Pattern avoidance in dynamical systems

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Abstract. Orbits generated by discrete-time dynamical systems have some interesting combinatorial properties. In this paper we address the existence of forbidden order patterns when the dynamics is generated by piecewise monotone maps on one-dimensional closed intervals. This means that the points belonging to a sufficiently long orbit cannot appear in any arbitrary order. The admissible patterns are then (the inverses of) those permutations avoiding the so-called forbidden root patterns in consecutive positions. The last part of the paper studies and enumerates forbidden order patterns in shift systems, which are universal models in information theory, dynamical systems and stochastic processes. In spite of their simple structure, shift systems exhibit all important features of low-dimensional chaos, allowing to export the results to other dynamical systems via order-isomorphisms. This paper summarizes some results from [1] and [11].

Résumé. Les orbites générées par des systèmes dynamiques à temps discret ont quelques propriétés combinatoires intéressantes. Dans cet article on adresse l'existence de motifs d'ordre exclus quand la dynamique est générée par des applications monotones à parts sur des intervalles fermés en une dimension. Ceci signifie que les points appartenant à une orbite suffisamment longue ne peuvent pas apparaître dans un ordre arbitraire. Les motifs admissibles sont alors (les inverses de) ces permutations qui évitent les motifs exclus fondamentaux en positions consécutives. La derniere partie de l'article étudie et énumère les motifs exclus dans les systèmes de déplacement, qui sont des modèles universels dans la théorie de l'information, les systèmes dynamiques et les processus stochastiques. Malgré leur structure simple, les systèmes de déplacement manifestent toutes les propriétés importantes du chaos en basse dimension, permettant exporter les résultats aux autres systèmes dynamiques via des isomorphismes d'ordre. Cet article résume quelques résultats de [1] et [11].

Keywords: order patterns, deterministic and random sequences, permutations avoiding consecutive patterns, time series analysis, dynamical systems, shift maps

1 Introduction

In this paper we present an application of permutations avoiding generalized patterns to dynamical systems generated by the iteration of interval maps. Given a finite orbit of such a discrete-time dynamical system, we can associate to it a so-called *order pattern*, which transcribes the order of the points along the orbit. It turns out that, under some mild mathematical assumptions, not all order patterns can be materialized by the orbits of a given, one-dimensional dynamic. Furthermore, if an order pattern of a given length is 'forbidden', i.e., cannot occur, its absence pervades all longer patterns in form of more missing order patterns. This cascade of outgrowth forbidden patterns grows super-exponentially (in fact, factorially) with the length, all its patterns sharing a common structure. Of course, forbidden and allowed order patterns can be viewed as permutations; allowed patterns are then those permutations avoiding the so-called forbidden root patterns and their shifted patterns (see Sect. 4 for an exact formulation). Permutations avoiding generalized and consecutive patterns is a popular topic in combinatorics (see, e.g., [5, 9, 10]). It is in this light that we approach order patterns in the present paper. In fact, the measure-theoretical aspects of the underlying dynamical system play no role in the combinatorial properties of the order patterns defined by its orbits and hence will be only considered when necessary. Also for this reason we will not dwell on the dynamical properties of shift systems and their role as prototypes of chaotic maps once endowed with appropriate invariant measures; see [6, 8] for readable accounts.

Order relations belong rather to discrete than continuous mathematics. Only in the standard real line, order and metric are coupled, leading to such interesting results as Sarkovskii's theorem [13, 12]. But even in this special though important framework, order fails to be preserved by isomorphisms, that consistently only address dynamical properties such as invariant measures, periodicity, mixing properties, etc., and this reduces its applicability. Yet, order relations have been successfully applied in discrete dynamical systems and information theory, e.g., to evaluate the measure-theoretic and topological entropies [7, 2]. This paper is an extension of those investigations. Isomorphisms that preserve the possibly existing order relations of the dynamical systems they identify, are called order-isomorphisms.

Forbidden order patterns, the only ones we will consider in this paper, should not be mistaken for other sorts of forbidden patterns that may occur in dynamics with constraints. Forbidden patterns in symbol sequences occur, e.g., in Markov subshifts of finite type and, more generally, in random walks on oriented graphs. On the contrary, the existence of forbidden *order* patterns does not entail necessarily any restriction on the patterns of the corresponding symbolic dynamic: the variability of *symbol* patterns is given by the statistical properties of the dynamic. As a matter of fact, the symbolic dynamic of one-dimensional chaotic maps are used to generate pseudo-random sequences, although all such maps used in practice have forbidden order patterns. In general it is very difficult to work out the specifics of the forbidden patterns of a given map, but we will see that shifts on finite-symbol sequence spaces are an important exception: the detailed analysis of the forbidden patterns of these transformations is precisely the topic of this paper.

The existence of forbidden patterns is a hallmark of deterministic orbit generation and thus it can be used to discriminate deterministic from random time series. Indeed, thanks to the super-exponentially growing trail of outgrowth forbidden patterns, the probability of a false forbidden pattern in a truly stochastic process vanishes very fast with the pattern length and, consequently, a time series with missing order patterns of moderate length can be promoted to deterministic with virtually absolute confidence. The quantitative details depend, of course, on the specificities of the process (probability distribution, correlations, etc.). Only those chaotic maps with all forbidden patterns of exceedingly long length seem to be intractable from the practical point of view. Besides, applications need to address some key issues, such as the robustness of the forbidden patterns against observational noise, and the existence of false forbidden patterns in *finite*, random time series. We refer to [4] for these issues.

This paper is organized as follows. In Sect. 2 we briefly recall the basics of shift systems and symbolic dynamics. The concepts and notation introduced in this section (including the examples) will be used throughout. Order patterns and forbidden root patterns, together with the outgrowth forbidden patterns, are presented in Sect. 3. The structure of the outgrowth forbidden patterns and their asymptotic growth with the length are discussed in Sect. 4. Finally, Sect. 5 is devoted to the structure of allowed patterns and the existence of root forbidden patterns in one-sided shift systems. In the examples we present some interesting by-products of the theoretical results. This paper is an extended abstract of a presentation given at the FPSAC 2008 conference. It is essentially a condensed version of the full paper [1], with some additional unpublished results from [11].

2 Shift systems and symbolic dynamics

Let us start by recalling some basics of shift systems and symbolic dynamics. We set $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$.

