The quasiinvariants of the symmetric group

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Abstract. For m a non-negative integer and G a Coxeter group, we denote by $\mathbf{QI_m}(G)$ the ring of m-quasiinvariants of G, as defined by Chalykh, Feigin, and Veselov. These form a nested series of rings, with $\mathbf{QI_0}(G)$ the whole polynomial ring, and the limit $\mathbf{QI}_\infty(G)$ the usual ring of invariants. Remarkably, the ring $\mathbf{QI_m}(G)$ is freely generated over the ideal generated by the invariants of G without constant term, and the quotient is isomorphic to the left regular representation of G. However, even in the case of the symmetric group, no basis for $\mathbf{QI_m}(G)$ is known. We provide a new description of $\mathbf{QI_m}(S_n)$, and use this to give a basis for the isotypic component of $\mathbf{QI_m}(S_n)$ indexed by the shape $[n-1, 1]$.

Résumé. Pour m un entier positif ou nul et G un groupe de Coxeter, nous notons $QI_m(G)$ l'anneau des quasiinvariants définis par Chalykh, Feigin et Veselov. On obtient ainsi une série d'anneaux emboités, $QI_0(G)$ étant l'anneau des polynômes, et la limite $QI_\infty(G)$ l'anneau des invariants usuels. Il est remarquable que l'anneau $QI_m(G)$ est librement généré sur l'idéal engendré par les invariants de G sans terme constant, et le quotient est isomorphe à la représentation régulière à gauche de G. Cependant, même dans le cas du groupe symétrique, aucune base de $QI_m(G)$ n'est connue. Nous donnons une nouvelle description de $QI_m(G)$ et l'utilisons pour obtenir une base du composant isotypique de $QI_m(S_n)$ indexée par la partition $(n-1, 1)$.

Keywords: symmetric group, invariants, quasiinvariants

1 Introduction

A permutation $\sigma \in S_n$ acts on a polynomial in $\mathbf{R} = \mathbb{Q}[x_1, \ldots, x_n]$ by permutation of indices:

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \tag{1}$$

The S_n-invariant polynomials are known as symmetric functions, and denoted by Λ_n. It is well known (*e.g.,* [Sta99]) that the elements of Λ_n without constant term are generated by the elementary symmetric functions $\{e_1, \ldots, e_n\}$ where

$$e_j = \sum_{i_1 < i_2 < \cdots < i_j} x_{i_1} \ldots x_{i_j}. \tag{2}$$

†NSF grant DMS-0500557
‡NSF grant DMS-0500557

The ring of coinvariants of S_n is the quotient

$$\mathbf{R}/\langle e_1, \ldots, e_n \rangle. \tag{3}$$

As an S_n-module, the ring of coinvariants is known to be isomorphic to the left regular representation. It is also known that $\mathbf{R}$ is freely generated over this ideal which implies that if we choose a basis of minimal degree elements $\mathcal{B} = \{b_1, \ldots, b_{n!}\}$ for the ring of coinvariants, any element $P \in \mathbf{R}$ has a unique expansion

$$P = \sum_{i=1}^{n!} b_i g_i \tag{4}$$

with the $g_i \in \Lambda_n$. More information is given by the Hilbert series for the isotypic component of $\mathbf{R}$ corresponding to λ, namely

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda \, q^{cocharge(T)}}{(1-q)(1-q^2)\ldots(1-q^n)}. \tag{5}$$

Known bases for the ring of coinvariants with combinatorial descriptions include the Artin monomials and the Schubert polynomials.

In [CV90] and [FV02], Chalykh, Feigin and Veselov introduced a generalization of invariance known as "m-quasiinvariance". For the symmetric group, the m-quasiinvariants are the polynomials $P \in \mathbb{Q}[x_1, \ldots, x_n]$ which have the divisibility property

$$(x_i - x_j)^{2m+1} \,\Bigg|\, \Big(1 - (i,j)\Big) P \tag{6}$$

for every transposition (i, j). We set

$$\mathbf{QI}_m = \{m\text{-quasiinvariants of } S_n\}. \tag{7}$$

A straightforward calculation (which we do not do here) shows that the m-quasiinvariants of S_n form a ring and an S_n module, and that we have the following containments:

$$\mathbf{R} = \mathbf{QI}_0 \supset \mathbf{QI}_1 \supset \cdots \supset \mathbf{QI}_m \supset \cdots \supset \Lambda_n. \tag{8}$$

