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Tiling $\mathbb{Z}^2$ with translations of one set

Hui Rao$^1$ and Yu-Mei Xue$^2$

$^1$Mathematical Department, Tsinghua university, Beijing 100084, China
E-mail: hrao@math.tsinghua.edu.cn
$^2$Mathematical Department, Tsinghua university, Beijing 100084, China
E-mail: YXue@math.tsinghua.edu.cn


Let $A$ be a finite subset of $\mathbb{Z}^2$. We say $A$ tiles $\mathbb{Z}^2$ with the translation set $C$, if any integer $z \in \mathbb{Z}^2$ can be represented as $z_1 + z_2$, $z_1 \in A$, $z_2 \in C$ in a unique way. In this case we call $A$ a $\mathbb{Z}^2$-tile and write $A \oplus C = \mathbb{Z}^2$. A tile $A$ is said to be a normal $\mathbb{Z}^2$-tile if there exists a periodic set $C$ such that $A \oplus C = \mathbb{Z}^2$. We characterize all normal $\mathbb{Z}^2$-tiles with prime cardinality.

Keywords: tiling, periodicity

1 Introduction

Let $A$ be a finite subset of $\mathbb{Z}^n$. We denote by $\#A$ the cardinality of $A$. We say $A$ is a $\mathbb{Z}^n$-tile (or tile in short), if there is a set $C \subseteq \mathbb{Z}^n$ such that any element $z \in \mathbb{Z}^n$ can be represented uniquely in the form

$$z = z_A + z_C, \quad z_A \in A, \quad z_C \in C.$$ 

In this case, we say the pair $(A, C)$ is a translation tiling of $\mathbb{Z}^n$ and write $A \oplus C = \mathbb{Z}^n$.

An infinite subset $C$ of $\mathbb{Z}^n$ is periodic, if there is a vector $\lambda$ such that $C = C + \lambda$; $\lambda$ is said to be a period of $C$. A set $C$ is $k$-periodic if it has $k$ linearly independent periods.

The $\mathbb{Z}$-tiles have been studied by many authors ([New], [Sands], [Szabo], [Tij1], [Coven]). It is well known that if $A$ is finite and $A \oplus C = \mathbb{Z}$, then the translation set $C$ must be periodic ([Fuchs], [New]). Hence tiling problems are translated to problems of decompositions of the finite cyclic group $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n - 1\}$. Newman [New] determined all $\mathbb{Z}$-tiles such that $\#A$ is a prime power. Particularly, when $\#A$ is a prime number, it is shown that

**Proposition 1.1** ([New], [Sands]) Let $p$ be a prime number and $A = \{s_0, s_1, \ldots, s_{p-1}\}$ be a subset of $\mathbb{Z}$. Then $A$ is a $\mathbb{Z}$-tile if and only if

$$\left\{ \frac{s_0}{d}, \frac{s_1}{d}, \ldots, \frac{s_{p-1}}{d} \right\} \equiv \{0, 1, \ldots, p - 1\} \pmod{p}$$

where $d = \gcd\{s_0, s_1, \ldots, s_{p-1}\}$ is the greatest common divisor.

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Remark 1.2 The above result can be expressed in terms of cyclotomic polynomials as follows (Coven: Lemma 1.1). Let \( \Phi_n(z) \) denote the \( n \)-th cyclotomic polynomial. Then

\[
A \text{ is a } \mathbb{Z} \text{-tile if and only if there exists an integer } k \geq 1 \text{ such that } \Phi_p^k(z) \text{ divides the polynomial } A(z) = z^{s_0} + z^{s_1} + \cdots + z^{s_{p-1}}.
\]

Recently, based on works of Sands [Sands] and Tijdeman [Tij1], Coven and Meyerowitz [Coven] characterized all the \( \mathbb{Z} \)-tiles \( A \) such that \( \#A \) has at most two prime factors.

However, the study of \( \mathbb{Z}^n \)-tiles seems to be untouched except the work of Beauquier and Nivat [BN]. [BN] gives an elegant characterization of the polyomino tiles which are disk-like. This is a special case of \( \mathbb{Z}^2 \)-tiles.

For the study of \( \mathbb{Z}^n \)-tiles, the first difficulty is the periodicity. We call a tile \( A \) a normal tile, if there is a periodic translation set \( C \) such that \( A \oplus C = \mathbb{Z}^n \). Namely, a tile is normal if it can tile \( \mathbb{Z}^n \) periodically.

It has been conjectured that any translation tile of \( \mathbb{Z}^n \) is normal.

