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Chip-Firing and Rotor-Routing on $\mathbb{Z}^d$ and on Trees

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Abstract. The sandpile group of a graph G is an abelian group whose order is the number of spanning trees of G. We find the decomposition of the sandpile group into cyclic subgroups when G is a regular tree with the leaves are collapsed to a single vertex. This result can be used to understand the behavior of the rotor-router model, a deterministic analogue of random walk studied first by physicists and more recently rediscovered by combinatorialists. Several years ago, Jim Propp simulated a simple process called rotor-router aggregation and found that it produces a near perfect disk in the integer lattice $\mathbb{Z}^2$. We prove that this shape is close to circular, although it remains a challenge to explain the near perfect circularity produced by simulations. In the regular tree, we use the sandpile group to prove that rotor-router aggregation started from an acyclic initial condition yields a perfect ball.

Keywords: abelian sandpile, chip-firing game, regular tree, rotor-router model, sandpile group

In this extended abstract, we summarize recent progress in understanding the shapes of two combinatorial growth models, the rotor-router model and the chip-firing or abelian sandpile model. We touch on a few of the highlights and main ideas of the proofs. The details and proofs omitted here can be found in the references [13, 16, 18].

1 Chip-Firing and Rotor-Routing on $\mathbb{Z}^d$

Rotor-router walk is a deterministic analogue of random walk, first studied by Priezzhev et al. [22] under the name "Eulerian walkers." At each site in the integer lattice $\mathbb{Z}^2$ is a *rotor* pointing north, south, east or west. A particle starts at the origin; during each time step, the rotor at the particle's current location is rotated clockwise by 90 degrees, and the particle takes a step in the direction of the newly rotated rotor. In rotor-router aggregation, introduced by Jim Propp, we start with n particles at the origin; each particle in turn performs rotor-router walk until it reaches a site not occupied by any other particles. Let A_n denote the resulting region of n occupied sites. For example, if all rotors initially point north, the sequence will begin $A_1 = \{(0,0)\}$, $A_2 = \{(0,0),(1,0)\}$, $A_3 = \{(0,0),(1,0),(0,-1)\}$. The region $A_{1,000,000}$ is pictured in Figure 1. In higher dimensions, the model can be defined analogously by repeatedly cycling the rotors through an ordering of the $2d$ cardinal directions in $\mathbb{Z}^d$.

Jim Propp observed from simulations in two dimensions that the regions A_n are extraordinarily close to circular, and asked why this was so [12, 23]. The first result in this direction [17] says that if A_n is

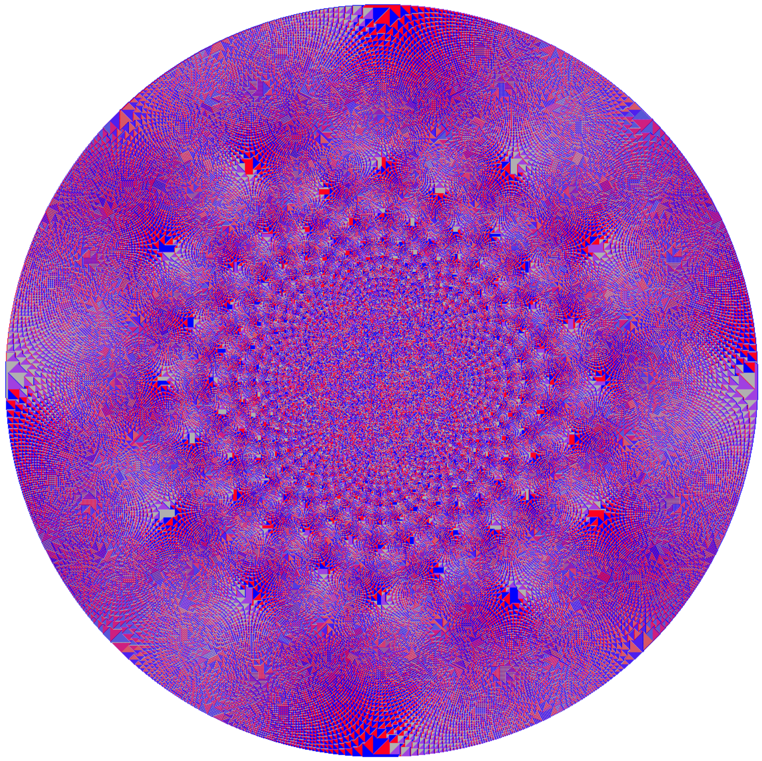

Fig. 1: Rotor-router aggregate of one million particles in $\mathbb{Z}^2$. Each site is colored according to the direction of its rotor.

