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Enumeration of orientable coverings of a non-orientable manifold

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Abstract. In this paper we solve the known V.A. Liskovets problem (1996) on the enumeration of orientable coverings over a non-orientable manifold with an arbitrary finitely generated fundamental group. As an application we obtain general formulas for the number of chiral and reflexible coverings over the manifold. As a further application, we count the chiral and reflexible maps and hypermaps on a closed orientable surface by the number of edges. Also, by this method the number of self-dual and Petri-dual maps can be determined. This will be done in forthcoming papers by authors.

Keywords: non-orientable manifold, fundamental group, conjugacy classes of subgroups, surface covering, enumeration, chiral pairs

1 Introduction

Let $\mathcal{M}$ be a connected non-orientable manifold. Denote by $\Gamma = \pi_1(\mathcal{M})$ the fundamental group and by $\tilde{\mathcal{M}}$ the universal covering of $\mathcal{M}$. Identify Γ with the group of covering transformations of $\tilde{\mathcal{M}} \to \mathcal{M}$. We note that $\tilde{\mathcal{M}}$ is always orientable and Γ acts on $\tilde{\mathcal{M}}$ as a group of homeomorphisms.

Denote by Γ^+ a subgroup of index two in Γ consisting of all orientation preserving homeomorphisms. Then $\mathcal{M}^+ = \tilde{\mathcal{M}} / \Gamma^+$ is an *orientable double* of $\mathcal{M}$ with fundamental group $\Gamma^+ = \pi_1(\mathcal{M}^+)$. The following facts from algebraic topology are well known (5).

Let $\pi : \mathcal{U} \to \mathcal{M}$ be an n-fold covering of $\mathcal{M}$. Then the fundamental group $K = \pi_1(\mathcal{U})$ is contained as a subgroup of index n in the group $\Gamma = \pi_1(\mathcal{M})$. Conversely, any subgroup of index n in Γ is the fundamental group of an n-fold covering of $\mathcal{M}$. Moreover, if $\mathcal{U}^+$ is orientable then the number $n = 2\,m$ is even and the group K is contained as a subgroup of index m in the group Γ^+. In this case, $\mathcal{U}^+$ is an m-fold covering of the manifold $\mathcal{M}^+$.

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Two coverings $\pi : \mathcal{U} \to \mathcal{M}$ and $\pi' : \mathcal{U}' \to \mathcal{M}$ are *equivalent* (or *isomorphic*) if there exists a homeomorphism $h : \mathcal{U} \to \mathcal{U}'$ such that $\pi = \pi' \circ h$. An orientable covering $\pi : \mathcal{U}^+ \to \mathcal{M}$ is called *reflexible* if there exists an orientation reversing homeomorphism $h : \mathcal{U}^+ \to \mathcal{U}^+$ such that $\pi \circ h = \pi$ and *irreflexible* (or *chiral*) otherwise. Irreflexible coverings are divided into chiral pairs of twins. Two twins are non-equivalent as coverings over $\mathcal{M}^+$, but have to be equivalent as coverings over $\mathcal{M}$.

Coverings $\pi : \mathcal{U} \to \mathcal{M}$ and $\pi' : \mathcal{U}' \to \mathcal{M}$ are equivalent if and only if the corresponding subgroups $\pi_1(\mathcal{U})$ and $\pi_1(\mathcal{U}')$ are conjugate in Γ. Hence, the number of non-equivalent n-fold coverings of $\mathcal{M}$ coincides with the number $c_\Gamma(n)$ of conjugacy classes of subgroups of index n in the group Γ.

Recall that the fundamental group of a bordered surface $\mathcal{S}$ is a free group $\Gamma = F_r$ of rank $r = 1-\chi(\mathcal{S})$, where $\chi(\mathcal{S})$ is the Euler characteristic of $\mathcal{S}$. It this case the number $c_\Gamma(n)$ was determined by V. Liskovets ((15), (16)), J. H. Kwak and J. Lee (12), and M. Hofmeister (6). For the fundamental group Γ of closed orientable and non-orientable surfaces the number $c_\Gamma(n)$ was given by A. Mednykh (19) and A. Mednykh and G. Pozdnyakova (23), respectively.