**Periodic Tiling Conjecture.** (Lagarias and Wang [LW]) Any \( \mathbb{Z}^n \)-tile is normal.

This conjecture is true for \( \mathbb{Z} \)-tiles as we have mentioned, but it is widely open for higher dimensions. For more details we refer to Tijdeman [Tij2].

In the present paper, our main purpose is to characterize the normal \( \mathbb{Z}^2 \)-tiles \( A \) with \( \#A \) a prime number. Our main result is the following theorem.

**Theorem 1.3** Let \( A = \{(s_0, t_0), \ldots, (s_{p-1}, t_{p-1})\} \) be a subset of \( \mathbb{Z}^2 \) where \( p \) is a prime number. Then \( A \) is a normal \( \mathbb{Z}^2 \)-tile if and only if there exist two integers \( a \) and \( b \), such that \( as_0 + bt_0, \ldots, as_{p-1} + bt_{p-1} \) are distinct and \( \{as_0 + bt_0, \ldots, as_{p-1} + bt_{p-1}\} \) is a \( \mathbb{Z} \)-tile.

Roughly speaking, \( A \) is a normal tile with prime cardinality if and only if a projection of \( A \) is a \( \mathbb{Z} \)-tile.

This paper is organized as follows. In Section 2 we give several interesting results on periodicity of tilings of normal tiles. In Section 3 we prove Theorem 1.3 by using cyclotomic polynomials. An algorithm is given in Section 4 to check whether a set \( A \) with prime cardinality is a normal tile.

The referee pointed out to us that Szegedy [Sze] has proved a more general result as follows.

**Theorem 1.4** (Szegedy [Sze]) Let \( A \) be a \( \mathbb{Z}^n \)-tile with \( \#A \) prime or \( \#A = 4 \), then \( A \) is normal.

He gives in both cases an algorithm to decide the tiling problem, and our result Theorem 1.3 is covered by Szegedy’s algorithm. However, while Szegedy’s approach is primarily group-theoretic, we use elementary congruences and cyclotomic polynomials, which have been used successfully to study \( \mathbb{Z} \)-tiles by previous authors. We hope that our method can give a clue how to tackle \( \mathbb{Z}^2 \)-tiles \( A \) with \( \#A = pq, \) where \( p, q \) are prime numbers.

## 2 Periodicity

In this section, we give some lemmas on the periodicity of tilings of normal tiles. Lemma 2.4 is an interesting generalization of a result of Tijdeman [Tij1].

Let \( A \oplus C = \mathbb{Z}^2 \), and let \( \phi \) be an invertible linear transformation from \( \mathbb{Z}^2 \) to \( \mathbb{Z}^2 \). We may regard \( \phi \) as an integral \( 2 \times 2 \) matrix with determinant \( \pm 1 \). It is obvious that \( A \oplus C = \mathbb{Z}^2 \) if and only if \( \phi(A) \oplus \phi(C) = \mathbb{Z}^2 \).

**Proposition 2.1** If \( A \) is a normal \( \mathbb{Z}^2 \)-tile, then there is a 2-periodic set \( C \) such that \( A \oplus C = \mathbb{Z}^2 \).
Proof: Suppose \( A \oplus C' = \mathbb{Z}^2 \) and \( C' \) is periodic. By applying a linear transformation, we may assume that a period of \( C' \) is \( (c, 0) \), and further we assume that \( c \) is larger than the diameter of \( A \). We divide the plane into squares of size \( c \times c \), and denote by \( S_{[a, b]} \) the square
\[
  S_{[a, b]} := [ac, (a + 1)c] \times [bc, (b + 1)c].
\]
Let us consider the intersection of the translation set \( C' \) and \( S_{[a, b]} \), and define
\[
  \mathcal{P}(a, b) = \{ (s, t) - (ac, bc); \ (s, t) \in C' \cap S_{[a, b]} \}.
\]
Now \( (c, 0) \) is a period of \( C' \) implies that \( \mathcal{P}(a, b) = \mathcal{P}(0, b) \).

Let \( b_1 \) and \( b_2 \) be two integers such that \( \mathcal{P}(0, b_1) = \mathcal{P}(0, b_2) \), this must happen since the number of the patterns \( \mathcal{P}(0, b), b \in \mathbb{Z} \), is finite. Let
\[
  C = \bigcup_{m \in \mathbb{Z}} (C' \cap (\mathbb{Z} \times [b_1c, b_2c])) + m\vec{v}
\]
where \( \vec{v} = (0, (b_2 - b_1)c) \). Then \( C \) is 2-periodic with periods \((c, 0)\) and \((0, (b_2 - b_1)c)\).