rescaled to have unit volume, the volume of the symmetric difference of A_n with a ball of unit volume tends to zero as a power of n, as $n \uparrow \infty$; the main outline of the argument is summarized in [19]. Fey and Redig [10] also show that A_n contains a diamond. However, these results do not rule out the possibility of "holes" in A_n far from the boundary or of long tendrils extending far beyond the boundary of the ball, provided the volume of these features is negligible compared to n.

More recently, we have obtained a stronger shape theorem which rules out the possibility of holes far from the boundary or of long tendrils in the rotor-router shape. For $r \geq 0$ let

$$B_r = \{x \in \mathbb{Z}^d \ : \ |x| < r\},$$

where $|x| = (x_1^2 + \ldots + x_d^2)^{1/2}$ is the usual Euclidean norm on $\mathbb{Z}^d$. Thus B_r consists of all the lattice points in an open ball of radius r. Our main result in $\mathbb{Z}^d$ is the following.

Theorem 1.1 *[18] Let A_n be the region formed by rotor-router aggregation in $\mathbb{Z}^d$ starting from n particles at the origin and any initial rotor state. There exist constants c, c' depending only on d, such that*

$$B_{r-c\log r} \subset A_n \subset B_{r(1+c'r^{-1/d}\log r)}$$

where $r = (n/\omega_d)^{1/d}$, and ω_d is the volume of the unit ball in $\mathbb{R}^d$.

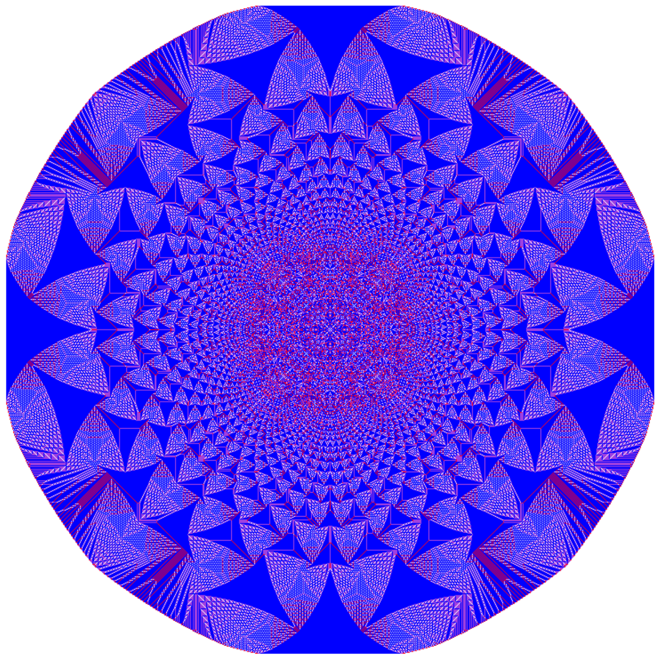

Fig. 2: Abelian sandpile aggregate of one million particles in $\mathbb{Z}^2$. Colors represent the number of grains at each site.

In the *abelian sandpile* or *chip-firing* model [1, 4], each site in $\mathbb{Z}^d$ has an integer number of grains of sand; if a site has at least $2d$ grains, it *topples*, sending one grain to each neighbor. If n grains of sand are started at the origin in $\mathbb{Z}^d$, write S_n for the set of sites that are visited during the toppling process; in particular, although a site may be empty in the final state, we include it in S_n if it was occupied at any time during the evolution to the final state.

