FPSAC 2008, Valparaiso-Viña del Mar, Chile DMTCS proc. **AJ**, 2008, 411–416

The Sorting Order on a Coxeter Group

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Abstract. Let (W, S) be an arbitrary Coxeter system. For each sequence $\omega = (\omega_1, \omega_2, \ldots) \in S^*$ in the generators we define a partial order—called the ω-sorting order—on the set of group elements $W_\omega \subseteq W$ that occur as finite subwords of ω. We show that the ω-sorting order is a supersolvable join-distributive lattice and that it is strictly between the weak and strong Bruhat orders on the group. Moreover, the ω-sorting order is a "maximal lattice" in the sense that the addition of any collection of edges from the Bruhat order results in a nonlattice.

Along the way we define a class of structures called supersolvable antimatroids and we show that these are equivalent to the class of supersolvable join-distributive lattices.

Keywords: Coxeter group, join-distributive lattice, supersolvable lattice, antimatroid, convex geometry

Extended Abstract

Let (W, S) be an arbitrary Coxeter system and let $\omega = (\omega_1, \omega_2, \ldots) \in S^*$ be an arbitrary sequence in the generators, called the sorting sequence. We will identify a finite subword $\alpha \subseteq \omega$ with the pair $(\alpha, I(\alpha))$, where $I(\alpha) \subseteq I(\omega) = \{1, 2, \ldots\}$ is the index set encoding the positions of the letters. Given a word $\alpha = (\alpha_1, \ldots, \alpha_k) \in S^*$, let

$$\langle\alpha\rangle = \alpha_1 \cdots \alpha_k \in W$$

denote the corresponding group element. The subsets of the ground set $I(\omega)$ are ordered lexicographically: if A and B are subsets of $I(\omega)$ we say that $A \leq_{\mathsf{lex}} B$ if the minimum element of $(A \cup B) \setminus (A \cap B)$ is contained in A.

Definition 1 *We say that a finite subword $\alpha \subseteq \omega$ of the sorting sequence is ω-sorted if*

1. *α is a reduced word,*
2. *$I(\alpha) = \min_{\leq_{\mathsf{lex}}}\{I(\beta) \subseteq I(\omega) : \langle\beta\rangle = \langle\alpha\rangle\}$.*

That is, α is ω-sorted if it is the lexicographically-least reduced word for $\langle\alpha\rangle$ among subwords of ω.

Let $W_\omega \subseteq W$ denote the set of group elements that occur as subwords of the sorting sequence. Then ω induces a canonical reduced word for each element of W_ω—its ω-sorted word. This, in turn, induces a partial order on the set W_ω by subword containment of sorted words.

Definition 2 *Given group elements $u, w \in W_\omega$, we write $u \leq_\omega w$ if the index set of ω-$\mathsf{sort}(u)$ is contained in the index set of ω-$\mathsf{sort}(w)$. This is called the ω-sorting order on W_ω.*

The sorting orders are closely related to other important orders on the group.

Theorem 1 *Let $\leq_{\mathsf{R}}$ denote the right weak order and let $\leq_{\mathsf{B}}$ denote the Bruhat order on W. For all $u, w \in W_\omega$ we have*

$$u \leq_{\mathsf{R}} w \quad \Rightarrow \quad u \leq_\omega w \quad \Rightarrow \quad u \leq_{\mathsf{B}} w.$$

For example, let $W = \mathfrak{S}_4$ be the symmetric group of permutations of $\{1,2,3,4\}$ with the generating set of adjacent transpositions

$$S = \{s_1 = (12), s_2 = (23), s_3 = (34)\}.$$

Figure 1 displays the Hasse diagrams of the weak order, $(s_1, s_2, s_3, s_2, s_1, s_2)$-sorting order and strong order on the symmetric group $\mathfrak{S}_4$—the weak order is indicated by the shaded edges; solid edges indicate the sorting order; solid and broken edges together give the Bruhat order.

