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Nested Hilbert Schemes and the nested q, t-Catalan series

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Abstract. In this paper we study the tangent spaces of the smooth nested Hilbert scheme $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$ of points in the plane, and give a general formula for computing the Euler characteristic of a $\mathbb{T}^2$-equivariant locally free sheaf on $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$. Applying our result to a particular sheaf, we conjecture that the result is a polynomial in the variables q and t with non-negative integer coefficients. We call this conjecturally positive polynomial as the "nested q, t-Catalan series," for it has many conjectural properties similar to that of the q, t-Catalan series.

Résumé. Dans cet article, nous étudions les espaces tangents du schéma de Hilbert emboité lisse $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$ de points du plan, et donnons une formule générale pour le calcul de la caractéristique d'Euler d'un faisceau $\mathbb{T}^2$-équivariant localement libre sur $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$. En appliquant notre resultat a un faisceau particulier, nous conjecturons que le résultat est un polynôme en q et t à coefficents positifs ou nuls. Nous appelons ce polynme conjecturalement positif la "serie de q, t-Catalan emboîtée", car il à de nombreuses propriétés (conjecturées) similaires à celles de la série de q, t-Catalan.

Keywords: Atiyah-Bott Lefschetz formula, (nested) Hilbert scheme of points, tangent spaces, diagonal coinvariants.

1 Extended Abstract

Let $\mathfrak{S}_n$ be the symmetric group. The Frobenius character $\mathcal{F}$ is a map from the Grothendieck group $Rep(\mathfrak{S}_n)$ of representations of $\mathfrak{S}_n$ into the ring of symmetric functions. It assigns to an irreducible representation V^λ the Schur function s_λ. Among the $\mathfrak{S}_n$-modules with an interesting Frobenius image is the ring of diagonal coinvariants: $\mathfrak{S}_n$ acts (diagonally) on the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, ..., x_n, y_n]$ by $\sigma(x_i) = x_j,\ \sigma(y_i) = y_j$. The ring of diagonal coinvariants is the quotient $R_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/I_+$ of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ by the ideal I_+ generated by the invariant polynomials without a constant term. R_n is a bigraded ring (by degree)

$$R_n = \bigoplus_{r,s} (R_n)_{r,s},$$

and the action of the symmetric group respects the bigrading. Therefore, the two variable Frobenius series

$$\mathcal{F}_{R_n}(q, t) = \sum_{r,s} \mathcal{F}((R_n)_{r,s}) q^r t^s$$

makes sense.

There is a compact way of writing $\mathcal{F}_{R_n}(q,t)$, however, it requires the theory of Macdonald polynomials. Let $\mathcal{Y}_n$ be the set of partitions of the positive integer n. The set $\{\widetilde{H}_\mu(X,q,t)\}_{\mu\in\mathcal{Y}_m}$ of (modified) Macdonald polynomials constitutes a vector space basis for the ring of symmetric functions $\Lambda^n_{\mathbb{Q}(q,t)}$ over the field of rational functions $\mathbb{Q}(q,t)$. The existence of this basis is proved by I. Macdonald (see (19) for details). A closed formula for an arbitrary $\widetilde{H}_\mu(X,q,t)$ has been conjectured by Haglund (11) and later proved by Haglund, Haiman and Loehr (13).

The Bergeron-Garsia operator $\nabla : \Lambda^n_{\mathbb{Q}(q,t)} \to \Lambda^n_{\mathbb{Q}(q,t)}$ is defined by setting

$$\nabla(\widetilde{H}_\mu(X,q,t)) = t^{n(\mu)}q^{n(\mu')}\widetilde{H}_\mu(X,q,t), \tag{1.1}$$

where $n(\mu) = \sum(i-1)\mu_i$ and μ' is the conjugate partiton to μ.

The following highly nontrivial result about the Frobenius series of R_n has been conjectured by Garsia and Haiman in (6) and finally been proved by Haiman in (16).

Theorem 1.1 *The Frobenius character $\mathcal{F}_{R_n}(q,t)$ of the ring of diagonal coinvariants is equal to $\nabla(e_n)$, where e_n is the n'th elementary symmetric function.*

The proof of this theorem involves the study of deep geometric properties of the Hilbert scheme of n points in the plane, and nontrivial manipulations involving Macdonald polynomials. The reader who wishes to see a brief survey about the proof may wish to check (21).