For all m, the ring $\mathbf{QI}_m$ was conjectured in [FV02] to be freely generated over the ideal generated by symmetric functions without constant term. The corresponding quotient $\mathbf{QI}_m/\langle e_1, \ldots, e_n \rangle$ was conjectured to be isomorphic as an S_n module to the left regular representation. These facts were proved in [EG02]. Further information was given in [FV03], where they showed the Hilbert series of the isotypic component of $\mathbf{QI}_m$ indexed by λ is given by

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda \, q^{m\left(\binom{n}{2} - content(\lambda(T))\right) + cocharge(T)}}{(1-q)(1-q^2)\ldots(1-q^n)}. \tag{9}$$

Here f_λ is the number of standard tableaux of shape λ and *content* and *cocharge* are two statistics on tableaux–we will not need the precise definitions, though we note that *content* only depends on the shape of T and thus is actually a statistic on partitions. We should note that the definitions, theorems and conjectures cited above were all phrased in terms of general Coxeter groups; it is only for simplicity that we have restricted our attention to S_n.

In light of the combinatorially interesting bases for the coinvariants in the classical (*i.e.,* $m = 0$) case, the authors have looked for a basis for larger m. In [FV02], and later in [BM05], a basis was given for the case $n = 3$. (The work [FV02] specifically described the quasiinvariants for dihedral groups; in particular for $D_3 \cong S_3$.) Further, in [FV03], Felder and Veselov provide integral expressions, $\phi^{(j)}(x)$ for $2 \leq j \leq n$, for the lowest degree non-symmetric m-quasiinvariants, i.e. those of degree $mn + 1$. In the present work, we give a complete basis of the isotypic component indexed by the partition $[n-1, 1]$. This is accomplished by means of a new characterization of $\mathbf{QI}_m$:

Theorem 1 *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \left(\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}\right)$$

where $ST(n)$ is the set of standard tableaux of size n, γ_T is a projection operator due to Young (defined in full detail in the next section) and V_T is the polynomial given by the product over the columns of T of the associated "Vandermonde determinants" (this is also defined in detail below). This characterization is proved using completely elementary methods. We then give a basis for the $[n-1, 1]$ isotypic component, using the previous characterization to show that our basis does, in fact, consist of quasiinvariants. Precisely, for T a standard Young tableau of shape $[n-1, 1]$ with j the entry in the second row, we set

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m dt. \tag{10}$$

With this definition, we have

Theorem 2 *The set*

$$\{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \ldots, Q_T^{n-2,m}\} \tag{11}$$

is a basis for $\gamma_T\left(\mathbf{QI}_m / \langle e_1, \ldots, e_n \rangle\right)$.

The fact $\mathbf{QI}_m$ is freely generated over $\langle e_1, \ldots, e_n \rangle$ then gives us that the set

$$\{Q_T^{k,m} e_1^{p_1} e_2^{p_2} \ldots e_n^{p_n}\} \qquad 0 \leq k \leq n-2, \qquad p_i \in \mathbb{N}$$

is a basis for $\mathbf{QI}_m$.

2 Acknowledgements

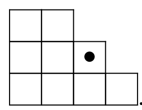
The authors are grateful for Adriano Garsia's guidance in this project, as well as for the support of the NSF.

3 Definitions and notation

Throughout this work, we will write elements of the symmetric group S_n using cycle notation. We will perform computations in the group algebra of S_n, and as such it will be useful to have shorthand notation for many commonly occurring elements. For a given subgroup A of S_n, we set

$$[A] = \sum_{\sigma \in A} \sigma \qquad \text{and}$$
$$[A]' = \sum_{\sigma \in A} \operatorname{sgn}(\sigma)\sigma.$$

The Young diagram of a partition λ consists of a collection of λ_i left-justified boxes in the i^{th} row of the positive integer lattice. These boxes are indexed by ordered pairs (i, j), where i is the row index and j is the column index. For example, in the following Young diagram of $[4, 3, 2]$, the cell $(2, 3)$ is marked:



A *tableau* of shape $\lambda \vdash n$ is a function from the cells of the Young diagram of λ to the set $\{1, \ldots, n\}$. We write the $T(i, j)$ for the value of T at the cell (i, j). For example, if T is the following tableau, $T(2, 3) = 8$:

6	7		
4	5	8	
1	2	3	9

We call a tableau *standard* if it is injective and the entries increase across the rows and up the columns. For example, the tableau above is standard. We denote the set of standard tableaux of shape λ by $ST(\lambda)$ and the set of all standard tableaux with n boxes by $ST(n)$.