It turns out that the collection of ω-sorted words has a remarkable structure. Given a ground set E and a collection of finite subsets $\mathscr{F} \subseteq 2^E$, the pair $(E, \mathscr{F})$ is called a set system. A set system $(E, \mathscr{F})$ is called an antimatroid (see (11)) if it satisfies

- For all nonempty $A \in \mathscr{F}$ there exists $x \in A$ such that $A \setminus \{x\} \in \mathscr{F}$,
- For all $A, B \in \mathscr{F}$ with $B \not\subseteq A$ there exists $x \in B \setminus A$ such that $A \cup \{x\} \in \mathscr{F}$.

Equivalently, $\mathscr{F}$ is the collection of open sets for a closure operator $\tau : 2^E \to 2^E$ that satisfies the anti-exchange property:

- If $x, y \notin \tau(A)$ then $x \in \tau(A \cup \{y\})$ implies $x \notin \tau(A \cup \{x\})$.

Such an operator τ models the notion of "convex hull", and so it is called a convex closure. Furthermore, we say that a lattice L is join-distributive if it satisfies:

- For each $x \in L$, the interval $[x, y]$, where y is the join of elements that cover x, is a boolean algebra.

Edelman (5) proved that a finite lattice is join-distributive if and only if it arises as the lattice of open sets of a convex closure. We will generalize Edelman's characterization to the case of supersolvable join-distributive lattices.

Definition 3 *Consider a set system $(E, \mathscr{F})$ on a totally ordered ground set $(E, \leq_E)$. We say that $(E, \mathscr{F})$ is a* supersolvable antimatroid *if it satisfies:*

- $\emptyset \in \mathscr{F}$.
- *For all $A, B \in \mathscr{F}$ with $B \not\subseteq A$ and $x = \min_{\leq_E} B \setminus A$ we have $A \cup \{x\} \in \mathscr{F}$.*

Theorem 2 *A (possibly infinite) lattice P is join-distributive and every interval in P is supersolvable if and only if P arises as the lattice of feasible sets of a supersolvable antimatroid.*

Our main result is the following.