Fix $N \geq 2$ and consider the measurable space $(\Omega, \mathcal{P}(\Omega))$, where $\Omega = \{0, 1, ..., N-1\}$ and $\mathcal{P}(\Omega)$ is the family of all subsets of Ω. Let $(\Omega^{\mathbb{N}_0}, \mathcal{B})$ denote the product space $\Pi_0^\infty(\Omega, \mathcal{P}(\Omega))$, i.e., $\Omega^{\mathbb{N}_0}$ is the space of (*one-sided*) *sequences* taking on values in the 'alphabet' Ω,

$$\Omega^{\mathbb{N}_0} = \{\omega = (\omega_n)_{n\in\mathbb{N}_0} : \omega_n \in \Omega\},$$

and $\mathcal{B}$ is the sigma-algebra generated by the *cylinder sets*

$$C_{a_0,...,a_n} = \{\omega \in \Omega^{\mathbb{N}_0} : \omega_k = a_k, 0 \le k \le n\}.$$

The elements of Ω are called *symbols* or *letters*. Segments of symbols of length L, like $\omega_k\omega_{k+1}\ldots\omega_{k+L-1}$, will be sometimes shortened $\omega_{[k,k+L-1]}$.

Furthermore, let $\Sigma : \Omega^{\mathbb{N}_0} \to \Omega^{\mathbb{N}_0}$ denote the (one-sided) *shift transformation* defined as

$$\Sigma : (\omega_0, \omega_1, \omega_2, ...) \mapsto (\omega_1, \omega_2, \omega_3, ...). \tag{1}$$

All probability measures on $(\Omega^{\mathbb{N}_0}, \mathcal{B})$ which make Σ a measure-preserving transformation are obtained in the following way [14]. For any $n \ge 0$ and $a_i \in \Omega$, $0 \le i \le n$, let a real number $p_n(a_0, ..., a_n)$ be given such that $p_n(a_0, ..., a_n) \ge 0$, $\sum_{a_0\in\Omega} p_0(a_0) = 1$, and $p_n(a_0, ..., a_n) = \sum_{a_{n+1}\in\Omega} p_{n+1}(a_0, ..., a_n, a_{n+1})$. If we define now

$$m(C_{a_0,...,a_n}) = p_n(a_0, ..., a_n),$$

then m can be extended to a probability measure on $(\Omega^{\mathbb{N}_0}, \mathcal{B})$. The resulting dynamical system, $(\Omega^{\mathbb{N}_0}, \mathcal{B}, m, \Sigma)$, is called the *one-sided shift space*.

Example 1. (a) Let $\mathbf{p} = (p_0, p_1, ..., p_{N-1})$, $N \ge 2$, be a probability vector with non-zero entries (i.e., $p_i > 0$ and $\sum_{i=0}^{N-1} p_i = 1$). Set $p_n(a_0, a_1, ..., a_n) = p_{a_0}p_{a_1}...p_{a_n}$. The resulting measure-preserving shift transformation is called the one-sided **p**-*Bernoulli shift*.

(b) Let $\mathbf{p} = (p_0, p_1, ..., p_{N-1})$ be a probability vector as in (a) and $P = (p_{ij})_{0\le i,j\le N-1}$ an $N \times N$ stochastic matrix (i.e., $p_{ij} \ge 0$ and $\sum_{i,j=0}^{N-1} p_{ij} = 1$) such that $\sum_{i=0}^{N-1} p_ip_{ij} = p_j$. Set then $p_n(a_0, a_1, ..., a_n) = p_{a_0}p_{a_0a_1}p_{a_1a_2}...p_{a_{n-1}a_n}$. The resulting measure-preserving shift transformation is called the one-sided $(\mathbf{p}, P)$-*Markov shift*.

(c) Let $\mathbf{S} = (S_n)_{n=0}^{\infty}$ be a discrete-time stochastic process on a probability space $(X, \mathcal{F}, \mu)$ started at time $n = 0$ with finitely many outcomes $\{0, 1, ..., N-1\} = \Omega$. The realizations (or "sample paths") $\mathbf{S}(x) = (S_0(x), ..., S_n(x), ...)$ are viewed as elements of $\Omega^{\mathbb{N}_0}$ endowed with the induced measure $p_n(a_0, ..., a_n) = \mu(\{x \in X : S_0(x) = a_0, \ldots, S_n(x) = a_n\}) \equiv \Pr\{S_0 = a_0, ..., S_n = a_n\}$, the probability of the event $S_0 = a_0, ..., S_n = a_n$. The resulting measure on $\Omega^{\mathbb{N}_0}$ is shift invariant if the stochastic process $\mathbf{S}$ is stationary. □

There are several metrics compatible with the topology of $\Omega^{\mathbb{N}_0}$, the most popular being

$$d_K(\omega, \omega') = \sum_{n=0}^{\infty} \frac{\delta(\omega_n, \omega'_n)}{K^n}, \tag{2}$$

where $\delta(\omega_n, \omega'_n) = 1$ if $\omega_n \ne \omega'_n$, $\delta(\omega_n, \omega_n) = 0$ and $K > 2$. Observe that given $\omega \in C_{a_0,...,a_n}$, then $d_K(\omega, \omega') < \frac{1}{K^n}$ if $\omega' \in C_{a_0,...,a_n}$ and $d_K(\omega, \omega') \ge \frac{1}{K^n}$ if $\omega' \notin C_{a_0,...,a_n}$, so that $C_{a_0,...,a_n} = B_{d_K}(\omega; \frac{1}{K^n})$, the open ball of radius K^{-n} and center ω in the metric space $(\Omega^{\mathbb{N}_0}, d_K)$. Since the base of the measurable sets are open balls, we conclude that $\mathcal{B}$ is the Borel sigma-algebra in the topology defined by the metric (2).

Continuity will play a role below. Since $\Sigma^{-1}C_{a_0,...,a_n} = \cup_{a\in\Omega}C_{a,a_0,...,a_n}$, Σ is continuous in $(\Omega^{\mathbb{N}_0}, d_K)$, each point $\omega \in \Omega^{\mathbb{N}_0}$ having exactly N preimages under Σ. Regarding the forward dynamic, Σ has N fixed points: $\omega = (\bar{n})$, $0 \le n \le N-1$, where *the overbar denotes indefinite repetition throughout*.

