamma \delta^{-1})^{j-i}\) of all vertices being coloured differently.
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References


[9] T. P. Hayes and E. Vigoda, Coupling with the stationarity distribution and improved sampling from colorings and independent sets.