Until recently, the best known constraints on the shape of S_n in two dimensions were due to Le Borgne and Rossin [15], who proved that

$$\{x \in \mathbb{Z}^2 \,|\, x_1 + x_2 \leq \sqrt{n/12} - 1\} \subset S_n \subset \{x \in \mathbb{Z}^2 \,|\, x_1, x_2 \leq \sqrt{n}/2\}.$$

Fey and Redig [10] proved analogous bounds in higher dimensions. The following result improves on the bounds of [10] and [15].

Theorem 1.2 *[18] Let S_n be the set of sites that are visited by the classical abelian sandpile model in $\mathbb{Z}^d$, starting from n particles at the origin. Write $n = \omega_d r^d$. Then for any $\epsilon > 0$ we have*

$$B_{c_1 r - c_2} \subset S_n \subset B_{c_1' r + c_2'}$$

where

$$c_1 = (2d-1)^{-1/d}, \qquad c_1' = (d-\epsilon)^{-1/d}.$$

The constant c_2 depends only on d, while c_2' depends only on d and ϵ.

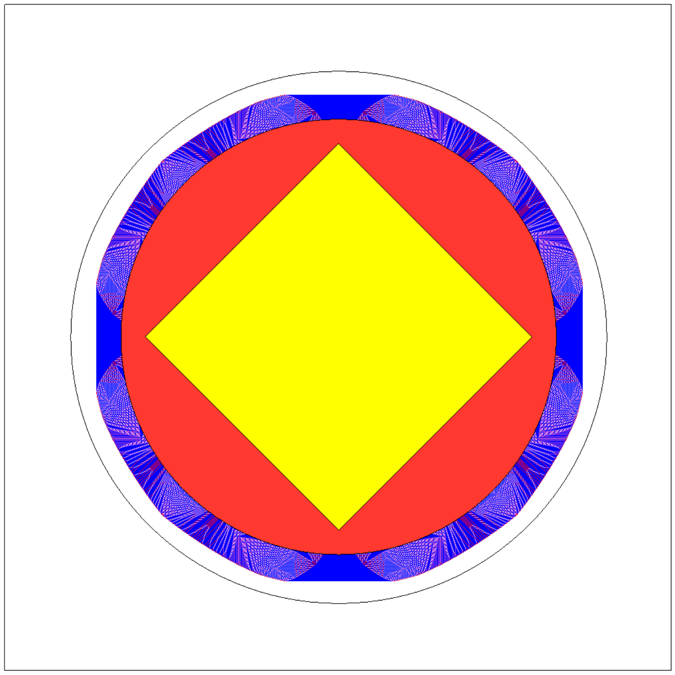

Fig. 3: Known bounds on the shape of the classical abelian sandpile in $\mathbb{Z}^2$. The inner diamond and outer square are due to Le Borgne and Rossin [15]; the inner and outer circles are those in Theorem 1.2.

Note that Theorem 1.2 does not settle the question of the asymptotic shape of S_n, and indeed it is not clear from simulations whether the asymptotic shape in two dimensions is a disc or perhaps a polygon (Figure 2). To our knowledge even the existence of an asymptotic shape is not known.

2 The Sandpile Group of a Tree

To any finite graph G is associated a finite abelian group $SP(G)$, the *sandpile group*, whose order is the number of spanning trees of G. This group was defined by Dhar [8], who was motivated by models in statistical physics, and independently by Lorenzini [20] in connection with arithmetic geometry. In the combinatorics literature, other common names for this group are the *critical group* [3] and the *Jacobian* [2]. If G has vertices $x_1, \ldots, x_n$, the sandpile group is defined as the quotient

$$SP(G) = \mathbb{Z}^n/\Delta$$

where $\Delta \subset \mathbb{Z}^n$ is the lattice

$$\Delta = \langle \Delta_{x_1}, \ldots, \Delta_{x_{n-1}}, \delta_{x_n} \rangle.$$

Here Δ_{x_i} is the vector in $\mathbb{Z}^n$ taking value 1 at each neighbor of x_i, value $-\deg(x_i)$ at x_i, and value 0 elsewhere; and δ_{x_n} is the elemenary basis vector taking value 1 at x_n and 0 elsewhere.