Let T be a measure preserving map on a probability space $(X, \mathcal{F}, \mu)$ and $\alpha = \{A_0, \ldots, A_{N-1}\}$ be a generating partition of the sigma-algebra $\mathcal{F}$ with respect to T, i.e., the subsets of the form $A_{a_0} \cap T^{-1}A_{a_1} \cap$

$... \cap T^{-n}A_{a_n}$ generate $\mathcal{F}$. Assume moreover that for every sequence $(A_{a_n})_{n\in\mathbb{N}_0}$, the set $\cap_{n=0}^{\infty} T^{-n}A_{a_n}$ contains at most one point of X; this assumption is fulfilled by any positively expansive continuous map or expansive homeomorphism on compact metric spaces (in particular, by the one-sided transformation we considered above) and implies that the coding map Φ to be defined below is one-to-one. Define now on the cylinder sets of $\Omega^{\mathbb{N}_0}$ the measure

$$m_T(C_{a_0,...,a_n}) = \mu(A_{a_0} \cap T^{-1}A_{a_1} \cap ... \cap T^{-n}A_{a_n}).$$

For $\omega \in \Omega^{\mathbb{N}_0}$ define the *coding map* $\Phi : X \to \Omega^{\mathbb{N}_0}$ by $\Phi(x) = (\omega_0, ..., \omega_n, ...)$, where $\omega_n = a_n \in \Omega$ if $T^n(x) \in A_{a_n}, n \geq 0$.

Then $\Phi : (X, \mathcal{F}, \mu) \to (\Omega^{\mathbb{N}_0}, \mathcal{B}, m_T)$ is measure-preserving (since, by definition, $\Phi^{-1}(C_{a_0,...,a_n}) = A_{a_0} \cap T^{-1}A_{a_1} \cap ... \cap T^{-n}A_{a_n}$) and, moreover,

$$\Phi \circ T = \Sigma \circ \Phi, \tag{3}$$

i.e., T and Σ are isomorphic and, hence, $(X, \mathcal{F}, \mu, T)$ and $(\Omega^{\mathbb{N}_0}, \mathcal{B}, m_T, \Sigma)$ are dynamically equivalent.

One interesting consequence of this construction is that the coded orbits of T contain any arbitrary pattern. Indeed, given any N-symbol *pattern* of length $L \geq 1$, $a_{[0,L-1]} := a_0a_1...a_{L-1}$ with symbols $a_n \in \{0, 1, ..., N-1\}$, choose

$$x_0 \in \bigcap_{n=0}^{L-1} T^{-n}A_{a_n}.$$

Then $\Phi(x_0) \in C_{a_0,...,a_{L-1}}$ and this for any $L \geq 1$. Letting $L \to \infty$, we conclude that the coding map Φ associates to each orbit $orb(x) = \{T^n(x) : n \geq 0\}$ a unique, infinitely long pattern of symbols from $\{0, 1, ..., N-1\}$, namely, $\Phi(x)$, for almost all $x \in X$.

Example 2. As a standard example (that it is going to be our workhorse), take $X = [0, 1]$, $\mathcal{F}$ the Borel sigma-algebra restricted to $[0, 1]$, $d\mu = \frac{1}{\pi\sqrt{x(1-x)}}dx$, $f(x) = 4x(1-x)$, the *logistic map*, and $\alpha = \{A_0 = [0, \frac{1}{2}), A_1 = [\frac{1}{2}, 1]\}$ (it is irrelevant whether the midpoint $\frac{1}{2}$ belongs to the left or to the right partition element). Then $\Phi(\frac{1}{4}) = (0, \bar{1})$, $\Phi(\frac{1}{2}) = (1, 1, \bar{0})$ and $\Phi(\frac{3}{4}) = (\bar{1})$. Observe for further reference that $\Phi(\frac{1}{4}) < \Phi(\frac{1}{2}) < \Phi(\frac{3}{4})$, where $<$ stands for the lexicographical order of $\{0, 1\}^{\mathbb{N}_0}$, but, e.g., $\Phi(\frac{1}{2}) > \Phi(1) = (1, \bar{0})$, hence the coding map $\Phi : [0, 1] \to \{0, 1\}^{\mathbb{N}_0}$ does not preserve the order structure. The fixed points of f are $0 = \Phi^{-1}((\bar{0}))$ and $\frac{3}{4} = \Phi^{-1}((\bar{1}))$. □

3 Forbidden order patterns

In the previous Section, we saw that the symbolic dynamic of a map defines any symbol pattern of any length, under rather general assumptions. In this Section we will see that the situation is not quite the same when considering order patterns.

Let $(X, <)$ be a totally ordered set and $T : X \to X$ a map. Given $x \in X$, the orbit of x is the set $\{T^n(x) : n \in \mathbb{N}_0\}$, where $T^0(x) \equiv x$ and $T^n(x) \equiv T(T^{n-1}(x))$. If x is not a periodic point of period less than $L \geq 2$, we can then associate with x an order pattern of length L, as follows. We say that x *defines the order pattern* $\pi = \pi(x) = [\pi_0, ..., \pi_{L-1}]$, where $\{\pi_0, ..., \pi_{L-1}\}$ is a permutation of $\{0, 1, \ldots, L-1\}$, if

$$T^{\pi_0}(x) < T^{\pi_1}(x) < ... < T^{\pi_{L-1}}(x).$$

Alternatively, we say that x *is of type* π or that π *is realized by* x. Thus, π is just a permutation on $\{0, 1, ..., L-1\}$, given by $0 \mapsto \pi_0, ..., L-1 \mapsto \pi_{L-1}$, that encapsulates the order of the points $x_n = T^n(x)$, $0 \leq n \leq L-1$. The set of order patterns of length L or, equivalently, the set of permutations on $\{0, 1, ..., L-1\}$ will be denoted by $\mathcal{S}_L$. Furthermore set

$$P_\pi = \{x \in X : x \text{ defines } \pi \in \mathcal{S}_L\}.$$

A plain difference between symbol patterns and order patterns of length L is their cardinality: the former grow exponentially with L (exactly as N^L, where N is the number of symbols) while the latter do super-exponentially. Specifically,

$$|\mathcal{S}_L| = L! \propto e^{L(\ln L-1)+(1/2)\ln 2\pi L} \tag{4}$$

(Stirling's formula), where, as usual, $|\cdot|$ denotes cardinality and $\propto$ means "asymptotically". Although one can construct functions whose orbits realize any possible order pattern (see below), numerical simulations support the conjecture that order patterns, like symbol patterns, grow only exponentially for 'well-behaved' functions [7]. In fact, if I is a closed interval of $\mathbb{R}$ and $f: I \to I$ is *piecewise monotone* (i.e., there is a finite partition of I into intervals such that f is continuous and strictly monotone on each of those intervals), then one can prove [7] that

$$|\{\pi \in \mathcal{S}_L : P_\pi \neq \varnothing\}| \propto e^{Lh_{top}(f)}, \tag{5}$$

where $h_{top}(f)$ is the topological entropy of f. From (4) and (5) we conclude:

Proposition 1. If f is a piecewise monotone self-map of a closed interval $I \subset \mathbb{R}$, then there are $\pi \in \mathcal{S}_L$, $L \geq 2$, such that $P_\pi = \varnothing$.