During the discussion at Dresden University between V. A. Liskovets and one of the authors in 1996 the following problem was stated:

Liskovets problem. *Find the number of non-equivalent n-fold orientable coverings of a given non-orientable manifold with a finitely generated fundamental group.*

The main purpose of the paper is to give a solution of the Liskovets problem (Theorem 3). As an application, we enumerate reflexible coverings and of chiral pairs of coverings over a non-orientable manifold with a finitely generated fundamental group. These results form a background for counting chiral pairs of maps and hypermaps on closed orientable surface. It will be done in a forthcoming paper by A. Breda, R. Nedela and A. Mednykh (1). The general formula for the number of reflexible coverings (Theorem 8) allows also to count self-dual and Petri-dual maps and other combinatorial objects.

2 Preliminaries

Following J. Širáň and M. Škoviera (24) we define a *group with sign structure* to be a pair (Γ, ω), where Γ is a group and $\omega : \Gamma \to \mathbb{Z}_2 = \{-1, 1\}$ is a homomorphism. Each $sign\ structure$ (or $orientation)$ ω is uniquely determined by a subgroup $H = \mathrm{Ker}\,(\omega)$ of index at most two in Γ. Equivalently, one can define a group with sign structure as a pair (Γ, H), where H is a subgroup of index one or two in Γ.

Let (Γ, ω) be a group with sign structure. The image of an element $g \in \Gamma$ under homomorphism ω will be referred as the $sign$ of g. Elements with sign $+1$ will be called $positive$ and those with sign -1 will be called $negative$. The set of positive elements of Γ forms a subgroup, denoted by Γ^+, of index one or two in Γ. The set of negative elements Γ^- is a coset of Γ^+ in Γ or empty. In the latter case the corresponding sign structure will be called $trivial$.

A subgroup $K < \Gamma$ is called to be *orientable* (with respect to ω) if $K \subset \mathrm{Ker}\,(\omega)$ and *non-orientable* otherwise. For any orientable and non-orientable subgroup $K < \Gamma$ the induced sign structure $\omega|_K : K \to \mathbb{Z}_2$ is trivial or non-trivial, respectively. The given definitions are justified by the following topological observations.

Let $\mathcal{M}$ be a manifold with the fundamental group $\Gamma = \pi_1(\mathcal{M})$. Then Γ acts as a homeomorphism group on the universal covering $\tilde{\mathcal{M}}$ of the manifold $\mathcal{M}$. Denote by Γ^+ the group of all orientation preserving homeomorphisms of Γ. The pair $(\Gamma,\, \Gamma^+)$ uniquely defines a sign structure $\omega : \Gamma \to \mathbb{Z}_2$ with $\mathrm{Ker}\,(\omega) = \Gamma^+$.

We note that the cyclic group $\Gamma = \mathbb{Z}_2$ is the fundamental group of a non-orientable projective plane $\mathcal{P}^2$ and of the orientable projective three-space $\mathcal{P}^3$. In the first case, $\Gamma^+ = \langle 1\rangle$ and (Γ, ω) is a non-orientable group. In the second, $\Gamma^+ = \mathbb{Z}_2$ and (Γ, ω) is an orientable group.

EXAMPLE 1 *Let $F_r = \langle x_1,\, x_2, \ldots, x_r\rangle$ be a free group of rank r and $\omega : F_r \to \mathbb{Z}_2 = \{-1,\, 1\}$ is the epimorphism defined by $\omega(x_i) = (-1)^{p_i},\ i = 1, 2, \ldots, r$, where at least one of the integers $p_1,\, p_2, \ldots, p_r$ is odd. Then $(F_r,\, \omega)$ be the group with sign structure.*

EXAMPLE 2 *Let $\Lambda_p = \langle x_1,\, x_2, \ldots, x_p\, :\, x_1^2 x_2^2 \cdots x_p^2 = 1\rangle$ be the fundamental group of a closed non-orientable surface of genus p and $\omega : \Lambda_p \to \mathbb{Z}_2 = \{1,\, -1\}$ be the epimorphism defined by $\omega(x_i) = (-1)^{q_i}$, $i = 1, \ldots, p$, where at least one of the integers $q_1,\, q_2, \ldots, q_p$ is odd. Then $(\Lambda_p,\, \omega)$ is a group with sign structure.*