The sandpile group can be understood combinatorially in terms of chip-firing. A nonnegative vector $u \in \mathbb{Z}^n$ may be thought of as a *chip configuration* on T with u_i chips at vertex x_i. A vertex $x \neq x_n$ is *unstable* if $u(x) \geq \deg(x)$. An unstable vertex may *fire*, sending one chip to each neighbor. Note that the operation of firing the vertex x corresponds to adding the vector Δ_x to u. The vertex x_n is not permitted to fire, and acts as a *sink*; we will often denote it s.

We say that a chip configuration u is *stable* if every non-sink vertex has fewer chips than its degree, so that no vertex can fire. If u is not stable, one can show that by successively firing unstable vertices, in finitely many steps we arrive at a stable configuration u°. Note that firing one vertex may cause other vertices to become unstable, resulting in a cascade of firings in which a given vertex may fire many times. The order in which firings are performed does not affect the final configuration u°; this is the "abelian property" of abelian sandpiles.

The operation $(u, v) \mapsto (u + v)^\circ$ gives the set of chip configurations the structure of a commutative monoid. The sandpile group is a subgroup of this monoid. A chip configuration u is called *recurrent* if there is a nonzero chip configuration v, such that $(u+v)^\circ = u$. One can show that every equivalence class mod Δ has a unique recurrent representative. Thus the sandpile group $SP(G)$ may be thought of as the set of recurrent configurations under the operation $(u, v) \mapsto (u + v)^\circ$ of addition followed by chip-firing. For proofs of these basic lemmas about recurrent configurations and the sandpile group, see, for example [7, 11].

The following result gives the decomposition of the sandpile group of a regular tree as a product of cyclic groups. We use the notation $\mathbb{Z}_p^q$ for the group $(\mathbb{Z}/p\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/p\mathbb{Z})$ with q summands.

Theorem 2.1 *[16] Let T_n be the regular tree of degree $d = a + 1$ and height n, with leaves collapsed to a single sink vertex and an edge joining the root to the sink. The sandpile group of T_n is given by*

$$SP(T_n) \simeq \mathbb{Z}_{1+a}^{a^{n-3}(a-1)} \oplus \mathbb{Z}_{1+a+a^2}^{a^{n-4}(a-1)} \oplus \ldots \oplus \mathbb{Z}_{1+a+\ldots+a^{n-2}}^{a-1} \oplus \mathbb{Z}_{1+a+\ldots+a^{n-1}}.$$

In the next section we give an application of this result to the rotor-router model on regular trees.

The presence of an edge connecting the the root to the sink is crucial in our analysis. It allows us to understand the group recursively by relating the sandpile groups of T_n and T_{n-1}. This approach is detailed below, beginning with Proposition 2.3. If one is interested in slightly coarser information about the group, however, it is often possible to remove this extra edge. Toumpakari [24] studied the sandpile group of the ball B_n inside the infinite d-regular tree. Her setup differs from ours in that there is no edge connecting the root to the sink. She found the rank, exponent, and order of $SP(B_n)$ and conjectured a formula for the ranks of its Sylow p-subgroups. In [16], we use Theorem 2.1 to give a proof of Toumpakari's conjecture.

We remark that Chen and Schedler [5] study the sandpile group of thick trees (i.e. trees with multiple edges) without collapsing the leaves to the sink. They obtain quite a different product formula in this setting.

The first step in proving Theorem 2.1 is to characterize explicitly the recurrent chip configurations on a tree. Our characterization applies to any finite tree T, not necessarily regular. We will always work with the "wired" tree formed from T by collapsing the leaves to a single sink vertex, and adding an edge from the root to the sink. Denote by $C(x)$ the set of children of a vertex $x \in T$. We will use the following recursive definition.

Defintion. A vertex $x \in T$ is *critical* for a chip configuration u if $x \neq s$ and

$$u(x) \leq \#\{y \in C(x) \mid y \text{ is critical}\}. \tag{1}$$

In particular, if x is a leaf, then x is critical if and only if $u(x) = 0$.