In this paper we are concerned with a combinatorially defined polynomial which arise from the study of the nested Hilbert scheme $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$. This is the parametrizing scheme of pairs (I_1, I_2) of 0-dimensional subschemes in the plane such that I_1 is a subscheme of I_2. We employ the Atiyah-Bott Lefschetz fixed point theorem to calculate the torus equivariant Euler characteristics of vector bundles on the nested Hilbert scheme. We conjecture that the torus equivariant Euler characteristic of a particular sheaf is a polynomial in the variables q and t with non-negative integer coefficients. We propose a combinatorial model for this polynomial.

Consider the subspace R_n^ε of $\mathfrak{S}_n$-alternating polynomials in R_n. This is the vector subspace generated by the images of those polynomials $f \in A$ with

$$\sigma \cdot f = (-1)^{sign(\sigma)} f.$$

There is a natural bigrading (by degree) on R_n^ε. So, we write $R_n^\varepsilon = \bigoplus_{r,s}(R_n^\varepsilon)_{r,s}$. Let

$$\mathcal{H}_{R_n^\varepsilon}(q,t) = \sum_{r,s} \dim(R_n^\varepsilon)_{r,s} q^r t^s \in \mathbb{Z}[q,t]$$

be its HIlbert series. In (14), it has been (empirically) observed by Haiman that the specialization $\mathcal{H}_{R_n^\varepsilon}(q,t)$ at $q=t=1$ gives the Catalan numbers

$$\mathcal{H}_{R_n^\varepsilon}(1,1) = \frac{1}{n+1}\binom{2n}{n}.$$

It is well known that the Catalan numbers count the "Dyck paths," which are the lattice paths in the xy-plane, starting at $(0,0)$ and ending at (n,n) with unit upward ($(0,1)-step$) and unit rightward ($(1,0)-step$) segments, and staying weakly above the diagonal $x=y$ (see Figure 1 below). Let $\mathcal{D}_n$ denote

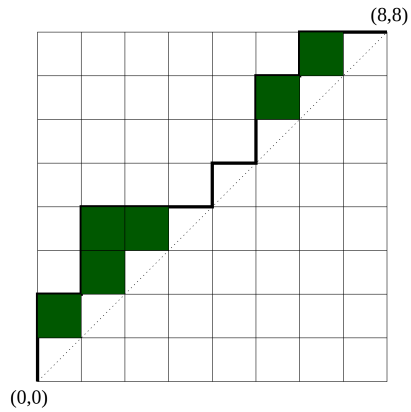

Fig. 1: A Dyck path of size 8 with $area(P) = 6$.

the set of all Dyck paths of size n. A Dyck path $P \in \mathcal{D}_n$ can be represented by a sequence $seq(P) = (a_1 a_2 \cdots a_{2n})$ of 0's and 1's, where a 0 represents an upward step and a 1 represents a rightward step. For example, the Dyck path in Figure 1 is represented by the sequence $seq(P) = (0010011101001011)$.

One can ask the following natural question: could we find a pair of functions $s_1, s_2 : \mathcal{D}_n \to \mathbb{N}$ such that

$$\mathcal{H}_{R_n^\varepsilon}(q,t) = \sum_{P \in \mathcal{D}_n} q^{s_1(P)} t^{s_2(P)}.$$

An answer to this question has been conjectured by Haglund (10), and proved later by Garsia and Haglund (5). For a basic introduction to the techniques of the proof and more, we recommend the recent book (12).

We describe the functions s_1 and s_2.

The first function s_1 is, in a sense very classical; given a Dyck path $P \in \mathcal{D}_n$, $s_1(P)$ is defined to be the "$area$" of P. This is the number of full cells below the path and above the main diagonal. It is not hard to see that $s_1(P) = area(P)$ is equivalent to the $coinv(seq(P)) - \binom{n+1}{2}$, where $coinv$ of a sequence $(a_1 \cdots a_{2n})$ is the number of pairs of indices (i, j) such that $i < j$ and $a_i < a_j$.

The value $s_2(P)$ at $P \in \mathcal{D}_n$ of the second function s_2 is called the $bounce$ number of the path $P \in \mathcal{D}_n$. It is defined algorithmically as follows.