It remains to show that \( A \oplus C = \mathbb{Z}^2 \). By the above construction,
\[
  \{(s, t) \in C'; \ b_1c \leq t < b_2c \} + A
\]
covers a strip of the plane. We shall show that this patch can be extended to a tiling. Notice that the configurations of \( C' \) in \( \mathbb{Z} \times [b_1c, (b_1 + 1)c] \) and \( \mathbb{Z} \times [b_2c, (b_2 + 1)c] \) are the same. Let \( \mathbb{Z} \times [b_2c, (b_2 + 2)c] \) have the same configuration as \( \mathbb{Z} \times [b_1c, (b_1 + 2)c] \), then the patch is extended with neither gap nor overlap. Repeating this procedure, the patch is extended to a tiling of the upper half plane. We can do the same for the lower half plane. Therefore we obtain a tiling and the translation set is \( C \) in \( \square \).

Lemma 2.2 If a set \( C \) is 2-periodic, then it has two periods \((M, 0)\) and \((0, N)\) for some integers \( M \) and \( N \).

Proof: Suppose \( \lambda_1 \) and \( \lambda_2 \) are two linear independent periods of \( C \). Then \( a\lambda_1 + b\lambda_2 \) is also a period of \( C \). Clearly we can choose \( a, b \) properly to have periods of the form \((M, 0)\) and \((0, N)\).

For a finite set \( A \), we define a polynomial \( A(x, y) \) as
\[
  A(x, y) := \sum_{(s, t) \in A} x^s y^t.
\]
For a polynomial \( P(x, y) = \sum x^s y^t \), define
\[
  P(x, y) \pmod{x^M - 1, y^N - 1} = \sum x^s \pmod{x^M} y^t \pmod{y^N}.
\]
Then as a corollary of Proposition 2.1 and Lemma 2.2 we have

Corollary 2.3 A finite set \( A \subset \mathbb{Z}^2 \) is a normal tile if and only if there exists a finite set \( B \subset \mathbb{Z}^2 \) and positive integers \( M, N \) such that \( \#A \#B = MN \) and
\[
  A(x, y)B(x, y) \equiv (1 + x + \cdots + x^{M - 1})(1 + y + \cdots + y^{N - 1}) \pmod{x^M - 1, y^N - 1}. \]
The following lemma is a generalization of a result of Tijdeman [Tijd], which concerns the \( \mathbb{Z} \)-tilings. Coven and Meyerowitz ([Coven]: Lemma 3.1) gave a nice proof of Tijdeman’s Lemma. Our proof is a generalization of the proof in [Coven].

**Lemma 2.4** Let \( A \) and \( B \) be finite subsets of \( \mathbb{Z}^2 \) with non-negative coordinates with corresponding polynomials \( A(x, y) \) and \( B(x, y) \) and let \( MN = \#A\#B \). If equation (2) holds for any \( i, j \) \( \equiv \) modulo \( p \). Since \( p \) is a prime which is not a factor of \( \#A \), then

\[
A(x^p, y^p)B(x, y) \equiv (1 + x + \cdots + x^{M-1})(1 + y + \cdots + y^{N-1}) \quad (\text{mod } x^M - 1, y^N - 1).
\]

**Proof:** Since \( p \) is prime, \( A(x^p, y^p) \equiv (A(x, y))^p \pmod{p} \), i.e., when the coefficients are reduced modulo \( p \). Let \( G_{M,N}(x, y) = (1 + x + \cdots + x^{M-1})(1 + y + \cdots + y^{N-1}) \). Then

\[
A(x^p, y^p)B(x, y) \equiv (A(x, y))^{p-1}A(x, y)B(x, y) \equiv (A(x, y))^{p-1}G_{M,N}(x, y),
\]

where \( \equiv \) means the exponents of \( x \) and \( y \) are reduced modulo \( M \) and \( N \) respectively, and then the coefficients are reduced modulo \( p \). Since

\[
x^i y^j G_{M,N}(x, y) \equiv G_{M,N}(x, y) \quad (\text{mod } x^M - 1, y^N - 1)
\]

holds for any \( i, j \), we have

\[
(A(x, y))^{p-1}G_{M,N}(x, y) \equiv (A(1, 1))^{p-1}G_{M,N}(x, y) \quad (\text{mod } x^M - 1, y^N - 1).
\]

Since \( p \) does not divide \( \#A \), Fermat’s Little Theorem yields \( (A(1, 1))^{p-1} = 1 \pmod{p} \). Therefore

\[
A(x^p, y^p)B(x, y) \equiv G_{M,N}(x, y),
\]

where the exponents of \( x \) and \( y \) are reduced modulo \( M \) and \( N \) respectively, and then the coefficients are reduced modulo \( p \).