EXAMPLE 3 *Let $\mathbb{Z}_\ell = \langle x :\ x^\ell = 1\rangle$ be a finite cyclic group of order ℓ. If ℓ is odd then there is only one (trivial) orientation on the group $\mathbb{Z}_\ell$ given by $\omega(x) = 1$. If ℓ is even there is only one non-trivial orientation defined by $\omega(x) = -1$. In both cases we will say that ω is the canonical orientation of $\mathbb{Z}_\ell$.*

Let $\Gamma = (\Gamma, \omega)$ and $A = (A, \eta)$ be two groups with sign structure. A homomorphism $\psi : \Gamma \to A$ is said to be *orientation preserving* if $\psi(\Gamma^+) \subseteq A^+$ and $\psi(\Gamma^-) \subseteq A^-$. An epimorphism $\psi : \Gamma \to A$ is orientation preserving if and only if its kernel is a subgroup of Γ^+. Note that the derived group $\Gamma' = [\Gamma,\, \Gamma] < \Gamma^+$. Denote by $\mathrm{Hom}^+(\Gamma, A)$, $\mathrm{Epi}^+(G, A)$ the respective sets of orientation preserving homomorphisms and epimorphisms $\Gamma \to A$. We consider $H_1(\Gamma) = \Gamma/\Gamma'$ as a group with sign structure whose positive elements are Γ^+/Γ'.

Thus we have the following auxiliary result.

Lemma 1 *Let $\Gamma = (\Gamma, \omega)$ and $A = (A, \eta)$ be groups with sign structure and let A be abelian. Then $|\mathrm{Hom}^+(\Gamma, A)| = |\mathrm{Hom}^+(H_1(\Gamma), A)|$.*

Let Γ and $\mathcal{A}$ be two finitely generated groups. Denote by $\mathrm{Hom}\,(\Gamma, \mathcal{K})$ (resp. $\mathrm{Epi}\,(\Gamma, \mathcal{K})$) the sets of homomorphisms (respectively, epimorphisms) from Γ to $\mathcal{K}$. By the Philip Hall inversion formula (3) we obtain

$$|\mathrm{Epi}\,(\Gamma, \mathcal{A})| = \sum_{\mathcal{K} \le \mathcal{A}} \mu(\mathcal{K}) |\mathrm{Hom}\,(\Gamma, \mathcal{K})|,$$

where μ is the Möbius function for the group $\mathcal{A}$ which assigns an integer $\mu(\mathcal{K})$ to each subgroup $\mathcal{K}$ of $\mathcal{A}$ by the recursive formula

$$\sum_{H \ge K} \mu(H) = \delta_{K,\mathcal{A}} = \begin{cases} 1 & \text{if } K = \mathcal{A}, \\ 0 & \text{if } K < \mathcal{A}. \end{cases}$$

Let $\Gamma = (\Gamma, \omega)$ and $\mathcal{A} = (\mathcal{A}, \eta)$ be groups with non-trivial sign structure. That is the groups $\Gamma^+ = \mathrm{Ker}\,\omega$ and $\mathcal{A}^+ = \mathrm{Ker}\,\eta$ are subgroups of index two in the groups Γ^+ and $\mathcal{A}^+$, respectively. We set also $\mathcal{K}^+ = \mathcal{K} \cap \Gamma^+$.

The following generalization of the P. Hall formula has been obtained in (13).

Proposition 1

$$|\mathrm{Epi}^{\,+}(\Gamma, \mathcal{A})| = \sum_{\substack{\mathcal{K} \le \mathcal{A} \\ \mathcal{K} \ne \mathcal{K}^+}} \mu(\mathcal{K}) |\mathrm{Hom}^{\,+}(\Gamma, \mathcal{K})|.$$

3 Counting conjugacy classes of subgroups

Consider a finitely generated group Γ. Let $\mathcal{P}$ be a property of subgroups of Γ, which is invariant under conjugation (for instance: to be normal, to be torsion free, to be orientable and so on). By a slight modification of arguments from the paper (21) we get the following result obtained earlier in (22)

Theorem 1 *Let Γ be a finitely generated group. Then the number of conjugacy classes of subgroups of index n in the group Γ having property $\mathcal{P}$ is given by the formula*

$$c_\Gamma^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{K <_m \Gamma} |\mathrm{Epi}^{\,\mathcal{P}}(K, \mathbb{Z}_\ell)|,$$

where the sum $\sum_{K <_m \Gamma}$ is taken over all subgroups K of index m in the group Γ and $\mathrm{Epi}^{\,\mathcal{P}}(K, \mathbb{Z}_\ell)$ is the set of epimorphisms of the group K onto the cyclic group $\mathbb{Z}_\ell$ whose kernel has the property $\mathcal{P}$.