Given $P \in \mathcal{D}_n$, starting at (n, n) we move leftward (in the negative x-axis direction) until encountering a lattice point (j_1, n) such that the line segment $\overline{(j_1, n)(j_1, n-1)}$ is an upward step of P. Next, we start at the lattice point (j_1, j_1) on the diagonal and move leftward once again until encountering a lattice point (j_2, j_1) such that the line segment $\overline{(j_2, j_1)(j_2, j_1 - 1)}$ is an upward step of P. Next, we start at (j_2, j_2) on the diagonal and move leftward until encountering a lattice point (j_3, j_2) such that the line segment $\overline{(j_3, j_2)(j_3, j_2 - 1)}$ is a rightward step of P. Once again we start at (j_3, j_3) and repeat the process. This continues until we reach to the point $(j_b, j_b) = (0, 0)$ on the diagonal.

Definition 1.2 *Let $\{(0, 0), (j_{b-1}, j_{b-1}), ..., (j_1, j_1)\}$ be the set of lattice points on the diagonal that we* *obtained by the above procedure. Let* $b(P)$ *be the Dyck path represented by the sequence*

$$seq(b(P)) = (\underbrace{0\cdots0}_{j_{b-1}}\underbrace{1\cdots1}_{j_{b-1}}\ \underbrace{0\cdots0}_{j_{b-2}-j_{b-1}}\ \underbrace{1\cdots1}_{j_{b-2}-j_{b-1}}\ \cdots\ \underbrace{1\cdots1}_{j_1-\sum_{k=2}^{b-1} j_{b_k}}\ ).$$

We call $b(P)$ *as the* $bounce\ path$ *of* P*. The part of* P *between the lattice points* (j_i, j_{i-1}) *and* (j_{i+1}, j_i) *is called the* $i'th\ bounce\ section$ *of* P*, and denoted by* $\mathbf{B}_i$ *(see figure 2 below). Finally, the bounce number,* $s_2(P)$ *is defined to be the sum*

$$s_2(P) = \sum_{i=1}^{b-1} n - j_i.$$

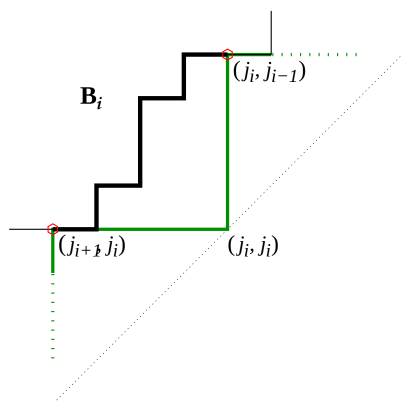

Fig. 2: i'th bounce region (between circles)

Example 1.3 *Consider the path given by the sequence* $seq(P) = (0011001001101011)$*, as in Figure 3 below. The sequence of the bounce path* $b(P)$ *is* $seq(b(P)) = (0011010001110011)$*. Then,* $j_1 = 6, j_2 = 3, j_3 = 2$*. Therefore, the bounce number of* P *is equal to* $s_2(P) = 2 + 5 + 6 = 13$*.*

Definition 1.4 *The* q,t*-Catalan series of size* n *is defined to be the summation*

$$C_n(q,t) = \sum_{P\in\mathcal{D}_n} q^{s_1(P)} t^{s_2(P)}, \tag{1.2}$$

where $s_1(P)$ *and* $s_2(P)$ *are, respectively, the area and the bounce of the path* P*, as described above.*

The stunning equality $\mathcal{H}_{R_n^{\varepsilon}}(q,t) = C_n(q,t)$ is one of the good reasons to study the "statistics" s_1 and s_2. Arguably, a more stunning fact is that the polynomial $C_n(q,t)$ can also be recovered by the Atiyah-Bott-Lefschetz formula applied to a certain locally free sheaf on the "zero fiber" of the Hilbert scheme of points in the plane. This is one of our main motivations for this article. Before we start with the algebraic-geometric details, we define another family of q,t-polynomials similar to $C_n(q,t)$, and present a conjecture of the author and J. Haglund.

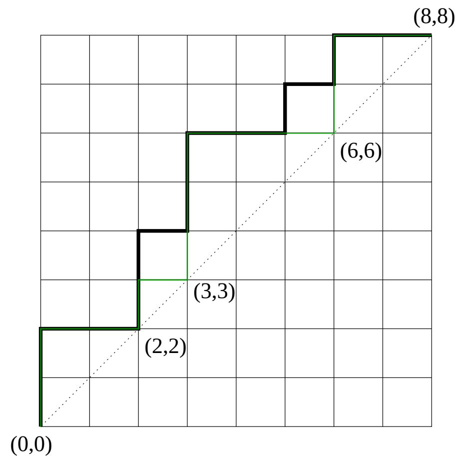

Fig. 3: A Dyck path with $s_2(P) = bounce(P) = 13$.