Order patterns that do not appear in any orbit of f are called *forbidden patterns*, at variance with the *allowed patterns*, for which there are intervals of points that realize them.

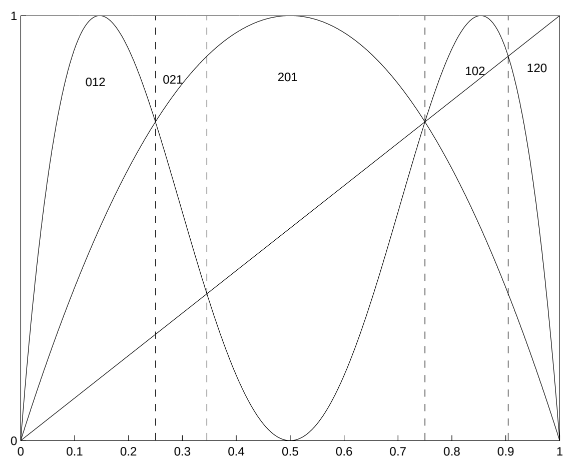

Fig. 1: The sets P_π, $\pi \in \sigma_3$, are graphically obtained by raising vertical lines at the crossing points of the curves $y = x$, $y = f(x)$, and $y = f^2(x)$. The three digits on the top are shorthand for order patterns (e.g., 012 stands for $[0,1,2]$). We see that $P_{[2,1,0]} = \varnothing$.

Example 3. As a simple illustration borrowed from [3], consider again the logistic map. For $L = 2$ we have

$$P_{[0,1]} = \left(0, \tfrac{3}{4}\right), \quad P_{[1,0]} = \left(\tfrac{3}{4}, 1\right).$$

Observe that the endpoints of P_π are period-1 (i.e., fixed) points (0 and $\frac{3}{4}$) or preimages of them ($f(1) = 0$). But already for $L = 3$ ($f^2(x) = -64x^4 + 128x^3 - 80x^2 + 16x$) there are permutations that are not realized (see Figure 1):

$$P_{[0,1,2]} = \left(0, \tfrac{1}{4}\right), \qquad P_{[0,2,1]} = \left(\tfrac{1}{4}, \tfrac{5-\sqrt{5}}{8}\right), \quad P_{[2,0,1]} = \left(\tfrac{5-\sqrt{5}}{8}, \tfrac{3}{4}\right),$$
$$P_{[1,0,2]} = \left(\tfrac{3}{4}, \tfrac{5+\sqrt{5}}{8}\right), \quad P_{[1,2,0]} = \left(\tfrac{5+\sqrt{5}}{8}, 1\right), \quad P_{[2,1,0]} = \varnothing.$$

When going from $\pi \in \mathcal{S}_2$ to $\pi \in \mathcal{S}_3$, we see that $P_{[0,1]}$ splits into the subintervals $P_{[0,1,2]}$, $P_{[0,2,1]}$ and $P_{[2,0,1]}$ at the eventually periodic point $\frac{1}{4}$ (preimage of $\frac{3}{4}$) and at the period-2 point $\frac{5-\sqrt{5}}{8}$. Likewise, $P_{[1,0]}$ splits into $P_{[1,0,2]}$ and $P_{[1,2,0]}$ at the period-2 point $\frac{5+\sqrt{5}}{8}$.

From a different perspective, as we move rightward in Figure 1 from the neighborhood of 0, where $x < f(x) < f^2(x)$, the curves $y = f(x)$ and $y = f^2(x)$ cross at $x = \frac{1}{4}$, what causes the first swap: $[0,1,2]$ transforms to $[0,2,1]$. In general, the crossings at $x = \frac{1}{4}, \frac{5-\sqrt{5}}{8}$ and $\frac{5+\sqrt{5}}{8}$ between $f^{\pi(i)}$ and $f^{\pi(i+1)}$ causes the exchange of $\pi(i)$ and $\pi(i+1)$ in the pre-crossing pattern. At $x = \frac{3}{4}$ all three curves cross and $[2,0,1]$ goes over to $[1,0,2]$.

The absence of $\pi = [2,1,0]$ triggers, in turn, an avalanche of longer missing patterns. To begin with, the pattern $[*,2,*,1,*,0,*]$ (where the wildcard $*$ stands eventually for any other entries of the pattern) cannot be realized by any $x \in [0,1]$ since the inequality

$$f^2(x) < f(x) < x \tag{6}$$

cannot hold true. By the same token, the patterns $[*,3,*,2,*,1,*]$, $[*,4,*,3,*,2,*]$, and, more generally, $[*,n+2,*,n+1,*,n,*] \in \mathcal{S}_L$, $0 \le n \le L-3$, cannot be realized either for the same reason (substitute x by $f^n(x)$ in (6)). $\square$

The same follows for the *tent map* $\Lambda : [0,1] \to [0,1]$,

$$\Lambda(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2-2x & \frac{1}{2} \le x \le 1 \end{cases} . \tag{7}$$

In fact, if λ is the Lebesgue measure, $d\mu = \frac{1}{\pi\sqrt{x(1-x)}}dx$ is (as in Example 2) the invariant measure of the logistic map $f(x) = 4x(1-x)$, and $\phi : ([0,1],\lambda) \to ([0,1],\mu)$ is the measure preserving isomorphism given by $\phi(x) = \sin^2(\frac{\pi}{2}x)$, then the dynamical systems $([0,1],\mathcal{B},\lambda,\Lambda)$ and $([0,1],\mathcal{B},\mu,f)$, where $\mathcal{B}$ is the Borel sigma-algebra restricted to the interval $[0,1]$, are *isomorphic* (or conjugate) by means of ϕ, i.e., $f \circ \phi = \phi \circ \Lambda$. Since, moreover, ϕ is strictly increasing, forbidden patterns for f correspond to forbidden patterns for Λ in a one-to-one way.

From the last paragraph it should be clear that isomorphic dynamical systems need not have the same forbidden patterns: the isomorphism (ϕ above) must also preserve the linear order of both spaces (supposing both spaces are linearly ordered), and this will be in general not the case. For example, the λ-preserving *dyadic map* $S_2 : x \mapsto 2x(\text{mod } 1)$, $0 \le x \le 1$, has no forbidden patterns of length 3, although it is isomorphic to the logistic and tent maps; the isomorphism with f, the logistic map, is proved via the semi-conjugacy $\varphi : ([0,1],\lambda) \to ([0,1],\mu)$, $\varphi(x) = \sin^2 \pi x$, which does not preserve order on account of being increasing on $(0,\frac{1}{2})$ and decreasing on $(\frac{1}{2},1)$. The same happens with the logistic map and the $(\frac{1}{2},\frac{1}{2})$-Bernoulli shift, a model for tossing of a fair coin, because, as we saw in Example 2, the corresponding isomorphism (actually, the coding map) $\Phi : [0,1] \to \{0,1\}^{\mathbb{N}_0}$ is not order-preserving.