From now on we suppose that $\Gamma = (\Gamma, \omega)$ is a finitely generated group with sign structure. The property $\mathcal{P} = \mathcal{P}^+$ or $\mathcal{P} = \mathcal{P}^-$ for the subgroups of Γ is "to be orientable" or "to be non-orientable", respectively. Applying Theorem 1 for $\mathcal{P} = \mathcal{P}^-$ we have the following result:

Theorem 2 *Let Γ be a finitely generated group with non-trivial sign structure. Then the number of conjugacy classes of non-orientable subgroups of index n in the group Γ is given by the formula*

$$c_\Gamma^{-}(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{K^- <_m \Gamma} |\mathrm{Epi}^{\,-}(K^-, \mathbb{Z}_\ell)|,$$

where the sum $\sum_{K^- <_m \Gamma}$ is taken over all non-orientable subgroups of index m in the group Γ and $\mathrm{Epi}^{\,-}(K, \mathbb{Z}_\ell)$ is the set of epimorphisms of the group K onto a cyclic group $\mathbb{Z}_\ell$ of order ℓ with non-orientable kernel.

Let $\mathcal{M}$ be a connected non-orientable manifold with a finitely generated group Γ. The group Γ acts by homeomorphisms on the universal covering $\tilde{\mathcal{M}}$ of the manifold $\mathcal{M}$. Denote by Γ^+ the subgroup of index two in Γ consisting of all orientation preserving homeomorphisms. Then Γ admits a non-trivial sign structure with the set of positive elements Γ^+. We identify the equivalency classes of orientable coverings of $\mathcal{M}$ with conjugacy classes of orientable subgroups in Γ. The following theorem gives a general solution of V. A. Liskovets problem

Theorem 3 *Let Γ be a finitely generated group with non-trivial sign structure. Denote by Γ^+ the group of positive elements of Γ. Then the number of conjugacy classes of orientable subgroups of index $2n$ in the group Γ is given by the formula*

$$c^+_\Gamma(2n) = \frac{1}{2n} \sum_{\substack{\ell \mid n \\ \ell\, m=n}} \left(\sum_{K^+ <_m \Gamma^+} |\text{Epi}\,(K^+,\, \mathbb{Z}_\ell)| + \sum_{K^- <_m \Gamma} |\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_{2\ell})| \right),$$

where the sum $\sum_{K^- <_m \Gamma}$ is taken over all non-orientable subgroups of index m in the group Γ, and $\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_{2\ell})$ *is the set of epimorphisms of the group K^- onto a cyclic group $\mathbb{Z}_{2\ell}$ with orientable kernel.*

Proof: By Theorem 1 for $\mathcal{P} = \mathcal{P}^+$ we obtain that the number of conjugacy classes of orientable subgroups of index $2n$ in the group Γ is given by the formula

$$c^+_\Gamma(2n) = \frac{1}{2n} \sum_{\substack{\ell \mid 2n \\ \ell\, m=2n}} \left(\sum_{K^+ <_m \Gamma} |\text{Epi}\,(K^+,\, \mathbb{Z}_\ell)| + \sum_{K^- <_m \Gamma} |\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_\ell)| \right),$$

where the sums $\sum_{K^+ \,<_m \Gamma}$ and $\sum_{K^- \,<_m \Gamma}$ are taken over all orientable and non-orientable subgroups of index m in the group Γ, respectively and $\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_\ell)$ is the set of epimorphisms of the group K^- onto a cyclic group $\mathbb{Z}_\ell$ of order ℓ with orientable kernel. We note that all orientable subgroups in Γ are of even index. Hence, the condition $\sum_{K^+ \,<_m \Gamma}$ can be rewritten in the form $\sum_{K^+ \,<_{\bar m} \Gamma^+}$, where $m = 2\,\bar m$ and $\ell\,\bar m = n$. By the same reason any orientable kernel of the epimorphism of K^- onto $\mathbb{Z}_\ell$ has an even index in K^-. Hence, ℓ is even and $\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_\ell)$ can be represented in the form $\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_{2\,\bar\ell})$, where $\bar\ell\, m = n$. After these remarks, by replacing $\bar\ell$ to ℓ an $\bar m$ to m we obtain the result. □