Note that there exists a unique Dyck path P_0 with $s_2(P_0) = 0$ and $s_1(P_0) = \binom{n}{2}$. This the path with n immediate upward steps. In other words,

$$seq(P_0) = (00\cdots 011\cdots 1). \tag{1.3}$$

Definition 1.5 *Let P_0 be as in in 1.3. Let $P \in \mathcal{D}^0_{n+1} = \mathcal{D}_{n+1} \setminus \{P_0\}$ be Dyck path defined by the sequence $seq(P) = (a_1a_2\cdots a_{2n+2})$. Let $\mathbf{B}_i$ be the ith bounce section of P as defined in 1.2, and let $v_i = v_i(P)$ be the number of 01's (a 0 immediately followed by 1) in the part of the sequence $seq(P)$ which belongs to the bounce section $\mathbf{B}_i$. Define $s_3 : \mathcal{D}^0_{n+1} \to \mathbb{N}$ by*

$$s_3(P) = \sum_{i=1}^{b-1} j_i - 1.$$

Finally we define the combinatorial nested q,t-Catalan series *$N_n(q,t)$ to be the summation*

$$N_n(q,t) = \sum_{P\in\mathcal{D}^0_{n+1}} q^{s_1(P)}t^{s_3(P)}(v_0 + v_1t + \cdots v_{b-1}t^{b-1}). \tag{1.4}$$

Example 1.6 *The smallest example is for $n = 2$. In that case, we have 4 Dyck paths of size 3 which are shown in figure 4 below. It is easy to check that*

$$\begin{aligned} N_2(q,t) &= q^0t\cdot(1+t) + qt\cdot 1 + qt^0\cdot 1 + q^2t^0\cdot 1 \\ &= q^2 + q + qt + t + t^2. \end{aligned}$$

Conjecture 1.7 *The combinatorial nested q,t-Catalan series $N_n(q,t)$ is symmetric in the variables q,t.*

The Hilbert schemes of points in the plane are, arguably, among celebrities of the smooth algebraic varieties (use http://www.ams.org/mathscinet/). Since their introduction, these spaces has been well studied

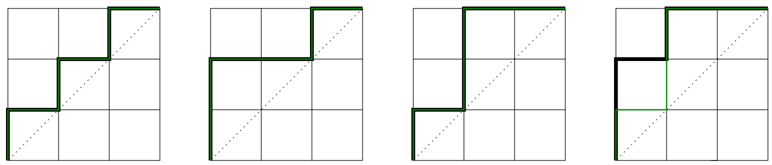

Fig. 4: The elements of $\mathcal{D}_3^0$ with their bounce paths.

and understood from different perspectives. For example, in (3), de Cataldo and Migliorini have computed the Chow motives and the Chow groups of Hilbert schemes of points. In (7), Gordon and Stafford have investigated the relationship between rational Cherednik algebras and the Hilbert schemes. In (2), Boissière investigates the relationship between McKay correspondence and K-theory of $\mathrm{Hilb}^n(\mathbb{A}^2)$, etc.. The literature on $\mathrm{Hilb}^n(\mathbb{A}^2)$ is vast, therefore, sadly we cannot avoid doing unjust towards authors whose work we can't mention here. Here are the few places that we have benefited from while trying to learn about these wonderful varieties. (18), (4), (8), (1), (20), (17). The reader who would like to see more on the literature may wish to check the references in (20).

We work over the field of complex numbers. Let $\mathbb{A}^2 = \mathbb{A}^2_{\mathbb{C}}$ denote the affine plane over $\mathbb{C}$. The Hilbert scheme of n points in $\mathbb{A}^2$ is denoted by $\mathrm{Hilb}^n(\mathbb{A}^2)$. It can be identified with (at least set theoretically) the set of ideals I in the polynomial ring $\mathbb{C}[x,y]$ such that the quotient space $\mathbb{C}[x,y]/I$ is of dimension n. Then, the nested Hilbert scheme $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$ can be identified with the pairs of ideals

$$(I_1, I_2) \in \mathrm{Hilb}^n(\mathbb{A}^2) \times \mathrm{Hilb}^{n-1}(\mathbb{A}^2),$$

such that $I_1 \subseteq I_2$. Both $\mathrm{Hilb}^n(\mathbb{A}^2)$ and $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$ are nonsingular and irreducible varieties.