Since \( A(1, 1)B(1, 1) = G_{M,N}(1, 1) = MN \), both \( A(x^p, y^p)B(x, y) \) and \( G_{M,N}(x, y) \) have nonnegative coefficients whose sum is \( MN \). Consider the following reductions.

(R1) \( A(x^p, y^p)B(x, y) \) is reduced modulo \( x^M - 1, y^N - 1 \), yielding a polynomial \( G^*(x, y) \).

(R2) The coefficients of \( G^*(x, y) \) are reduced modulo \( p \), yielding \( G_{M,N}(x, y) \).

Reduction (R1) preserves the sum of the coefficients, but (R2) reduces the sum by some nonnegative multiple of \( p \). Because the sum of the coefficients of both \( G^*(x, y) \) and \( G_{M,N}(x, y) \) is \( MN \), that multiple is 0. Therefore \( G^*(x, y) = G_{M,N}(x, y) \). \( \Box \)

The following theorem is a two dimensional generalization of Lemma 2.3 in [Coven]. We have shown that if \( A \) is a normal tile, then there is a translation set with periods \((M, 0)\) and \((0, N)\). Theorem 2.5 says further that we may assume that \( M \) and \( N \) have the same prime factors as \( \#A \).

**Theorem 2.5** If \( A \subset \mathbb{Z}^2 \) is a normal tile, then there exists a finite set \( B \subset \mathbb{Z}^2 \) and two integers \( M \) and \( N \) are products of prime factors of \( \#A \), such that

\[
A(x, y)B(x, y) \equiv (1 + x + \cdots + x^{M-1})(1 + y + \cdots + y^{N-1}) \quad (\text{mod } x^M - 1, y^N - 1).
\]
Let $A \oplus C = \mathbb{Z}^2$ be a tiling of periods $(M, 0)$ and $(0, N)$ and $r > 1$ is a prime factor of $M$ which does not divides $\#A$, then by Lemma 2.4 $rA \oplus C = \mathbb{Z}^2$. Therefore $rA \oplus C_0 = r\mathbb{Z}^2$ where $C_0 = \{(s, t) \in C; s \equiv 0, t \equiv 0 \pmod{r}\}$.

Since the periods $(M, 0)$ and $(0, rN)$ of $C$ are also periods of $C_0$, we conclude that $A \oplus C_0/r = \mathbb{Z}^2$ is a tiling with periods $(M/r, 0)$ and $(0, N)$. Continuing this procedure, we may remove any prime factor $r$ from $M$ which does not divide $\#A$. 

### 3 Tiles with prime cardinality

In this section, we prove Theorem 1.3, the main result of this paper.

**Proposition 3.1** Let $A = \{(s_0, t_0), (s_1, t_1), \ldots, (s_{n-1}, t_{n-1})\}$ be a subset of $\mathbb{Z}^2$. If there exist two integers $a$ and $b$ such that $as_0 + bt_0, as_1 + bt_1, \ldots, as_{n-1} + bt_{n-1}$ are distinct and $(as_0 + bt_0, as_1 + bt_1, \ldots, as_{n-1} + bt_{n-1})$ is a $\mathbb{Z}$-tile, then $A$ is a normal $\mathbb{Z}^2$-tile.

**Proof:** Let $k \in \mathbb{Z}$ and $k \neq 0$, then obviously $E$ is a $\mathbb{Z}$-tile if and only if $kE$ is a $\mathbb{Z}$-tile ([Coven]: Lemma 1.4).

**Case 1.** $a \neq 0$ and $b = 0$ (or, vice versa, $a = 0$ and $b \neq 0$). By assumption $\{as_0, \ldots, as_{n-1}\}$ is a $\mathbb{Z}$-tile, so $\{s_0, \ldots, s_{n-1}\}$ is also a $\mathbb{Z}$-tile. Hence there is a translation set $F$ such that

$$\{s_0, \ldots, s_{n-1}\} \oplus F = \mathbb{Z}. \quad (3)$$

Let $C = F \times \mathbb{Z}$. Clearly $A \oplus C = \mathbb{Z}^2$ and $C$ is periodic.