To make the above solution more explicit we have to calculate the numbers of epimorphisms $|\text{Epi}\,(K^+,\, \mathbb{Z}_\ell)|$ and $|\text{Epi}^{\,+}(K^-,\, \mathbb{Z}_{2\ell})|$. This will be done in the next section.

4 Counting orientation preserving homomorphisms

In this section we describe how to calculate orientation preserving epimorphisms of a given finitely generated group onto the cyclic group. We will follow the ideas given in papers (9), (13) and (14).

Let $\Gamma = (\Gamma,\, \omega)$ be a finitely generated group with sign structure. We represent the oriented homology group $(H_1(\Gamma),\, \bar\omega)$ of the group Γ in the form

$$H_1(\Gamma) = \mathbb{Z}^{\varepsilon_1}_{k_1} \oplus \mathbb{Z}^{\varepsilon_2}_{k_2} \oplus \cdots \oplus \mathbb{Z}^{\varepsilon_n}_{k_n},$$

where $k_j \in \{2, 3, \ldots, \infty\}$ and the notation $\mathbb{Z}^\varepsilon_k$ means that the corresponding cyclic group $\mathbb{Z}_k = \langle x \mid x^k = 1\rangle$ is generated by an element x with $\bar\omega(x) = \varepsilon$ for some $\varepsilon \in \{-1, 1\}$.

We accept the following conventions $\mathbb{Z}^{-1}_k = \mathbb{Z}^-_k,\ \ \mathbb{Z}^{+1}_k = \mathbb{Z}^+_k,\ \mathbb{Z}_\infty = \mathbb{Z},\ (-1)^\infty = 1,\ (k,\, \ell) = gcd(k,\, \ell)$ and $(\infty, \ell) = \ell$ for any positive integer ℓ.

EXAMPLE 4 *Let $F_r = \langle x_1,\, x_2, \ldots, x_r\rangle$ be the free group of rank r with a sign structure given by $\omega(x_i) = -1,\ i = 1, 2, \ldots r$. Then $H_1(F_r) = (\mathbb{Z}^-)^r$.*

EXAMPLE 5 *Let $\Lambda_p = \langle x_1,\, x_2, \ldots, x_p \,:\, x_1^2\, x_2^2 \cdots x_p^2 = 1\rangle$ be the fundamental group of a closed non-orientable surface of genus p with sign structure given by $\omega(x_i) = -1,\ i = 1, \ldots, p$. Then $H_1(\Lambda_p) = (\mathbb{Z}^-)^{(p-1)} \oplus \mathbb{Z}_2^{(-1)^p}$.*

The main result is given by the following theorem obtained in (14).

Theorem 4 *Let Γ be a finitely generated group with non-trivial sign structure and*

$$H_1(\Gamma) = \mathbb{Z}_{k_1}^{\varepsilon_1} \oplus \mathbb{Z}_{k_2}^{\varepsilon_2} \oplus \cdots \oplus \mathbb{Z}_{k_n}^{\varepsilon_n}$$

is the oriented homology group of Γ. Then $|\mathrm{Epi}^+(\Gamma, \mathbb{Z}_\ell)| = 0$ if ℓ is odd and

$$|\mathrm{Epi}^+(\Gamma, \mathbb{Z}_{2\ell})| = \prod_{j=1}^{n} \frac{1 + \varepsilon_j^{\frac{k_j}{(k_j,\, \ell)}}}{2} \sum_{\substack{m \\ \frac{\ell}{m}:\mathrm{odd}}} \mu(\frac{\ell}{m})(k_1,\, m)(k_2,\, m) \ldots (k_n,\, m).$$

We note that a similar formula for the number $|\mathrm{Epi}\,(\Gamma, \mathbb{Z}_\ell)|$ of all epimorphisms Γ onto Z_ℓ was obtained in (21). Then we also have

$$|\mathrm{Epi}^-(\Gamma, \mathbb{Z}_\ell)| = |\mathrm{Epi}\,(\Gamma, \mathbb{Z}_\ell)| - |\mathrm{Epi}^+(\Gamma, \mathbb{Z}_\ell)|.$$

5 Application to surface coverings

In this section we employ the above general theorems to count orientable coverings over bordered and and non-orientable surfaces. By straightforward calculations from Theorems 3 and 4 we obtain the following results.