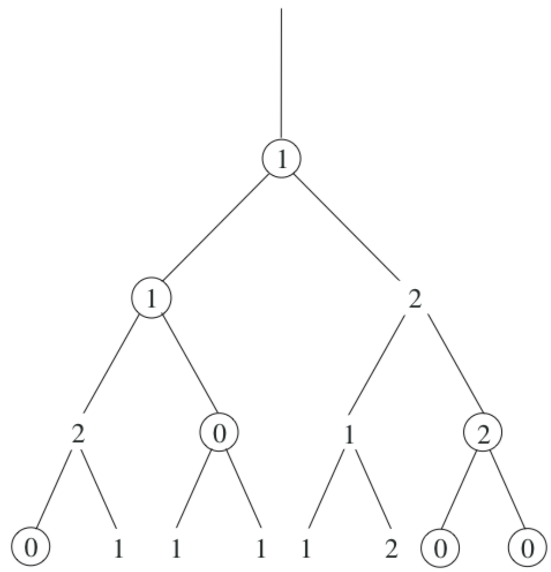

Fig. 4: A recurrent configuration on the ternary tree of height 5. Critical vertices are circled; if any of the circled vertices had fewer chips, the configuration would not be recurrent.

Lemma 2.2 *A configuration $u \in SP(T)$ is recurrent if and only if equality holds in (1) for every critical vertex x.*

Note that if v is a nonnegative configuration, its recurrent representative is given by

$$\hat{v} := (e+v)^\circ,$$

where e is the identity element of $SP(G)$ (the recurrent representative of 0); indeed, $\hat{v}$ is recurrent since e is recurrent, and $\hat{v} \equiv v \pmod{\Delta}$ since $e \in \Delta$.

We write δ_x for a single chip at a vertex x, and denote by $\hat{x} = (e + \delta_x)^\circ$ its recurrent form. Write $\langle \hat{x} \rangle$ for the cyclic subgroup of the sandpile group generated by $\hat{x}$.

The *principal branches* of T are the subtrees $T^1, \ldots, T^k$ rooted at the children $r_1, \ldots, r_k$ of the root of T. We include in T^i an edge from r_i to the sink; thus r_i has the same degree in T^i as in T, as the edge from r_i to r has been replaced by an edge from r_i to the sink.

Proposition 2.3 *Let T be a finite tree, and let $T^1, \ldots, T^k$ be its principal branches. Then*

$$SP(T)/\langle \hat{r} \rangle \simeq \bigoplus_{i=1}^{k} SP(T^i)/\langle (\hat{r}_1, \ldots, \hat{r}_k) \rangle \tag{2}$$

where r, r_i are the roots of T, T^i respectively.

The idea behind this result is that chips in different branches of T interact only via the root. Modding out by $\langle \hat{r} \rangle$ removes this interaction, so that the addition of chip configurations can be carried out independently

$\hat{r}$	$2\hat{r}$	$3\hat{r}$	$4\hat{r}$	$5\hat{r}$	$6\hat{r}$	$7\hat{r}$	$8\hat{r}$	$9\hat{r}$	$10\hat{r}$	$11\hat{r}$	$12\hat{r}$	$13\hat{r}$	$14\hat{r}$	$15\hat{r} = e$
2	0	1	2	0	1	2	2	2	0	1	2	0	1	2
0	1	1	1	2	2	2	2	0	1	1	1	2	2	2
2	2	2	2	2	2	2	0	1	1	1	1	1	1	1

Fig. 5: Multiples of the root $\hat{r}$ in the ternary tree of height 4. Each column vector represents a chip configuration which is constant on levels of the tree.

in each branch. Since firing the root in T sends a single chip to each r_i, we need to mod out on the right by the cyclic subgroup of $\bigoplus SP(T^i)$ generated by the element $(\hat{r}_1, \dots, \hat{r}_k)$.

Next we show that for regular trees, Proposition 2.3 can be strengthened to express $SP(T)$ as a direct sum. Let T_n be the wired regular tree of degree d and height n, i.e., the regular tree with leaves collapsed to a sink vertex and an edge added from the root to the sink. The chip configurations which are constant on the levels of T_n form a subgroup of $SP(T_n)$. If each vertex at height k has a_k chips, we can represent the configuration as a vector $(a_1, \dots, a_{n-1})$. If such a recurrent configuration is zero on a level, all vertices between that level and the root are critical, so by Lemma 2.2 they must have $d-1$ chips each. The recurrent configurations constant on levels are thus in bijection with integer vectors $(a_1, \dots, a_{n-1})$ with $0 \le a_i \le d-1$ subject to the constraint that if $a_i = 0$ then $a_1 = \ldots = a_{i-1} = d-1$.