Two isomorphic dynamical systems, whose phase spaces are linearly ordered, are called *order-isomorphic* if the isomorphism between them is also an order-isomorphism (i.e., it also preserves the order structure). It is obvious that two order-isomorphic systems (like those defined by the logistic and the tent map) have the same order patterns. The proof of the following proposition is omitted in this extended abstract, but it appears in [1].

Proposition 2. Given $X_1, X_2 \subset \mathbb{R}$ endowed with the standard Borel sigma-algebra $\mathcal{B}$, suppose that the dynamical systems $(X_1,\mathcal{B},\mu_1,f_1)$ and $(X_2,\mathcal{B},\mu_2,f_2)$ are isomorphic via a continuous map $\phi : X_1 \to X_2$. If f_1 is topologically transitive and, for all $x \in X_1$, both x and $\phi(x)$ define the same order patterns, then ϕ is order-preserving.

Finally, observe that the setting we are considering is more general than the setting of Kneading Theory since our functions need not be continuous (but only piecewise-continuous). Under some assumptions [12], the kneading invariants completely characterize the order-isomorphism of continuous maps.

4 Outgrowth forbidden patterns

According to Proposition 1, for every piecewise monotone interval map on $\mathbb{R}$, $f : I \to I$, there exist $\pi \in \mathcal{S}_L$, $L \geq 2$, which cannot occur in any orbit. We call them *forbidden patterns* for f and recall how their absence pervades all longer patterns in form of *outgrowth forbidden patterns* (see Example 3). Since $\pi = [\pi_0, ..., \pi_{L-1}]$ is forbidden for f, then the $2(L+1)$ patterns of length $L+1$,

$$[L, \pi_0, ..., \pi_{L-1}], [\pi_0, L, \pi_1, ..., \pi_{L-1}], ..., [\pi_0, ..., \pi_{L-1}, L],$$
$$[0, \pi_0 + 1, ..., \pi_{L-1} + 1], [\pi_0 + 1, 0, \pi_1 + 1, ..., \pi_{L-1} + 1], ..., [\pi_0 + 1, ..., \pi_{L-1} + 1, 0],$$

are also forbidden for f. The following form of the converse holds also true: if $[L, \pi_0, ..., \pi_{L-1}]$, $[\pi_0, L, ..., \pi_{L-1}]$, ..., $[\pi_0, ..., \pi_{L_0-1}, L] \in \mathcal{S}_{L+1}$ are forbidden, then $[\pi_0, ..., \pi_{L-1}] \in \mathcal{S}_L$ is also forbidden.

In general, all the forbidden patterns generated by π have the form

$$[*, \pi_0 + n, *, \pi_1 + n, *, ..., *, \pi_{L-1} + n, *] \in \mathcal{S}_N \tag{8}$$

with $n = 0, 1, ..., N-L$, where $N-L \geq 1$ is the number of wildcards $* \in \{0, 1, ..., n-1, L+n, ..., N-1\}$ (with $* \in \{L, ..., N-1\}$ if $n = 0$ and $* \in \{0, ..., N-L-1\}$ if $n = N-L$).

An upper bound on the number of outgrowth forbidden patterns of length N of π is obtained using the following reasoning. For fixed n, the number of outgrowth patterns of π of the form (8) is $N!/L!$. This is because out of all possible permutations of the numbers $\{0, 1, \ldots, N-1\}$, we only count those that have the entries $\{\pi_0 + n, \pi_1 + n, \ldots, \pi_{L-1} + n\}$ in the required order. Next, note that we have $N - L + 1$ choices for the value of n. Each choice generates a set of $N!/L!$ outgrowth patterns. These sets are not necessarily disjoint, but an upper bound on the size of their union, i.e., the set of all outgrowth forbidden patterns of length N of π, is given by $(N-L+1)\frac{N!}{L!}$.

Forbidden patterns that are not outgrowth patterns of other forbidden patterns of shorter length are called *forbidden root patterns* since they can be viewed as the root of the tree of forbidden patterns spanned by the outgrowth patterns they generate, branching taking place when going from one length (or generation) to the next.

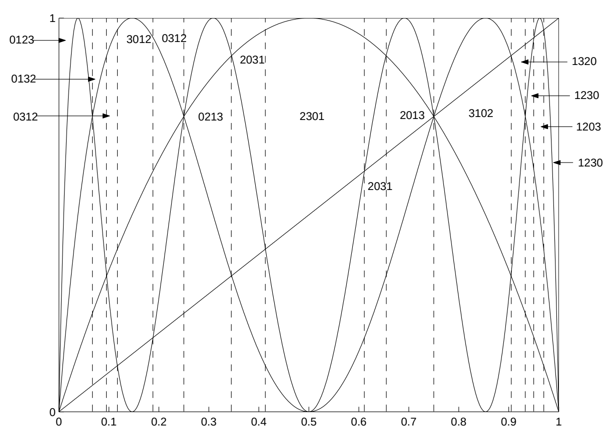

Fig. 2: The twelve allowed order patterns of length 4 for the logistic map. Note the two components of $P_{[0,3,1,2]}$, $P_{[2,0,3,1]}$ and $P_{[1,2,3,0]}$.

Example 4. Let f be the logistic map. In Figure 2, which is Figure 1 with the curve $y = f^3(x)$ superimposed, we can see the 12 allowed patterns of length 4 of the logistic map. Since there are 24 possible patterns of length 4, we conclude that 12 of them are forbidden. The outgrowth patterns of

$[2,1,0]$, the only forbidden pattern of length 3, are (see (8)):

$$\begin{array}{ll} (n=0) & [3,2,1,0],[2,3,1,0],[2,1,3,0],[2,1,0,3] \\ (n=1) & [0,3,2,1],[3,0,2,1],[3,2,0,1],[3,2,1,0] \end{array}.$$

Observe that the pattern $[3,2,1,0]$ is repeated. Therefore, the remaining five forbidden patterns of length 4, namely,

$$[0,2,3,1],[1,0,2,3],[1,0,3,2],[1,3,0,2],[3,1,2,0],$$

are root patterns.

Given the permutation $\sigma \in \mathcal{S}_N$, we say that σ contains the *consecutive pattern* $\tau = [\tau_0, \tau_1, \dots, \tau_{L-1}] \in \mathcal{S}_L$, $L < N$, if it contains a consecutive subsequence order-isomorphic to τ. Alternatively, we say that σ avoids the *consecutive pattern* τ if it contains no consecutive subsequence order-isomorphic to τ [9].