Given a tableau T we let C_i be the set of elements in the i^{th} column and we define R_i similarly for the rows. We also set

$$C(T) = \bigcup_i \{(a, b) : a, b \in C_i\} \tag{12}$$
$$N(T) = \prod_i [C_i]' \tag{13}$$
$$P(T) = \prod_i [R_i] \tag{14}$$
$$\gamma_T = \frac{f_\lambda \, N(T)P(T)}{n!} \tag{15}$$
$$\lambda(T) = \text{ the shape of tableau } T. \tag{16}$$

Finally, we define the following useful polynomial associated with a tableau T:

$$V_T = \prod_{(i,j) \in C(T)} (x_i - x_j). \tag{17}$$

4 Useful Facts From Representation Theory

The following is a fundamental fact about the representation theory of the symmetric group.

Proposition 1 *For W a finite dimensional S_n-module,*

$$W \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_\lambda} \tag{18}$$

where the V_λ are the irreducible representations of S_n and the m_λ are non-negative integers.

The vector space and S_n-module $V_\lambda^{\oplus m_\lambda}$ is known as the *isotypic component* of V indexed by λ. Now, $\mathbf{QI}_m$is infinite dimensional, but it is the direct sum of homogeneous components, each of which are finite dimensional. So we have that each homogeneous component of $\mathbf{QI}_m$decomposes into the direct sum of irreducibles. The direct sum of all copies of V_λ occurring in this decomposition is still itself an S_n-module, and is still referred to as the isotypic component indexed by λ. However, we will find the following decomposition of V more useful.

Proposition 2 *On any S_n module W, the group algebra elements $\{\gamma_T\}_{T\in ST(n)}$ act as projection operators. In symbols, we have the conditions*

1. $\gamma_T^2 = \gamma_T$
2. $W = \bigoplus_{T\in ST(n)} \gamma_T W$.

Note that in this decomposition, unlike the previous one, the direct summands are not themselves S_n-modules. We do have the following proposition, however, nicely relating the previous two.

Proposition 3 *For any S_n module W,*

$$\bigoplus_{T\in ST(\lambda)} \gamma_T W \tag{19}$$

is the isotypic component of W indexed by λ.

In the case of the quasiinvariants, we have the following

Proposition 4 *The $\mathbb{Q}$-vector space of m-quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T\in ST(n)} \gamma_T \mathbf{QI}_m.$$

Our goal will be to use the decomposition $\mathbf{QI}_m/\langle e_1, \dots e_n\rangle = \bigoplus_T \gamma_T \left(\mathbf{QI}_m/\langle e_1, \dots e_n\rangle\right)$ to find a partial basis for this space.

5 A New Characterization of S_n-Quasiinvariants

In this section we prove the following theorem:

Theorem 1 *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T\in ST(n)} \left(\gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}\right).$$

We will prove this by showing

$$\gamma_T \mathbf{QI}_m = \gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}. \tag{20}$$

Combining (20) with Proposition 4 will prove the theorem. We note first that one containment of (20) is easy:

$$\gamma_T \mathbf{QI}_m \subseteq \gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}. \tag{21}$$

By definition, any element of $\gamma_T \mathbf{QI}_m$ is also in $\gamma_T \mathbf{R}$. Thus, we must only show that for $Q \in \mathbf{QI}_m$ we have

$$\gamma_T Q \in V_T^{2m+1}\mathbf{R}. \tag{22}$$

Let $P = \gamma_T Q = N(T)Q'$. P must be anti-symmetric with respect to all transpositions in $C(T)$ since it is in the image of $N(T)$. Thus, for any $(a,b) \in C(T)$, $\Big(1-(a,b)\Big)P = 2P$. Hence $(x_a - x_b)^{2m+1}$ divides $2P$ (and also P) for all $(a,b) \in C(T)$. This establishes equation (21).