Theorem 5 *Let $\mathcal{B}$ be a bordered non-oriented surface with the fundamental group $\pi_1(\mathcal{B}) = F_r$. Then the number of orientable $2n$-fold coverings of $\mathcal{B}$ is given by the formula*

$$c^+_{F_r}(2n) = \frac{1}{2n} \sum_{\substack{\ell | n \\ \ell\, m = n}} (\varphi_{2\,m(r-1)+1}(\ell)\, s_{F_{2\,r-1}}(m) + \varphi^{odd}_{m(r-1)+1}(\ell)\, s^-_{F_r}(m)),$$

where $\varphi_r(\ell)$ and $\varphi_r^{odd}(\ell)$ are the Jordan functions, $s_{F_r}(m)$ is the numbers s of subgroups of index m, and $s^-_{F_r}(m)$ is the numbers of non-orientable subgroups of index m in the group F_r.

We note that $c^+_{F_r}(2n)$ coincides with the number of balanced $2n$-fold coverings of a unbalanced graph B_r obtained in (2).

Theorem 6 *Let $\mathcal{N}_p$ be a closed non-oriented surface of genus p with the fundamental group $\pi_1(\mathcal{N}_p) = \Lambda_p$. Then the number of orientable $2n$-fold coverings of $\mathcal{N}_p$ is given by the formula*

$$c^+_{\Lambda_r}(2n) = \frac{1}{2n} \sum_{\substack{\ell \mid n \\ \ell\, m = n}} \left(\varphi_{2\,s+1}(\ell)\, s_{\Lambda_{2(p-1)}}(m) + \frac{1 + (-1)^{(s-1)(\ell-1)}}{2}\, (2,\ell)\, \varphi^{odd}_s(\ell)\, s^-_{\Lambda_p}(m) \right),$$

where $s = m(p-2)+1$, $\varphi_s(\ell)$ and $\varphi^{odd}_s(\ell)$ are the Jordan functions, $s_{\Lambda_p}(m)$ is the number of subgroups of index m, and $s^-_{\Lambda_p}(m)$ is the number of non-orientable subgroups of index m in the group Λ_p.

The last theorem was obtained earlier in (2) by a combinatorial argument.

Let $\mathcal{K}$ be a Klein bottle, that is a closed non-oriented surface of genus 2. It was shown in (20) and (10) that the number $c^-_{\mathcal{K}}(n)$ of n-fold coverings of $\mathcal{K}$ is expressed in terms of classical number-theoretical functions. The number $c^-_{\mathcal{K}}(n)$ of non-orientable n-fold coverings of $\mathcal{K}$ was calculated in (17) by a complicated method based on analysis of the subgroup structure in the fundamental group of $\mathcal{K}$.

Now, as a corollary of Theorem 2 we obtain a simple formula for $c^-_{\mathcal{K}}(n)$.

Recall that any positive integer n can be uniquely represented in the form $n = 2^s \cdot n^-$, where $s \geq 0$ and n^- is an odd number. We call $n^+ = 2^s$ and n^- by *even* and *odd* part of n, respectively.

Theorem 7 *Let $\mathcal{K}$ be a Klein bottle. Then the number of non-orientable n-fold coverings of $\mathcal{K}$ is given by the formula*

$$c^-_{\mathcal{K}}(n) = (2,n)\, d(n^-),$$

where n^- is the odd part of n and $d(n)$ is the number of positive divisors of n.

6 Reflexible coverings and chiral pairs

Let $\mathcal{M}$ be a non-orientable manifold or orbifold. An orientable coverings $\pi : \mathcal{U}^+ \to \mathcal{M}$ is called *reflexible* if there exists an orientation reversing homeomorphism $h : \mathcal{U}^+ \to \mathcal{U}^+$ such that $\pi \circ h = \pi$ and *irreflexible* (or *chiral*) otherwise. All irreflexible coverings are divided into chiral pairs of twins. Two twins are non-equivalent as coverings over $\mathcal{M}^+$, but have to be equivalent as coverings over $\mathcal{M}$.