The following lemma uses the lexicographic order given by $a < b$ if for some k we have $a_{n-1} = b_{n-1}, \dots, a_{k+1} = b_{k+1}$ and $a_k < b_k$. In the *cyclic lexicographic order* on recurrent vectors we have also $(d-1, \dots, d-1) < (d-1, \dots, d-1, 0)$.

Lemma 2.4 *If u, v are recurrent configurations on T_n that are constant on levels, write $u \rightsquigarrow v$ if v follows u in the cyclic lexicographic order on the set of recurrent vectors. Then for every integer $k \ge 0$, we have*

$$k\hat{r} \rightsquigarrow (k+1)\hat{r}.$$

Figure 4 illustrates Lemma 2.4 for the 15 recurrent vectors on the ternary tree of height 4.

Lemma 2.5 *Let T_n be the wired regular tree of degree d and height n, and let $R(T_n)$ be the subgroup of $SP(T_n)$ generated by $\hat{r}$. Then $R(T_n)$ consists of all recurrent configurations that are constant on levels, and its order is*

$$\#R(T_n) = \frac{(d-1)^n - 1}{d-2}. \tag{3}$$

Proposition 2.6 *Let T_n be the wired regular tree of degree d and height n, and let $R(T_n) = (\hat{r})$ be the subgroup of $SP(T_n)$ generated by the root. Then*

$$SP(T_n) \simeq R(T_n) \oplus \frac{SP(T_{n-1}) \oplus \ldots \oplus SP(T_{n-1})}{(R(T_{n-1}), \dots, R(T_{n-1}))}$$

with $d-1$ summands of $SP(T_{n-1})$ on the right side.

Proposition 2.6 is proved by defining a projection map p of the full group $SP(T_n)$ onto the subgroup $R(T_n)$. Since $R(T_n)$ consists of the configurations constant on levels, we can define such a map by symmetrizing:

$$p(u)(x) = (d-1)^{n-|x|} \sum_{|y|=|x|} u(y). \tag{4}$$

By construction, $p(u)$ is constant on levels, so the image of p lies in $R(T_n)$ by Lemma 2.5. If u is already constant on levels, then since there are $(d-1)^{|x|}$ terms in the sum (4), we obtain

$$p(u) = (d-1)^n u = u$$

where the second inequality follows from (3).

We remark that Proposition 2.6 fails for general trees. For example, if T is the tree consisting of a root with 2 children each of which have 3 children, then $SP(T) \simeq \mathbb{Z}/40\mathbb{Z}$, and $R(T) \simeq \mathbb{Z}/10\mathbb{Z}$ is not a summand.

Theorem 2.1 follows directly from Lemma 2.5 and Proposition 2.6 by induction.

3 The Rotor-Router Model on Regular Trees

Let T be the infinite d-regular tree. In *rotor-router aggregation*, we grow a cluster of points in T by repeatedly starting chips at the origin o and letting them perform rotor-router walk until they exit the cluster. Beginning with $A_1 = \{o\}$, define the cluster A_n inductively by

$$A_n = A_{n-1} \cup \{x_n\}, \qquad n > 1$$

where $x_n \in T$ is the endpoint of a rotor-router walk started at o and stopped on first exiting A_{n-1}. We do not change the positions of the rotors when adding a new chip. Thus the sequence $(A_n)_{n\geq 1}$ depends only on the choice of the initial rotor configuration.

A rotor configuration is *acyclic* if the rotors form no oriented cycles. Since we are working on a tree, this is equivalent to the requirement that for any two neighboring vertices x and y, if the rotor at x points to y, then the rotor at y does not point to x.