It remains to show that for all standard tableaux T and all $m \geq 0$, we have the following containment of vector spaces:

$$\gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R} \subseteq \gamma_T \mathbf{QI}_m. \tag{23}$$

Since γ_T is an idempotent, it suffices to show that for any polynomial P in the ideal $V_T^{2m+1}\mathbf{R}$, $\gamma_T P = P$ implies that P is m-quasiinvariant. With this in mind, we let P be such that $V_T^{2m+1}|P$ and $\gamma_T P = P$. We wish to show that $\Big(1-(a,b)\Big)P$ is divisible by $(x_a - x_b)^{2m+1}$ for all transpositions (a,b). We first consider the case where a and b are in the same column of T. In this case we have

$$(a,b)N(T) = -N(T)$$

and so

$$(a,b)P = (a,b)\gamma_T P = -\gamma_T P = -P.$$

Thus

$$\Big(1-(a,b)\Big)P = 2P \in V_T^{2m+1}\mathbf{R}$$

which is divisible by the required factor of $(x_a - x_b)^{2m+1}$.

The remaining case, showing that $\Big(1-(a,b)\Big)P$ is divisible by $(x_a - x_b)^{2m+1}$ when a and b are in different columns, is the only difficult part of the proof. Suppose without loss that a is in column i, and is to the left of b, which is in column j. We define $\alpha_{i,b} \in \mathbb{Q}[S_n]$ to be the sum of all transpositions (c,b) where c is an element of column i. The key property of this is element is the following:

Lemma 1 *The operator $\alpha_{i,b}$ preserves γ_T. In symbols,*

$$\alpha_{i,b}\gamma_T = \gamma_T.$$

The proof of Lemma 1, though completely elementary, is lengthy and is omitted here.

With this fact in hand, it is not difficult to complete the proof. By Lemma 1, P is preserved by $\alpha_{i,b}$:

$$\alpha_{i,b}P = \alpha_{i,b}\gamma_T P = \gamma_T P = P. \tag{24}$$

This fact and the definition of $\alpha_{i,b}$ gives

$$\begin{aligned}\Big(1-(a,b)\Big)P &= P-(a,b)P &(25)\\ &= \alpha_{i,b}P-(a,b)P &(26)\\ &= \sum_{\substack{c\in C_i\\ c\neq a}}(c,b)P. &(27)\end{aligned}$$

Since $P \in V_T^{2m+1}\mathbf{R}$, for any $c \in C_i$ with $c \neq a$ we can rewrite P as

$$P = (x_c - x_a)^{2m+1}(\text{other factors}). \tag{28}$$

Thus

$$(c,b)P = (x_b - x_a)^{2m+1}(\text{other factors}) \tag{29}$$

and we have

$$(x_b - x_a)^{2m+1} \text{ divides } (c,b)P \text{ for every } c \in C_i \text{ with } c \neq a. \tag{30}$$

Hence $(x_b - x_a)^{2m+1}$ divides the right-hand side of equation 27, which completes the proof.

6 A Basis For The Isotypic Component $\lambda(T) = [n-1,1]$

In this section, we refer to the quotient $\mathbf{QI}_m/\langle e_1,\ldots,e_n\rangle$ by the symbol $\mathbf{QI}_m^*$. Our object here is to describe a basis for $\gamma_T\mathbf{QI}_m^*$ when T has a hook shape of the form $[n-1,1]$. Until otherwise specified, let λ be the partition $[n-1,1]$ and let T be one of the $(n-1)$ standard tableaux of shape λ. In fact T is uniquely defined by the lone entry of the second row. Suppose it's $j \in \{2,3,\ldots,n\}$. We define

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t-x_i)^m dt. \tag{31}$$

Theorem 2 *The polynomials $\{Q_T^{k,m}\}_{k=0}^{n-2}$ are a set of representatives for a basis of $\gamma_T\mathbf{QI}_m^*$.*

By equation (9) we know this consists of the right number of elements, and they are of the right degree. Thus we are faced with two tasks: showing the elements $Q_T^{k,m}$ are, in fact, m-quasiinvariant, and showing that they are linearly independent in $\mathbf{QI}_m^*$. To show the m-quasiinvariance of $Q_T^{k,m}$ we must, by Theorem 1, show these polynomials are preserved by γ_T and divisible by V_T^{2m+1}.