**Case 2.** $ab \neq 0$. We may assume that $a$ and $b$ are coprime. Let $u, v$ be two integers such that $au - bv = 1$, and let

$$\phi = \begin{pmatrix} a & b \\ v & u \end{pmatrix}.$$ 

In the following, we will regard $\phi$ as a linear operator from $\mathbb{Z}^2$ to $\mathbb{Z}^2$. Then $\phi$ gives a one-to-one map from $\mathbb{Z}^2$ to $\mathbb{Z}^2$. Since

$$\phi(A) = \{(as_0 + bt_0, vs_0 + ut_0), \ldots, (as_{n-1} + bt_{n-1}, vs_{n-1} + ut_{n-1})\}$$

and $(as_0 + bt_0, as_1 + bt_1, \ldots, as_{n-1} + bt_{n-1})$ is a $\mathbb{Z}$-tile, so by the conclusion of Case 1 (by choosing $a = 1, b = 0$ there), $\phi(A)$ is a normal tile. Hence $A$ is also a normal tile. 

From now on, we prove the other direction of Theorem 1.3 Suppose $A$ is a normal tile and $\#A = p$ is a prime. Then by the discussion of Section 2, there is a translation set $C$ with periods $(p^m, 0)$ and $(0, p^m)$ for some positive integer $m$. Let

$$D_1 = \{(1, b); b = 0, 1, \ldots, p^m - 1\}, \quad D_2 = \{(a, 1); a = 0, p, 2p, \ldots, p^m - p\},$$

and set $D = D_1 \cup D_2$. If $p^k$ divides $x$ but $p^{k+1}$ does not divide $x$, then we write $v_p(x) = k$. For two integers $s, t$ we denote $v_p(s, t) := v_p(gcd\{s, t\})$. First we establish two lemmas.
Lemma 3.2 Let \((s, t)\) be a point of \(\mathbb{Z}^2\) with \(v_p(s, t) = k\). Then the constant term of
\[
\sum_{(a, b) \in D} z^{as+bt} \pmod{z^{p^m} - 1}
\]
is \(p^k\) when \(k < m\), is \(p^m(1 + \frac{1}{p})\) when \(k \geq m\).

**Proof:** When \(v_p(s, t) \geq m\), each term in \((4)\) is 1, and hence the constant term of \((4)\) is \#\(D\) = \(p^m(1 + \frac{1}{p})\).

So let us assume that \(v_p(s, t) < m\).

Without loss of generality, let us assume that \(v_p(s) \geq v_p(t)\), which implies \(v_p(t) = k\). There are exactly \(p^k\) elements (in sense of a multiple set) in \(\{s + bt; 0 \leq b \leq p^m - 1\}\) which can be divided by \(p^m\).

Hence the constant term of
\[
\sum_{(a, b) \in D_1} z^{as+bt} \pmod{z^{p^m} - 1} = \sum_{0 \leq b \leq p^m - 1} z^{s+bt} \pmod{z^{p^m} - 1}
\]
is \(p^k\). Clearly the constant term of
\[
\sum_{(a, b) \in D_2} z^{as+bt} \pmod{z^{p^m} - 1}
\]
is 0. The lemma is proved.

**Lemma 3.3** Let \(B\) be a subset of \(\mathbb{Z}^2\) with \#\(B\) = \(p^n\), and let \(B(x, y)\) be the corresponding polynomial. If for any non-negative coprime integers \(a, b\), holds
\[
B(z^a, z^b) \equiv p^{n-m}(1 + z + \cdots + z^{p^m-1}) \pmod{z^{p^m} - 1},
\]
then \(n \geq 2m\) and
\[
B(x, y) \equiv p^{n-2m}(1 + x + \cdots + x^{p^m-1})(1 + y + \cdots + y^{p^m-1}) \pmod{x^{p^m} - 1, y^{p^m} - 1}.
\]

**Proof:** We say \(B\) is equally-distributed in \(\{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\}\) if
\[
\#\{(g, h) \in B; g \equiv s, h \equiv t \pmod{p^n}\} = \#B/p^{2m}
\]
for any \((s, t) \in \{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\}\). Let \(c_i\) be the number of points \((g, h)\) in \(B\) with \(v_p(g, h) \geq i\); we shall call \(c_0 = c_0(B), c_1 = c_1(B), \ldots, c_m = c_m(B)\) the indices of the set \(B\). We claim that:

**Claim.** For a set \(B \subset \mathbb{Z}^2\) with \#\(B\) = \(p^n \geq p^{2m}\), there exists a translation \(B^* = B + (s^*, t^*)\) such that
\[
c_0(B^*) = p^n, c_1(B^*) \geq p^{n-2}, c_2(B^*) \geq p^{n-4}, \ldots, c_m(B^*) \geq p^{n-2m}.
\]

Moreover, if \(B\) is not equally-distributed in \(\{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\}\), then at least one of the inequalities is strict.
We prove this claim by induction on $m$. When $m = 1$, the claim is obviously true.
Clearly $c_0(B + (s, t)) = \#B = p^n$ for any $(s, t)$.
When $(s, t)$ runs over $\{0, 1, \ldots, p - 1\} \times \{0, 1, \ldots, p - 1\}$, we have
$$
\sum c_1(B + (s, t)) = \#B = p^n,
$$
hence there exists $(s_1, t_1) \in \{0, 1, \ldots, p - 1\} \times \{0, 1, \ldots, p - 1\}$ such that
$$
c_1(B + (s_1, t_1)) \geq p^{n-2}
$$
by the pigeon-hole principle. Let
$$
B_1 := \{(g, h) \in B + (s_1, t_1); v_p(g, h) \geq 1\},
$$
then $\#B_1 \geq p^{n-2}$. Let $\tilde{B}_1$ be any subset of $B_1$ with cardinality $\#B_1 = p^{n-2}$ and set $B_2 = \tilde{B}_1/p$.

By induction hypothesis, there exists $(s_2, t_2)$ such that for $B_2' = B_2 + (s_2, t_2)$, it holds that
$$
c_0(B_2') = p^{n-2}, c_1(B_2') \geq p^{n-4}, c_2(B_2') \geq p^{n-6}, \ldots, c_{m-1}(B_2') \geq p^{n-2m}.
$$
Set $(s^*, t^*) = (s_1, t_1) + p(s_2, t_2)$ and
$$
B^* = B + (s^*, t^*) = B + (s_1, t_1) + p(s_2, t_2).
$$
Then:

(i) $c_0(B^*) = p^n$ as we have mentioned before.

(ii) $c_1(B^*) = c_1(B + (s_1, t_1)) \geq p^{n-2}$ by the choice of $(s_1, t_1)$.

(iii) For $i \geq 2$, notice that
$$
B^* \supset B_1 + p(s_2, t_2) \supset \tilde{B}_1 + p(s_2, t_2) = pB_2 + p(s_2, t_2) = pB_2',
$$
therefore $c_i(B^*) \geq c_{i-1}(B_2') \geq p^{n-2i}$. The first assertion of the claim is proved.

Suppose $B$ is not equally-distributed in $\{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\}$. If $c_1(B + (s, t)) > p^{n-2}$, then the claim already holds. So we assume that $c_1(B + (s, t)) = p^{n-2}$ for all $(s, t) \in \{0, 1, \ldots, p - 1\} \times \{0, 1, \ldots, p - 1\}$. Choose $(s_1, t_1)$ such that $B_2 = (B + (s_1, t_1))/p$ is not equally-distributed in $\{0, 1, \ldots, p^{m-1} - 1\} \times \{0, 1, \ldots, p^{m-1} - 1\}$. Again we get the desired inequality by the induction hypothesis. Our claim is proved.

Now we return to the proof of the lemma. Let us first assume that $p^n \geq p^{2m}$.

If $B$ is equally-distributed in $\{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\}$, then obviously the lemma holds. Let us assume that $B$ is not equally-distributed in $\{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\}$.

Notice that if $B$ satisfies the conditions of the lemma, then any translation of $B$, particularly $B^*$ in the Claim, also satisfies the conditions of the lemma. Let us consider the polynomial
$$
\sum_{(a, b) \in D} B^*(z^a, z^b) = \sum_{(a, b) \in D} \sum_{(s, t) \in B^*} z^{as+bt} = \sum_{(s, t) \in B^*} \sum_{(a, b) \in D} z^{as+bt}.
$$
Under the assumptions of the lemma, an easy calculation shows that the constant term of
\[ \sum_{(a, b) \in D} B^*(z^a, z^b) \pmod{z^{p^m} - 1} \]  
(6)
is \( p^n(1 + \frac{1}{p}) \). On the other hand, by Lemma 3.2, the constant term of (6) is
\[ (c_0^* - c_1^*) + (c_1^* - c_2^*)p + \cdots + (c_{m-1}^* - c_m^*)p^{m-1} + c_m^*(p^m + p^{m-1}) \]
(\( c_i^* = c_i(B^*) \) are the indices of \( B^* \). This together with (5) implies that
\[ c_0^* = p^n, c_1^* = p^{n-2}, c_2^* = p^{n-4}, \ldots, c_m^* = p^{n-2m}. \]
This contradicts the second assertion of the Claim. The lemma is proved in the case \( n \geq 2m \).