In particular, any regular covering π is reflexible. The multiplicity of any finite sheeted reflexible covering is an *even* number.

The following theorem is a consequence of Theorem 1 and Theorem 2:

Theorem 8 *Let $\mathcal{M}$ be a connected non-orientable manifold with a finitely generated fundamental group Γ. Then the number of $2n$-fold reflexible coverings of $\mathcal{M}$ is given by the formula*

$$a_\Gamma(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell\, m = n}} \sum_{K^- <_m \Gamma} |\mathrm{Epi}^+(K^-, \mathbb{Z}_{2\ell})|,$$

where the sum $\sum_{K^- <_m \Gamma}$ is taken over all non-orientable subgroups of index m in the group Γ and $Epi^+(K, \mathbb{Z}_\ell)$ is the set of epimorphisms of the group K onto a cyclic group $\mathbb{Z}_\ell$ of order ℓ with orientable kernel.

Proof: Let Γ^+ be the positive subgroup of Γ and $\Gamma = \Gamma^+ + \sigma\,\Gamma^+$ is its coset decomposition. Let K be a subgroup of Γ^+. Denote by $[K]_{\Gamma^+}$ and $[K]_\Gamma$ the conjugacy class of K in Γ^+ and Γ, respectively.

There are two kinds of subgroups K in Γ^+. Either *reflexible* with the property $[K]_{\Gamma^+} = [K^\sigma]_{\Gamma^+} = [K]_\Gamma$ or *twin* with $[K]_{\Gamma^+} \neq [K^\sigma]_{\Gamma^+}$ and $[K]_\Gamma = [K]_{\Gamma^+} \cup [K^\sigma]_{\Gamma^+}$. By definition, the set of all orientable subgroups is the disjoint union of reflexible and twin subgroups. Denote by $a_\Gamma(n)$ and $t_\Gamma(n)$ the numbers of conjugacy classes of reflexible and twin subgroups of index $2n$ in the group Γ, respectively. Now we calculate the numbers of orientable subgroups of index $2n$ up to conjugacy in Γ and Γ^+. We get

$$\begin{aligned} c_\Gamma^+(2n) &= a_\Gamma(n) + t_\Gamma(n) \\ c_{\Gamma^+}(n) &= a_\Gamma(n) + 2\,t_\Gamma(n). \end{aligned}$$

From Theorem 3 we have

$$c_\Gamma^+(2n) = \frac{1}{2}(\,c_{\Gamma^+}(n) + \,I(n)),$$

where

$$I(n) = \frac{1}{n}\sum_{\substack{\ell \mid n \\ \ell\, m = n}} \sum_{K^- <_m \Gamma} |\mathrm{Epi}^{\,+}(K^-, \mathbb{Z}_{2\ell})|.$$

Hence,

$$a_\Gamma(n) = 2c_\Gamma^+(2n) - c_{\Gamma^+}(n) = (c_{\Gamma^+}(n) + I(n)) - c_{\Gamma^+}(n) = I(n).$$

□

Also, since $t_\Gamma(n) = \dfrac{1}{2}(c_{\Gamma^+}(n) - a_\Gamma(n))$ we obtain the following proposition.

Proposition 2 *Let $\mathcal{M}^+$ be the orientable double of a non-orientable manifold $\mathcal{M}$. Then the number of chiral pairs of n-fold coverings of $\mathcal{M}^+$ is given by the formula*

$$t_\Gamma(n) = \frac{c_{\Gamma^+}(n) - a_\Gamma(n)}{2},$$

where Γ^+ is the fundamental group of $\mathcal{M}^+$ and $c_{\Gamma^+}(n)$ and $a_\Gamma(n)$ are determined by Theorems 1 and 8, respectively.