Suppose now $\sigma \in \mathcal{S}_N$, $\pi \in \mathcal{S}_L$, $L < N$, and

$$\begin{array}{llll} \pi(p_0) = 0, & \pi(p_1) = 1, & \dots, & \pi(p_{L-1}) = L-1, \\ \sigma(s_0) = n & \sigma(s_1) = 1+n, & \dots, & \sigma(s_{L-1}) = L-1+n, \end{array}$$

with $n \in \{0,1,...,N-L\}$. Then, the sequences $p_0, p_1, \dots, p_{L-1}$ and $s_0, s_1, \dots, s_{L-1}$ are consecutive subsequences of π^{-1} and σ^{-1} (starting at positions 0 and n), respectively. If, moreover, σ is an outgrowth pattern of π (see (8)), then $s_0, s_1, \dots, s_{L-1}$ is order-isomorphic to $p_0, p_1, \dots, p_{L-1}$. It follows that $\sigma \in \mathcal{S}_N$ is an outgrowth pattern of $\pi = [\pi_0, \dots, \pi_{L-1}]$ if σ^{-1} contains π^{-1} as a consecutive subsequence. Hence, the allowed patterns for f are the permutations that avoid all such consecutive subsequences for every forbidden root pattern of f.

Example 5. Take $\pi = [2,0,1]$ to be a forbidden pattern for a certain function f. Then $\sigma = [4,2,1,5,3,0]$ is an outgrowth pattern of π because it contains the subsequence $4,2,3$ ($n=2$). Equivalently, $\sigma^{-1} = [5,2,1,4,0,3]$ contains the consecutive pattern $1,4,0$ (starting at location σ_2^{-1}), which is order-isomorphic to $\pi^{-1} = [1,2,0]$. □

Let out(π) denote the family of outgrowth patterns of the forbidden pattern π,

$$\text{out}_N(\pi) = \text{out}(\pi) \cap \mathcal{S}_N = \{\sigma \in \mathcal{S}_N : \sigma^{-1} \text{ contains } \pi^{-1} \text{ as a consecutive pattern}\}, \text{ and}$$

$$\text{avoid}_N(\pi) = \mathcal{S}_N \backslash \text{out}_N(\pi) = \{\sigma \in \mathcal{S}_N : \sigma^{-1} \text{ avoids } \pi^{-1} \text{ as a consecutive pattern}\}.$$

where $\backslash$ stands for set difference. The fact that some of the outgrowth patterns of a given length will be the same and that this depends on π, makes the analytical calculation of $|\text{out}_N(\pi)|$ extremely complicated. Yet, from [9] we know that there are constants $0 < c, d < 1$ such that $c^N N! < |\text{avoid}_N(\pi)| < d^N N!$ (for the first inequality, $L \geq 3$ is needed). This implies that

$$(1-d^N)N! < |\text{out}_N(\pi)| < (1-c^N)N!. \tag{9}$$

This factorial growth with N can be exploited in practical applications to tell random from deterministic time series with, in principle, arbitrarily high probability. As said in the Introduction, these practical aspects are beyond the scope of this paper, but let us bring up here the following, related point. In the case of real (hence, *finite*) randomly generated sequences, a given order pattern $\pi \in \mathcal{S}_L$ can be missing with nonvanishing probability. We call *false forbidden patterns* such missing order patterns in finite random sequences without constraints, to distinguish them from the 'true' forbidden patterns of deterministic (finite or infinite) sequences. True and false forbidden patterns of self maps on one-dimensional intervals have been studied in [4].

5 Order patterns and one-sided shifts

The general study of order patterns and forbidden patterns is quite difficult. Analytical results seem to be only feasible for particular maps. In this section we consider one-sided shifts Σ on N symbols since, owing to their simple structure, they can be analyzed with greater detail. As we saw in Sect. 2, shifts are continuous maps on compact metric spaces $(\{0, 1, ..., N-1\}^{\mathbb{N}_0}, d_K)$ that can be lexicographically ordered:

$$\omega < \omega' \;\Leftrightarrow \begin{cases} \omega_0 < \omega_0' \\ \text{or} \\ \omega_0 = \omega_0', ..., \; \omega_{n-1} = \omega_{n-1}' \text{ and } \omega_n < \omega_n' \; (n \geq 1) \end{cases},$$

If $\mathcal{N}$ denotes the countable, dense and Σ-invariant set of ω eventually terminating in an infinite string of $(N-1)$s except the sequence $(\overline{N-1})$, then the map $\psi : \{0, 1, ..., N-1\}^{\mathbb{N}_0} \backslash \mathcal{N} \to [0, 1]$ defined by

$$\psi : (\omega_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \omega_n N^{-(n+1)}. \tag{10}$$

is one-to-one and *order-preserving*; moreover, ψ^{-1} is also order-preserving. As a matter of fact, the lexicographical order in $\{0, 1, ..., N-1\}^{\mathbb{N}_0} \backslash \mathcal{N}$ corresponds via ψ to the standard order (induced by the positive numbers) in the interval $[0, 1]$. Let us point out that ψ is continuous while ψ^{-1} is not. Since the map

$$S_N = \psi \circ \Sigma \circ \psi^{-1} : [0, 1] \to [0, 1], \tag{11}$$

is piecewise linear, it follows (Proposition 1) that Σ will have forbidden order patterns —although Σ has no forbidden *symbol* pattern, see Sect. 2. In particular, if Σ is the $(\frac{1}{N}, ..., \frac{1}{N})$-Bernoulli shift, then S_N is the Lebesgue-measure preserving *sawtooth map* $S_N : x \mapsto Nx (\text{mod } 1)$. Observe that only sequences that are not eventually periodic define order patterns of any length.

What is the structure of the allowed order patterns? It is easy to convince oneself (see Example 6 below) that, given $\omega = (\omega_0, ..., \omega_{L-1}, ...) \in \{0, 1, ..., N-1\}^{\mathbb{N}_0}$ of type $\pi \in \mathcal{S}_L$, π can be decomposed into, in general, N blocks,

$$[\pi_0, ..., \pi_{k_0-1}; \pi_{k_0}, ..., \pi_{k_0+k_1-1}; ...; \pi_{k_0+...+k_{N-2}}, ..., \pi_{k_0+...+k_{N-2}+k_{N-1}-1}], \tag{12}$$

with at most $N-1$ semicolons separating the different blocks, where $k_n \geq 0$, $0 \leq n \leq N-1$, is the number of symbols $n \in \{0, 1, \ldots, N-1\}$ in $\omega_{[0,L-1]}$ ($k_n = 0$ if none, with the corresponding block missing) and $k_0 + ... + k_{N-1} = L$. Moreover, these blocks obey the following basic restrictions.