Lemma 2 *The polynomials $Q_T^{k,m}$ are preserved by the operator γ_T. That is,*

$$\gamma_T Q_T^{k,m} = Q_T^{k,m}. \tag{32}$$

Proof: We will prove this inductively, by showing that the $Q_T^{k,m}$ satisfy a recursion that respects the action of γ_T. In what follows, e_i will denote the ith elementary symmetric function in the variables $x_1, \ldots, x_n$, with the convention that $e_0 = 1$. We first state for reference a classical symmetric function identity:

$$\prod_{i=1}^{n}(t - x_i) = \sum_{i=0}^{n}(-1)^i e_i t^{n-i}. \tag{33}$$

We now show that for $m > 1$, we have the recurrence:

$$Q_T^{k,m} = \sum_{i=0}^{n}(-1)^i e_i \, Q_T^{n-i+k,m-1}. \tag{34}$$

This can be seen by first expanding the definition of $Q_T^{k,m}$ to get

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left(\prod_{i=1}^{n}(t - x_i)\right) t^k \prod_{l=1}^{n}(t - x_l)^{m-1} dt. \tag{35}$$

Substituting (33) into (35) and pulling out the factors not involving t gives

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left(\sum_{i=0}^{n}(-1)^i e_i t^{n-i}\right) t^k \prod_{l=1}^{n}(t - x_l)^{m-1} dt \tag{36}$$

$$= \sum_{i=0}^{n}(-1)^i e_i \int_{x_1}^{x_j} t^{n-i+k} \prod_{l=1}^{n}(t - x_l)^{m-1} dt \tag{37}$$

$$= \sum_{i=0}^{n}(-1)^i e_i \, Q_T^{n-i+k,m-1}. \tag{38}$$

We now proceed with a proof of Lemma 2 by induction on m. From the definition of $Q_T^{k,m}$ we have

$$Q_T^{k,0} = \int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t - x_i)^0 dt \tag{39}$$

$$= \frac{1}{k+1}\left(x_j^{k+1} - x_1^{k+1}\right). \tag{40}$$

That $x_j^{k+1} - x_1^{k+1}$ is preserved by γ_T is a straightforward calculation which we omit here.

We can now complete the proof by using equation (34), the inductive hypothesis, and the fact that symmetric functions commute with group algebra elements:

$$\gamma_T Q_T^{k,m} = \sum_{i=0}^{n} (-1)^i \gamma_T \big(e_i Q_T^{n-i+k,m-1}\big) \tag{41}$$

$$= \sum_{i=0}^{n} (-1)^i e_i \big(Q_T^{n-i+k,m-1}\big) \tag{42}$$

$$= Q_T^{k,m}. \tag{43}$$

□

In order to complete our task of showing that $Q_T^{k,m} \in \mathbf{QI}_m$, we must show that $(x_j - x_1)^{2m+1}$ divides $Q_T^{k,m}$. We do so by proving the following stronger statement:

Lemma 3 *For all k,*

$$\lim_{x_1 \to x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \frac{(-1)^m m!^2}{(2m+1)!} x_j^k \prod_{\substack{i=2 \\ i \neq j}}^{n} (x_j - x_i)^m.$$

Proof: This is a straightforward calculation. Expanding $Q_T^{k,m}$ according to the definition gives

$$\lim_{x_j \to x_1} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \lim_{x_j \to x_1} \frac{\int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m dt}{(x_j - x_1)^{2m+1}} \tag{44}$$

which is an indeterminate expression of the form $\frac{0}{0}$. Repeatedly applying L'Hopital's rule and evaluating the numerator with Leibniz's rule for differentiation under the integral gives that the expression in (44) equals

$$\lim_{x_j \to x_1} \frac{(-1)^m \cdot m! \int_{x_1}^{x_j} t^k \prod_{\substack{i=1 \\ i \neq j}}^{n} (t - x_i)^m dt}{(2m+1)(2m)(2m-1)\cdots(m+2)(x_j - x_1)^{m+1}}.$$