As GL_2 acts on $\mathbb{A}^2$, so does the maximal torus of the diagonal invertible matrices. We shall identify $\mathbb{T}^2$ with $\mathbb{C}^* \times \mathbb{C}^*$. Its action on $\mathbb{A}^2$ passes onto $\mathrm{Hilb}^n(\mathbb{A}^2)$, and onto $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$.

The set of fixed points of the torus $\mathbb{T}^2$ on $\mathrm{Hilb}^n(\mathbb{A}^2)$ is indexed by the partitions μ of n. We use French notation for partitions (the Young diagram of a partition is left justified, and decreases from bottom to top).The fixed point set on $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$ is indexed by the pairs of partitions $(\mu,\nu) \in \mathcal{Y}_n \times \mathcal{Y}_{n-1}$ such that the Young diagram of ν is contained in that of μ. We denote the set of all such pairs of partitions by $\mathcal{Y}_{n,n-1}$. The arm $a(\alpha)$ of a cell $\alpha \in \mu$ is the number of cells in the row of α that are on the right hand side of α. Similarly, the leg $l(\alpha)$ of $\alpha \in \mu$ is the number of cells in the column of α which lie above. $\mathcal{R}ow(\alpha)$ and $\mathcal{C}ol(\alpha)$ stand for the row and column of the cell $\alpha \in \mu$.

Theorem 1.8 *Let M be a $\mathbb{T}^2$-equivariant vector bundle on $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$, and $\mathcal{H}_{M(I_\mu,I_\nu)}(q,t)$ denote the Hilbert series of the fiber $M(I_\mu,I_\nu)$ of M at the torus fixed point $(I_\mu,I_\nu) \in \mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$, where ν is a partition induced from μ by taking off a corner cell $\zeta \in \mu$ (we write $\nu = \mu \setminus \{\zeta\}$). Then, the equivariant Euler characteristic of M is*

$$\chi_M(q,t) = \sum_{(\mu,\nu)\in\mathcal{Y}_{n,n-1}} \frac{\mathcal{H}_{M(I_\mu,I_\nu)}(q,t)}{(1-t)(1-q)P_1(\mu,\nu)P_2(\mu,\nu)P_3(\mu,\nu)}, \tag{1.5}$$

wheret P_1, P_2 and P_3 are given by

$$\begin{aligned}
P_1(\mu,\nu) &= \prod_{\alpha\in\mu\setminus(\mathcal{R}ow(\zeta)\cup\mathcal{C}ol(\zeta))} (1-t^{1+l(\alpha)}q^{-a(\alpha)})(1-t^{-l(\alpha)}q^{1+a(\alpha)}),\\
P_2(\mu,\nu) &= \prod_{\alpha\in\mathcal{R}ow(\zeta)} (1-t^{1+l(\alpha)}q^{-a(\alpha)})(1-t^{-l(\alpha)}q^{a(\alpha)}),\\
P_3(\mu,\nu) &= \prod_{\alpha\in\mathcal{C}ol(\zeta)} (1-t^{-l(\alpha)}q^{1+a(\alpha)})(1-t^{l(\alpha)}q^{-a(\alpha)}),
\end{aligned}$$

if $\nu = \mu\setminus\{\zeta\}$.

To prove this result we use the techniques developed by Haiman in (16).

Let $\Lambda^n_{\mathbb{Q}(q,t)}$ be the space of homogeneous symmetric functions of degree n on a set of algebraically independent variables $\{x_1, x_2, x_3 ...\}$ over the field of rational functions $\mathbb{Q}(q,t)$. As vector space, $\Lambda^n_{\mathbb{Q}(q,t)}$ has quite a few different distinguished bases. One of them is the basis $\{e_\mu\}_{\mu\in\mathcal{Y}_n}$ of elementary symmetric functions, defined as follows. Suppose $\mu\in\mathcal{Y}_n$ is a partition with parts $(\mu_1, ..., \mu_k)$. Then, for $i = 1...k$ define

$$e_{\mu_i} = \sum_{1\le j_1<\cdots<j_{\mu_i}} x_{j_1}x_{j_2}\cdots x_{j_{\mu_i}}, \text{ and } e_\mu = e_{\mu_1}\cdots e_{\mu_k}.$$

Another basis for $\Lambda^n_{\mathbb{Q}(q,t)}$ is given by the (modified) Macdonald polynomials $\widetilde{H}_\mu(X,q,t)$, whose existence is proved by I. Macdonald (see (19) for details). A closed formula for these polynomials has been conjectured by Haglund (11) and later proved by Haglund, Haiman and Loehr (13).