(R1) Every block $\pi_{k_0+...+k_{j-1}}, ..., \pi_{k_0+...+k_{j-1}+k_j-1}$, $1 \leq j \leq N-2$, contains the locations of the js in $\omega_{[0,L-1]}$. Each j-run contained in or intersected by $\omega_{[0,L-1]}$, if any, contributes a subsequence of the same length as the run, that is increasing $(\pi_{k_0+...+k_{j-1}+i}, \pi_{k_0+...+k_{j-1}+i} + 1, \ldots)$ if the run is followed by a symbol $> j$, or *decreasing* $(\pi_{k_0+...+k_{j-1}+i}, \pi_{k_0+...+k_{j-1}+i} - 1, \ldots)$ if the run is followed by a symbol $< j$. These subsequences may be intertwined with other entries of the same block.

(R2) If the entries $\pi_m \leq L-2$ and $\pi_n \leq L-2$ belong to the *same block of* $\pi \in \mathcal{S}_L$, and π_m appears on the left of π_n (i.e., $0 \leq m < n \leq L-1$), then $\pi_m + 1$ appears also on the left of $\pi_n + 1$ (i.e., $\pi_m + 1 = \pi_{m'}$, $\pi_n + 1 = \pi_{n'}$ and $0 \leq m' < n' \leq L-1$).

In (R2), $\pi_m + 1$ and $\pi_n + 1$ may appear in the same block or in different blocks. Let us mention at this point that (R2) implies some simple consequences for the relative locations of increasing and decreasing subsequences within the same block and their continuations (if any) outside the block. In particular:

(A) If $\pi_i, \pi_i+1, ..., \pi_i+l-1$, $1 \leq l \leq L-1$, is an increasing subsequence within the same block of $\pi \in \mathcal{S}_L$ with $\pi_i + l < L$, then $\pi_i + l$ is on the right of $\pi_i + l - 1$ (i.e., $\pi_i + l - 1 = \pi_m$, $\pi_i + l = \pi_n$, and $m < n$).

(B) If $\pi_i, \pi_i-1, ..., \pi_i-l+1$, $1 \leq l \leq L-1$, is a decreasing subsequence within the same block of $\pi \in \mathcal{S}_L$ with $\pi_i < L-1$, then $\pi_i + 1$ is on the left of π_i (i.e., $\pi_i + 1 = \pi_j$ with $j < i$).

(C) If $\pi_i, \pi_i \pm 1, ..., \pi_i \pm l \mp 1$ and $\pi_j, \pi_j \pm 1, ..., \pi_j \pm h \mp 1$, $1 \leq l, h \leq L-1$, are two subsequences with the same monotony (upper signs for increasing, lower signs for decreasing subsequences) within the same block of $\pi \in \mathcal{S}_L$, then they are fully separated or, if intertwined, then it cannot happen that two or more entries of one of them are between two entries of the other.

Example 6. Take in $\{0,1,2\}^{\mathbb{N}_0}$ the sequence

$$\omega = (2_0\ 1_1\ 1_2\ 1_3\ 2_4\ 2_5\ 0_6\ 0_7\ 1_8\ 1_9\ 0_{10}\ 0_{11}\ 2_{12}\ 2_{13}\ 2\ 1\ \ldots), \tag{13}$$

where b_k indicates that the entry $b \in \{0,1,2\}$ is at place k. Then ω defines the order pattern

$$\pi = [6, 10, 7, 11; 9, 8, 1, 2, 3; 5, 0, 4, 13, 12] \in \mathcal{S}_{14},$$

where the first block, $\pi_{[0,3]} = 6, 10, 7, 11$, is set by the $k_0 = 4$ symbols 0 in $\omega_{[0,13]}$, which appear grouped in two runs, $\omega_{[6,7]}$ and $\omega_{[10,11]}$ (note the two increasing subsequences $6, 7$ and $10, 11$ in this block); the intermediate block, $\pi_{[4,8]} = 9, 8, 1, 2, 3$, comes from the $k_1 = 5$ symbols 1 in $\omega_{[0,13]}$, grouped also in two runs, $\omega_{[1,3]}$, followed by the symbol $2 > 1$, and $\omega_{[8,9]}$, followed by the symbol $0 < 1$ (note the corresponding increasing subsequence $1, 2, 3$, and decreasing subsequence $9, 8$, in this block); finally, the last block $\pi_{[9,13]} = 5, 0, 4, 13, 12$ accounts for the $k_2 = 5$ appearances of the symbol 2 in $\omega_{[0,13]}$ (the decreasing subsequences $5, 4$ and $13, 12$ come from the runs $\omega_{[4,5]}$ and $\omega_{[12,13]}$, respectively, where $\omega_{[12,13]}$ is the intersection within $\omega_{[0,13]}$ of a longer 2-run). (R2) is easily checked to be fulfilled. □

Proposition 3. The one-sided shift on $N \geq 2$ symbols has no forbidden patterns of length $L \leq N + 1$.

The proof can be found in [1]. Next we are going to show that the one-sided shift on N symbols has forbidden patterns (more specifically, forbidden *root* patterns) of any length $L \geq N + 2$. In order to construct explicit instances, we need first to introduce some notation and definitions.

Consider a partition of the sequence $0, 1, ..., L-1$ of the form

$$p_1 < p_2 < ... < p_d < ... < p_D, \tag{14}$$

where

$$p_d = e_d, e_d + 1, ..., e_d + h_d - 1, \tag{15}$$

$1 \leq d \leq D$, $D \geq 2$, with (i) $h_d \geq 1$, $h_1 + ... + h_D = L$, (ii) $e_1 = 0$, $e_D + h_D - 1 = L - 1$, and (iii) $e_d + h_d = e_{d+1}$ for $1 \leq d \leq D-1$, i.e., the *follower* of p_d, $e_d + h_d$, $d \leq D-1$, is the first element of p_{d+1}, e_{d+1}. We call (14) a partition of $0, 1, ..., L-1$ in D segments, (15) being an *increasing segment*, and denote by $\overleftarrow{p_d}$ the *decreasing* or *reversed segment* $\overleftarrow{p_d} = e_d + h_d - 1, ..., e_d + 1, e_d$. We also call e_d the first element of $\overleftarrow{p_d}$ and e_{d+1} the follower of $\overleftarrow{p_d}$.