Let

$$B_r = \{x \in T \,:\, |x| \leq r\}$$

be the ball of radius r centered at the origin $o \in T$. Here $|x|$ is the length (in edges) of the shortest path from o to x. Write

$$b_r = \#B_r = 1 + d\frac{(d-1)^r - 1}{d-2}.$$

Theorem 3.1 *[13] Let A_n be the region formed by rotor-router aggregation on the infinite d-regular tree, starting from n chips at o. If the initial rotor configuration is acyclic, then*

$$A_{b_r} = B_r.$$

Thus, provided we start with an acyclic configuration of rotors, the occupied cluster A_n is a perfect ball for suitable values of n. It follows that at all other times the cluster is as close as possible to a ball: if $b_r < n < b_{r+1}$, then $B_r \subset A_n \subset B_{r+1}$.

The proof of Theorem 3.1 uses the sandpile group of a finite regular tree, whose structure is given by Theorem 2.1. We first define the *rotor-router group* of a graph and show that it is isomorphic to the sandpile group. Although our application is to trees, this construction works for any graph. Let G be a strongly connected finite directed graph without loops. Fix a sink vertex s in G. Given a configuration of rotors R, write $e_x(R)$ for the rotor configuration resulting from starting a chip at x and letting it walk according to the rotor-router rule until it reaches the sink. (Note that if the chip visits a vertex infinitely often, it visits all of its neighbors infinitely often; since G is strongly connected, the chip eventually reaches the sink.) We view R as a subgraph of G in which every vertex except the sink has out-degree one.

We write $Rec(G)$ for the set of *oriented spanning trees* of G rooted at the sink, that is, acyclic spanning subgraphs of G in which every vertex except the sink has out-degree one.

Lemma 3.2 *If $R \in Rec(G)$, then $e_x(R) \in Rec(G)$.*

Lemma 3.3 *If $R_1, R_2 \in Rec(G)$ and $e_x(R_1) = e_x(R_2)$, then $R_1 = R_2$.*

Thus for any vertex x of G, the operation e_x of adding a chip at x and routing it to the sink acts invertibly on the set of states $Rec(G)$ whose rotors form oriented spanning trees rooted at the sink. In this sense the oriented spanning trees are analogous to the recurrent states in chip-firing. We define the *rotor-router group* $RR(G)$ as the subgroup of the permutation group of $Rec(G)$ generated by $\{e_x\}_{x\in G}$. One can show that the operators e_x commute, so the group $RR(G)$ is abelian; for a general discussion and proof of this property, which is shared by a wide class of models including the abelian sandpile and the rotor-router, see [9].

Lemma 3.4 *$RR(G)$ acts transitively on $Rec(G)$.*

Theorem 3.5 *Let G be a strongly connected finite directed graph without loops, let $RR(G)$ be its rotor-router group, and $SP(G)$ its sandpile group. Then $RR(G) \simeq SP(G)$.*

This isomorphism is mentioned in the physics literature [21, 22], although the proof is not recorded there. For a proof, see [13].

A function H on the vertices of a directed graph G is *harmonic* if

$$H(x) = \frac{1}{\text{outdeg}(x)} \sum_{y \leftarrow x} H(y)$$

for all vertices x.

Lemma 3.6 *Let H be a harmonic function on the vertices of G. Suppose chips on G can be routed, starting with $u(x)$ chips at each vertex x and ending with $v(x)$ chips at each vertex x, in such a way that the initial and final rotor configurations are the same. Then*

$$\sum_{x\in V(G)} H(x)u(x) = \sum_{x\in V(G)} H(x)v(x).$$

Let T_n be the regular tree of degree d and height n, with an edge added from the root r to the sink o. Denote by $(X_t)_{t\geq 0}$ the simple random walk on T_n, and let τ be the first hitting time of the set consisting of the leaves and the sink. Fix a leaf z of T_n, and let

$$H(x) = \mathbb{P}_x(X_\tau = z) \tag{5}$$

be the probability that random walk started at x and stopped at time τ stops at z. A standard result about gambler's ruin gives

$$H(r) = \frac{a-1}{a^n-1}, \tag{6}$$

where $a = d-1$.

Applying Lemma 3.6 with H given by (5), and using the isomorphism of the rotor-router and sandpile groups, we can prove the following lemma.