Let ∇ be the Bergeron-Garsia operator on the space of symmetric functions defined as in (1.1). The following theorem, which we address by the "q,t-Catalan theorem" is a culmination of the works of many mathematicians.

Theorem 1.9 *(Garsia, Haglund, Haiman, (5),(6), (15), (16)) For all $n\ge 1$, the Hilbert series $\mathcal{H}_{R_n^\varepsilon}(q,t)$ of the space of diagonal alternating coinvariants R_n^ε is equal to each of the following quantities below:*

1. *The multiplicity $\langle \mathcal{F}_{R_n}(q,t), e_n\rangle = \langle \nabla e_n, e_n\rangle$ of the sign character $s_{1^n} = e_n$ in the $\mathfrak{S}_n$-module R_n, where $\langle,\rangle$ is the Hall scalar product on the space of symmetric functions.*
2. *The q,t-Catalan series*
$$C_n(q,t) = \sum_{P\in\mathcal{D}_n} q^{s_1(P)}t^{s_2(P)},$$
where s_1 and s_2 are the area and bounce functions on $\mathcal{D}_n$.
3. *The rational function*
$$\sum_{\mu\in\mathcal{Y}_n} \frac{t^{n(\mu)}q^{n(\mu')}(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)}{\prod_{\zeta\in\mu}(1-t^{1+l(\zeta)}q^{-a(\zeta)})(1-t^{-l(\zeta)}q^{1+a(\zeta)})},$$
where $B_\mu(q,t) = \sum_{(r,s)\in\mu} t^r q^s$, $\Pi_\mu(q,t) = \prod_{(r,s)\in\mu\setminus(0,0)}(1-t^r q^s)$, and $a(\zeta), l(\zeta)$ are the lengths of the arm and leg (respectively) of the cell $\zeta\in\mu$.

4. *The Euler characteristic of a certain $\mathbb{T}^2$-equivariant sheaf supported on the zero fiber Z_n of the Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{A}^2)$.

In (15), Haiman showed that $\mathrm{Hilb}^n(\mathbb{A}^2)$ can be realized as a blow up of the n'th symmetric product $(\mathbb{A}^2)^n/\mathfrak{S}_n$ of the plane $\mathbb{A}^2$ along a particular subscheme. Let $\mathcal{O}(1)$ be the ample sheaf on $\mathrm{Hilb}^n(\mathbb{A}^2)$ arising from the Proj construction of the blow up. Then the suitable sheaf giving the fourth item in the q,t-Catalan theorem is $M := \mathcal{O}(1) \otimes \mathcal{O}_{Z_n}$, where Z_n is the zero fiber in $\mathrm{Hilb}^n(\mathbb{A}^2)$ (the subscheme consisting of all points that are supported at the origin of $\mathbb{A}^2$).

Based on the computer experiments we conjecture that

Conjecture 1.10 *Let* $\eta : \mathrm{Hilb}^{n,n-1}(\mathbb{A}^2) \to \mathrm{Hilb}^n(\mathbb{A}^2)$ *be the projection sending* (I_1, I_2) *to* I_1. *Then for every* $m \geq 1$*, the Euler characteristic*

$$\chi_{\eta^*(\mathcal{O}(m)\otimes\mathcal{O}_{Z_n})}(q,t) \tag{1.6}$$

of the pull back of the sheaf $\mathcal{O}(m) \otimes \mathcal{O}_{Z_n}$ *on* $\mathrm{Hilb}^{n,n-1}(\mathbb{A}^2)$ *is a polynomial in q and t with nonnegative integer coefficients. Furthermore, letting $m = 1$ we conjecture that*

$$\chi_{\eta^*(\mathcal{O}(1)\otimes\mathcal{O}_{Z_n})}(q,t) = N_n(q,t), \tag{1.7}$$

where the right hand side is given by 1.4 above.

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