Since a segment p_d is nothing else but a special case of a subsequence $\pi_i, \pi_i + 1, ..., \pi_i + l - 1$, where $0 \leq \pi_i = e_d \leq L - l$ if p_d is increasing and $l + 1 \leq \pi_i = e_d \leq L - 1$ if p_d is decreasing, and $\pi_i \pm 1 = \pi_{i+1}, ..., \pi_i \pm l \mp 1 = \pi_{i+l-1}$, respectively, the consequences (A)-(C) of the restriction (R2) apply as well. In the proof of the existence of forbidden root patterns below (Lemma 1 and 2, and Proposition 4) we are going to use (A) and (B) in the following, particularized version (that will be also referred to as (R2)): *The follower (if any) of an increasing segment p_n (correspondingly, decreasing segment $\overleftarrow{p_n}$) in an allowed pattern π appears always to the right of p_n (correspondingly, to the left of $\overleftarrow{p_n}$).*

Definition. Consider the partition (14) of $0, 1, ..., L-1$ in segments.

1. We call $\pi = [p_1, p_3, ..., \overleftarrow{p_4}, \overleftarrow{p_2}]$ and $\pi_{mirrored} = [p_2, p_4, ..., \overleftarrow{p_3}, \overleftarrow{p_1}]$ a *tent pattern* of length L.
2. We call $\pi = [..., \overleftarrow{p_3}, \overleftarrow{p_1}, p_2, p_4, ...]$ and $\pi_{mirrored} = [..., \overleftarrow{p_4}, \overleftarrow{p_2}, p_1, p_3, ...]$ a *spiralling pattern* of length L.

Observe that the relation between partitions of $0, 1, ..., L-1$ in segments and spiralling patterns of length L is one-to-one except when $p_1 = 0$ ($h_1 = 1$). In this case, $\overleftarrow{p_1}, p_2 = 0, 1, ..., e_2 + h_2 - 1$ can be taken for $p_1' \equiv 0, 1, ..., e_2 + h_2 - 1$ ($h_1' = h_2 + 1$).

Lemma 1. If $N \geq 2$ is the number of symbols and π is a tent pattern with D segments, then π is forbidden if and only if $D \geq N + 2$.

Lemma 2. If $N \geq 2$ is the number of symbols and π is a spiralling pattern with D segments and $h_1 \geq 2$ (i.e., $p_1 = 0, 1, ...$), then

1. π is forbidden if and only if (a) $D = N$ and $h_D \geq 2$, or (b) $D \geq N + 1$;
2. π is allowed if and only if (a') $D < N$, or (b') $D = N$ and $h_D = 1$.

The constructive procedure used in the proofs of Lemmas 1 and 2, which we omit in this extended abstract, can be used in general to decompose any ordinal pattern into well-formed (i.e., complying with (R1),(R2)) blocks. For instance, one could start with the leftmost entry and move on rightward one entry at a time, introducing a semicolon between the current and the last entry whenever necessary to enforce the restrictions (R1),(R2). The following proposition is proved in [1].

Proposition 4. The following patterns of length $L \geq N + 2$, together with their corresponding mirrored patterns, are forbidden root patterns.

1. The tent patterns with $N+2$ segments $[0, p_3, ..., p_N, L-1, \overleftarrow{p_{N+1}}, ..., \overleftarrow{p_2}]$ if N is odd, or $[0, p_3, ..., p_{N+1}, L-1, \overleftarrow{p_N}, ..., \overleftarrow{p_2}]$ if N is even. Here $p_1 = 0$, and $p_{N+2} = L - 1$.
2. The spiralling pattern with $N + 1$ segments $[L-2, \overleftarrow{p_{N-2}}, ..., \overleftarrow{p_3}, 1, 0, p_2, ..., p_{N-1}, L-1]$ if N is odd, or $[L-1, \overleftarrow{p_{N-1}}, ..., \overleftarrow{p_3}, 1, 0, p_2, ..., p_{N-2}, L-2]$ if N is even. Here $p_1 = 0, 1$, $p_N = L - 2$, and $p_{N+1} = L - 1$.
3. The spiralling pattern with N segments $[L-1, L-2, \overleftarrow{p_{N-2}}, ..., \overleftarrow{p_3}, 1, 0, p_2, ..., p_{N-1}]$ if N is odd, or $[\overleftarrow{p_{N-1}}, ..., \overleftarrow{p_3}, 1, 0, p_2, ..., p_{N-2}, L-2, L-1]$ if N is even. Here $p_1 = 0, 1$, and $p_N = L-2, L-1$.

Example 8. For $N = 2n$, Proposition 4 provides the following six forbidden patterns of minimal length $L = N + 2$: $[0, 2, ..., 2n, 2n+1, 2n-1, ..., 3, 1]$, $[2n+1, 2n-1, ..., 3, 1, 0, 2, ..., 2n-2, 2n]$, $[2n-1, ..., 1, 0, 2, ..., 2n-2, 2n, 2n+1]$, and their corresponding mirrored patterns.

Corollary 1. For every $K \geq 2$ there are maps on $[0, 1]$ without forbidden patterns of length $L \leq K$.

Proof: Let $S_N = \psi \circ \Sigma \circ \psi^{-1} : [0, 1] \to [0, 1]$ be the map (11). Since ψ is an order-isomorphism, S_N and Σ, the shift on N symbols, have the same forbidden patterns. Therefore, if $N + 1 \leq K$, then S_N has no forbidden patterns of length $L \leq K$ because of Proposition 3. □

It follows that *there are interval maps on* $\mathbb{R}^n$ *without forbidden patterns*. For example, one can decompose $[0, 1]$ in infinite many half-open intervals (of vanishing length), $[0, 1] = \cup_{N=2}^{\infty} I_N$ and define on each I_N a properly scaled version of S_N, $\tilde{S}_N : I_N \to I_N$. In $\mathbb{R}^2$ one can perform the said decomposition along the 1-axis and define on $I_N \times [0, 1]$ the function ($\tilde{S}_N$, Id). Now, Eq. (5) shows that adding some natural assumption, like piecewise monotony, can make all the difference.

In [11] it is proved that the six patterns given by Proposition 4 are the only forbidden patterns of minimal length $L = N + 2$. We conclude with a more general result, proved in [11], which provides the enumeration of forbidden patterns of the shift on N symbols.

Theorem 1. Let $N \geq 2$ be the number of symbols, and let $L \geq N + 2$. The number of forbidden patterns of length L is $a_{L,N+1} + a_{L,N+1} + \cdots + a_{L,L-1}$, where

$$a_{L,M} = \sum_{i=0}^{M-2} (-1)^i \binom{L}{i} \left((M-i-2)(M-i)^{L-2} + \sum_{t=1}^{L-1} \psi_{M-i}(t)(M-i)^{L-t-1} \right),$$

and $\psi_M(k)$ denotes the number of primitive words of length k over an M-letter alphabet.

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