Lemma 3.7 *Let $a = d-1$. If the initial rotor configuration on T_n is acyclic, then starting with $\frac{a^n-1}{a-1}$ chips at the root, and stopping each chip when it reaches a leaf or the sink, exactly one chip stops at each leaf, and the remaining $\frac{a^{n-1}-1}{a-1}$ chips stop at the sink. Moreover the starting and ending rotor configurations are identical.*

Proof of Theorem 3.1: Define a modified aggregation process A'_n as follows. Stop the n-th chip when it either exits the occupied cluster A'_{n-1} or returns to o, and let

$$A'_n = A'_{n-1} \cup \{x'_n\}$$

where x'_n is the point where the n-th chip stops. By relabeling the chips, this yields a time change of the original process, i.e. $A'_n = A_{f(n)}$ for some sequence $f(1), f(2), \ldots$. Thus it suffices to show $A'_{c_\rho} = B_\rho$ for some sequence $c_1, c_2, \ldots$. We will show by induction on ρ that this is the case for

$$c_\rho = 1 + (a+1)\sum_{t=1}^{\rho} \frac{a^t-1}{a-1},$$

and that after c_ρ chips have stopped the rotors are in their initial state. For the base case $\rho = 1$, we have $c_1 = a+2 = d+1$. The first chip stops at o, and the next d stop at each of the neighbors of o, so $A'_{d+1} = B_1$. Since the rotor at o has performed one full turn, it is back in its initial state.

Assume now that $A'_{c_{\rho-1}} = B_{\rho-1}$ and that the rotors are in their initial acyclic state. Starting with $c_\rho - c_{\rho-1}$ chips at o, let each chip in turn perform rotor-router walk until either returning to o or exiting the ball $B_{\rho-1}$. Then each chip is confined to a single principal branch of the tree, and each branch receives $\frac{a^\rho-1}{a-1}$ chips. By Lemma 3.7, exactly one chip will stop at each leaf $z \in B_\rho - B_{\rho-1}$, and the remainder will stop at o. Thus $A'_{c_\rho} = B_\rho$. Moreover, by Lemma 3.7, once all chips have stopped, the rotors are once again in their initial state, completing the inductive step.

Much previous work on the rotor-router model has taken the form of comparing the behavior of rotor-router walk with the expected behavior of random walk. For example, Cooper and Spencer [6] show that for any configuration of chips on even lattice sites in $\mathbb{Z}^d$, letting each chip perform rotor-router walk for n steps results in a configuration that differs by only constant error from the expected configuration had the chips performed independent random walks. We continue in this vein by investigating the recurrence and transience of rotor-router walk on trees. If a walk started at the origin never returns to the origin, we say it "escapes to infinity." Such a walk visits each vertex only finitely many times, so the positions of the rotors after such a walk are well-defined.

Start with n chips at the origin in the regular ternary tree, and let them perform rotor-router walks one by one, stopping the walks if they return to the origin. For each chip, record whether it returns to the origin or escapes to infinity. We say that a binary word $a_1 \dots a_n$ is an *escape sequence* if there exists an initial rotor configuration so that the k-th chip escapes to infinity if and only if $a_k = 1$. The following result characterizes all possible escape sequences on the ternary tree.

Theorem 3.8 *[13] Let $a = a_1 \dots a_n$ be a binary word. For $j \in \{1, 2, 3\}$ write $a^{(j)} = a_j a_{j+3} a_{j+6} \dots$. Then a is an escape sequence for some rotor configuration on the infinite ternary tree if and only if for each j and all $k \geq 2$, every subword of $a^{(j)}$ of length $2^k - 1$ contains at most 2^{k-1} ones.*

In particular, the all zeros sequence is an escape sequence. In [13] we strengthen this slightly, showing that there exists an initial rotor configuration on the infinite ternary tree which makes rotor-router walk recurrent.

While Theorem 3.8 completely characterizes the possible escape sequences for rotor-router walk on the infinite ternary tree, we know nothing about the possible escape sequences for rotor-router walk on another natural class of transient graphs, namely $\mathbb{Z}^d$ for $d \geq 3$. We conclude with the following open question: does there exist a rotor configuration on $\mathbb{Z}^d$ for $d \geq 3$ which makes rotor-router walk recurrent? We remark that Jim Propp has found such a configuration on $\mathbb{Z}^2$.

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