We illustrate results of the last section by the following example. Let $\mathcal{M} = \mathcal{K}$ be a Klein bottle with fundamental group

$$\Gamma = \pi_1(\mathcal{K}) = \langle x,\, y \mid xyxy^{-1} = 1\rangle.$$

The orientable double of $\mathcal{K}$ is a torus $\mathcal{T}$ with fundamental group

$$\Gamma^+ = \pi_1(\mathcal{T}) = \langle a,\, b \mid [a,\, b] = 1\rangle,$$

where $a = x$ and $b = y^2$. We note that $\Gamma = \Gamma^+ + \Gamma^+ y,\ \ yay^{-1} = a^{-1}$, and $yby^{-1} = b$.

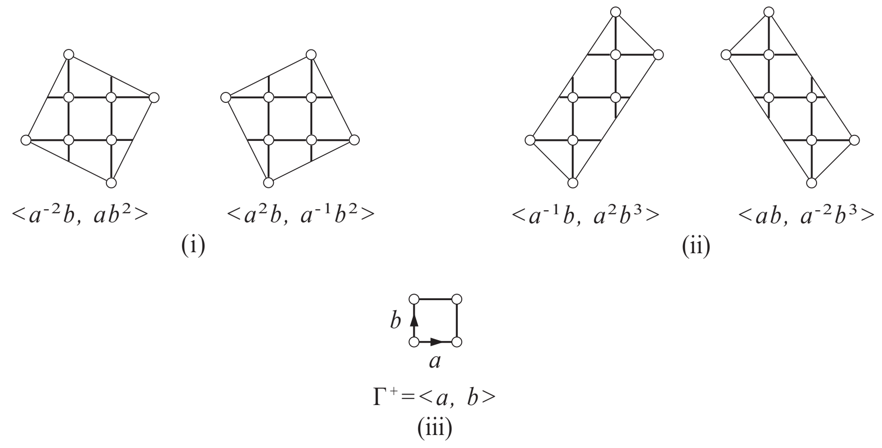

Fig. 1

The chiral pairs of coverings

Set $n = 5$. Then, by direct calculation from Theorems 1 and 6 we obtain the following equations $c_\Gamma^+(10) = a_\Gamma(5) + t_\Gamma(5) = 4$ and $c_{\Gamma^+}(5) = a_\Gamma(5) + 2\,t_\Gamma(5) = 6$. Hence, $a_\Gamma(5) = t_\Gamma(5) = 2$.

That is the group Γ^+ has six (conjugacy classes of) subgroups producing coverings. Two of them, $\langle a^5, b\rangle$ and $\langle a, b^5\rangle$ are reflexible and other four are divided into chiral pairs. They are $\langle a^2b, a^{-1}b^2\rangle$, $\langle a^{-2}b, ab^2\rangle$ and $\langle ab, a^{-2}b^3\rangle$, $\langle a^{-1}b, a^2b^3\rangle$, respectively. (See Fig. 1(i), (ii)).
The twin subgroups in chiral pairs are not conjugate in Γ^+ but are conjugate by element y in the group Γ.

7 Numerical tables

n	Total number	Non-orientable	Orientable	Reflexible	Chiral pairs
01	1	1	0	0	0
02	3	2	1	1	0
03	2	2	0	0	0
04	5	2	3	3	0
05	2	2	0	0	0
06	7	4	3	2	1
07	2	2	0	0	0
08	8	2	6	5	1
09	3	3	0	0	0
10	8	4	4	2	2
11	2	2	0	0	0
12	13	4	9	6	3

Table 1. The number of n-fold coverings of genus 2 non-orientable surface (Klein bottle)

n	Total number	Non-orientable	Orientable
01	1	1	0
02	7	6	1
03	14	14	0
04	89	80	9
05	264	264	0
06	1987	1930	57
07	11086	11086	0
08	93086	92240	846
09	779579	779579	0
10	7753717	7738414	15303
11	82768656	82768656	0
12	990033082	989476104	556978

Table 2. The number of n-fold coverings of genus 3 non-orientable surface

n	Total number	Non-orientable	Orientable
01	1	1	0
02	15	14	1
03	82	82	0
04	1583	1544	39
05	30242	30242	0
06	1110935	1109452	1483
07	51724322	51724322	0
08	3296935883	3296581880	354003
09	264893065443	264893065443	0
10	26438156395760	26437948184476	208211284
11	3193922150539634	3193922150539634	0
12	459644128922143309	459643859463061456	269459081853

Table 3. The number of n-fold coverings of genus 4 non-orientable surface

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