Finally we show that \( n \geq 2m \) must hold. Otherwise, let \( B \) be a multi-set which has the same elements as \( B \), but the multiplicity of each element multiplied by a factor \( p^{n'} \) so that \( \#B = p^n + n' \geq p^{2m} \). It is seen that \( B \) also satisfies the conditions of the lemma. Therefore, \( B \) is equally-distributed in \( \{0, 1, \ldots, p^m - 1\} \times \{0, 1, \ldots, p^m - 1\} \) and so that \( B \) is also equally-distributed. It follows that \( n \geq 2m \). The lemma is proved. \( \square \)

**Proof of Theorem 1.3.** One direction is proved by Proposition 3.1 we prove the other direction in the following.

Let \( A = \{(s_0, t_0), (s_1, t_1), \ldots, (s_{p-1}, t_{p-1})\} \) be a normal tile of \( \mathbb{Z}^2 \) with \( \#A = p \). We may assume that \( (s_0, t_0) = (0, 0) \). Then according to Theorem 2.5, there exist a set \( B \) and \( M = p^m \), such that
\[ A(x, y)B(x, y) \equiv (1 + x + \cdots + x^{M-1})(1 + y + \cdots + y^{M-1}) \pmod{x^M - 1, y^M - 1}. \]  
(7)
(Note that \( \#B = p^{2m-1} \). Suppose for any non-negative coprime integers \( a, b \) the set
\[ \{as_0 + bt_0, as_1 + bt_1, \ldots, as_{p-1} + bt_{p-1}\} \]
is not a \( \mathbb{Z} \)-tile. Then by Remark 1.2 for any integer \( k \geq 1 \), \( \Phi_p(z) = 1 + z^p + \cdots + z^{(p-1)p^{k-1}} \) does not divide
\[ A(z^a, z^b) = z^{as_0 + bt_0} + z^{as_1 + bt_1} + \cdots + z^{as_{p-1} + bt_{p-1}}, \]
where \( \Phi_n(x) \) denotes the \( n \)-th cyclotomic polynomial. From (7), we have
\[ A(z^a, z^b)B(z^a, z^b) \equiv p^m(1 + z + \cdots + z^{p^m-1}) \pmod{z^{p^m} - 1}. \]
Hence \( \Phi_p(z) \) \((1 \leq k \leq m)\) must be factors of \( B(z^a, z^b) \), which implies that \( B(z^a, z^b) \) is a multiple of \( \Phi_p(z) \Phi_p(z) \cdots \Phi_p(z) \equiv 1 + z + \cdots + z^{p^m-1} \). Hence for any non-negative coprime integers \( a, b \),
\[ B(z^a, z^b) \equiv p^m(1 + z + \cdots + z^{p^m-1}) \pmod{z^{p^m} - 1}. \]
Now by Lemma 3.3, we have that \( \#B \geq p^{2m} \) which contradicts with \( \#B = p^{2m-1} \). This contradiction proves the theorem. \( \square \)
4 Algorithm

In this section, we give an algorithm to check the conditions of Theorem 1.3. We note that our algorithm is essentially identical with the algorithm given by Szegedy [Sze].

4.1 Span of $A$.

Let $L(A)$ be the set of $\mathbb{Z}$-linear combinations of vectors in $A$, which is a sublattice of $\mathbb{Z}^2$. If the rank of $L(A)$ is 1, then $A = \{a_0v, a_1v, \ldots, a_{p-1}v\}$ for some vector $v \in \mathbb{Z}^2$; in this case, $A$ is a tile if and only if $\{a_0, a_1, \ldots, a_{p-1}\}$ is a $\mathbb{Z}$-tile. So we assume that $L(A)$ is a full-rank lattice. Then there is an integral matrix $\phi$ such that $L(A) = \phi(\mathbb{Z}^2)$.