One more application of L'Hopital's rule, evaluated this time with the Fundamental Theorem of Calculus, yields

$$\lim_{x_j \to x_1} \frac{(-1)^m \cdot m! \cdot x_j^k \prod_{\substack{i=1 \\ i \neq j}}^{n} (x_j - x_i)^m}{(2m+1)(2m)(2m-1)\cdots(m+1)(x_j - x_1)^m}. \tag{45}$$

We now cancel the term $(x_j - x_1)^m$ from both numerator and denominator and the Lemma is proven. □

The polynomiality of $\lim_{x_1 \to x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}}$ immediately gives that $(x_j - x_1)^{2m+1}$ divides $Q_T^{k,m}$. Thus we have established that $Q_T^{k,m}$ is m-quasiinvariant.

We need one more lemma to establish Theorem 2.

Lemma 4 *The set* $\{Q_T^{k,m}\}_{k=0}^{n-2}$ *is linearly independent in* $\mathbf{QI}_m^*$.

Proof: As these polynomials are of different degrees, it suffices to show that they are non-zero in $\mathbf{QI}_m^*$. Put another way, we must show that $Q_T^{k,m}$ is not in the ideal of $\gamma_T \mathbf{QI}_m$ generated by $\langle e_1, \ldots, e_n \rangle$. Equivalently, we must show that polynomials of the form

$$P_k = Q_T^{k,m} + A_1 Q_T^{k-1,m} + \cdots + A_{k-1} Q_T^{1,m} + A_k Q_T^{0,m} \tag{46}$$

(where the A_i are symmetric functions of degree i) can only equal 0 if $k \geq n-1$. To accomplish this, we use the explicit formulas for $\lim_{x_j \to x_1} Q_T^{k,m}/V_T^{2m+1}$ given by Lemma 3 to show the stronger statement

$$\lim_{x_j \to x_1} P_k / V_T^{2m+1} = 0 \implies k \geq n-1 \tag{47}$$

regardless of the choice of the symmetric functions. Letting $\widetilde{A_i}$ denote the limit $x_j \to x_1$ applied to the symmetric function A_i, and assuming without loss of generality that $j = 2$, we have

$$\lim_{x_2 \to x_1} P_k / V_T^{2m+1} = 0 \tag{48}$$

$$\implies \left(\frac{(-1)^m m!^2}{(2m+1)!} \prod_{i=3}^{n} (x_1 - x_i)^m \right) \left(x_1^k + \widetilde{A_1} x_1^{k-1} + \cdots + \widetilde{A_{k-1}} x_1 + \widetilde{A_k} \right) = 0 \tag{49}$$

$$\implies x_1^k + \widetilde{A_1} x_1^{k-1} + \cdots + \widetilde{A_{k-1}} x_1 + \widetilde{A_k} = 0 \tag{50}$$

$$\implies \lim_{x_2 \to x_1} \left(x_1^k + A_1 x_1^{k-1} + \cdots + A_k \right) = 0. \tag{51}$$

We set

$$Q(x_1, \ldots, x_n) = x_1^k + A_1 x_1^{k-1} + \cdots + A_k \tag{52}$$

and now equation (51) implies that

$$Q(x_1, \ldots, x_n) = (x_2 - x_1) \cdot R(x_1, \ldots, x_n). \tag{53}$$

However, Q must be symmetric with respect to all pairs of variables not involving x_1. Thus, for any $\sigma \in S_{\{2,3,\ldots,n\}}$, $\sigma Q = Q$ and so

$$Q(x_1, \ldots, x_n) = \sigma Q(x_1, \ldots, x_n) = (x_{\sigma(2)} - x_1) \cdot \sigma R(x_1, \ldots, x_n). \tag{54}$$

Hence $\prod_{i=2}^{n} (x_i - x_1)$ divides $Q(x_1, \ldots, x_n)$, and k, which is the degree of $Q(x_1, \ldots, x_n)$, must be greater than or equal to $n-1$. □

The proof of Theorem 2 now follows immediately from Lemmas 2, 3, and 4.

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