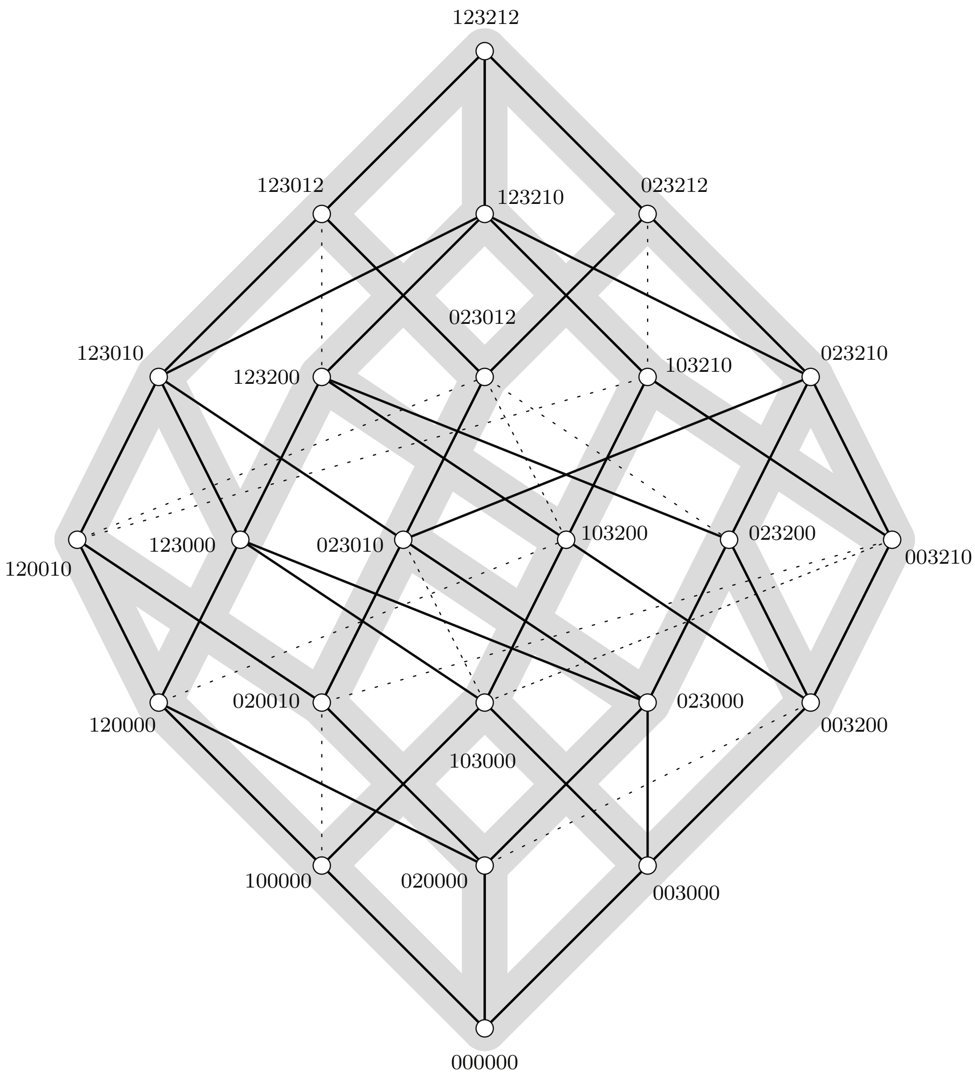

Fig. 1: The weak order, 123212-sorting order and strong order on $\mathfrak{S}_4$

Theorem 3 *Let (W,S) be an arbitrary Coxeter system and consider an arbitrary sequence $\omega \in S^*$. The collection of index sets of ω-sorted subwords*

$$\mathscr{F} = \{I(\alpha) \subseteq I(\omega) : \alpha \text{ is } \omega\text{-sorted } \}$$

is a supersolvable antimatroid with respect to the natural order on the ground set $E = I(\omega)$.

Corollary 1 *The ω-sorting order is a join-distributive lattice in which every interval is supersolvable and it is graded by the usual Coxeter length function $\ell : W \to \mathbb{Z}$.*

Note that this holds even when infinitely many group elements occur as subwords of the sorting sequence. This is remarkable because the weak order on an infinite group is *not* a lattice. Indeed, we do not know of any other natural source of lattice structures on the elements of an infinite Coxeter group.

We also have

Corollary 2 *There exists a* reduced *sequence $\omega' \subseteq \omega$ (that is, every prefix of ω' is a reduced word) such that the ω'-sorting order coincides with the ω-sorting order.*

That is, we may assume that the sorting sequence is reduced. Finally, we have

Lemma 1 *If ω and ζ are sequences that differ by the exchange of adjacent commuting generators, then the ω-sorting order coincides with the ζ-sorting order.*

In summary, for each commutation class of reduced sequences we obtain a supersolvable join-distributive lattice that is strictly between the weak and Bruhat orders. This is particularly interesting in the case that ω represents a commutation class of reduced words for the longest element $w_\circ$ in a finite Coxeter group.

We end by noting an important special case. Let (W,S) be a Coxeter system with generators $S = \{s_1,\ldots,s_n\}$. Any word of the form $(s_{\sigma(1)},\ldots,s_{\sigma(n)})$—where $\sigma \in \mathfrak{S}_n$ is a permutation—is called Coxeter word, and the corresponding element $\langle c\rangle \in W$ is a Coxeter element. We say that a cyclic sequence is any sequence of the form

$$c^\infty := ccc\ldots.$$

The case of c^∞-sorted words was first considered by Reading (see (13; 14)), and this is the main motivation behind our work. However, Reading did not consider the structure of the collection of sorted words nor did he consider the sorting order.

Acknowledgements

The author gratefully acknowledges helpful conversations with Nathan Reading, Vic Reiner, David Speyer and Hugh Thomas.

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