The matrix $\phi$ can be obtained in the following way. All the vectors in $A$ form a $p \times 2$ matrix

$$
\begin{pmatrix}
  s_0 & t_0 \\
  s_1 & t_1 \\
  \vdots & \vdots \\
  s_{p-1} & t_{p-1}
\end{pmatrix}
$$

By applying elementary row operators, the matrix can be reduced to the form

$$
\begin{pmatrix}
  s & t \\
  0 & t' \\
  0 & 0 \\
  \vdots & \vdots \\
  0 & 0
\end{pmatrix}
$$

Clearly $L(A) = L\{ (s, t), (0, t') \} = \phi(\mathbb{Z}^2)$ where

$$
\phi = \begin{pmatrix}
  s & 0 \\
  t & t'
\end{pmatrix}.
$$

The linear span $L(A) \neq \mathbb{Z}^2$ if and only if $|\det \phi| > 1$.

Let $A' = \phi^{-1}(A)$. We claim that $A$ tiles $\mathbb{Z}^2$ if and only if $A'$ tiles $\mathbb{Z}^2$. Suppose $A$ tiles $\mathbb{Z}^2$, i.e., $A \oplus C = \mathbb{Z}^2$. Let $C_0 = C \cap \phi(\mathbb{Z}^2)$. Then $A \oplus C_0 = \phi(\mathbb{Z}^2)$, $A' \oplus \phi^{-1}(C_0) = \mathbb{Z}^2$ and so that $A'$ tiles $\mathbb{Z}^2$. On the other hand, suppose $A'$ tiles $\mathbb{Z}^2$, i.e., $A' \oplus C' = \mathbb{Z}^2$. Let $R$ be a complete representative system of residues $\mathbb{Z}^2/\phi(\mathbb{Z}^2)$. Then $A \oplus \phi(C') = \phi(\mathbb{Z}^2)$, and so that $A \oplus C = \mathbb{Z}^2$ where $C = \phi(C') \oplus R$.

4.2 Algorithm

So, to check whether $A$ is a normal tile, it suffices to check whether $A'$ is a normal tile. Hence, from now on, we assume that $L(A) = \mathbb{Z}^2$.

Proposition 4.1 If $L(A) = \mathbb{Z}^2$, then there exist integers $a, b$ satisfying the conditions of Theorem 1.3 if and only if there exist integers $a, b \in \{0, 1, \ldots, p-1\}$ satisfying these conditions.
Proof: As we have pointed out, we may assume that $(a, b) = 1$. Hence at least one of $a, b$ is coprime to $p$, let us say, $(a, p) = 1$. The integers $a, b$ satisfying the conditions of Theorem 1.3 means that there exists an integer $m$ such that

$$as_i + bt_i \equiv l_i p^m \pmod{p^{m+1}}$$

and $\{l_0, \ldots, l_{p-1}\}$ is a complete representative system modulo $p$. If $m = 0$, we may choose $a \pmod{p}, b \pmod{p}$ instead of $a, b$, and the proposition is proved.

Suppose $m \geq 1$. Let $c$ be an integer such that $ac \equiv 1 \pmod{p^{m+1}}$. Then

$$acs_i + bct_i \equiv l_i c p^m \pmod{p^{m+1}},$$

$$s_i + bct_i \equiv l_i c p^m \pmod{p^{m+1}},$$

where $\{cl_0, \ldots, cl_{p-1}\}$ is still a complete representative system modulo $p$. Write $s_i + bct_i = L_i p^m$, let $\phi$ be the matrix

$$\phi = \begin{pmatrix} p & -bc \\ 0 & 1 \end{pmatrix}.$$ 

Then $(s_i, t_i) = \phi(L_i p^{m-1}, t_i)$ and $\det \phi = p$, which contradicts with $L(A) = \mathbb{Z}^2$. \hfill \Box

Algorithm:

Step 1. Find the matrix $\phi$ such that $L(A) = \phi(\mathbb{Z}^2)$. If $\det \phi = 0$, then the problem is reduced to a $\mathbb{Z}$-tiling problem; otherwise set

$$A' = \phi^{-1}(A) = \{(s'_0, t'_0), (s'_1, t'_1), \ldots, (s'_{p-1}, t'_{p-1})\}.$$ 

Step 2. Check whether there exist integers $a, b \in \{0, 1, \ldots, p - 1\}$ such that $as'_0 + bt'_0, as'_1 + bt'_1, \ldots, as'_{p-1} + bt'_{p-1}$ are distinct and form a $\mathbb{Z}$-tile.

4.3 \#A is a prime power.

We remark that the conclusion of Theorem 1.3 is false even for normal tiles $A$ with $\#A = p^2$. For example, let $p = 3$ and

$$A = \{(0, 0), (0, 2), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (3, 1)\}.$$ 

Clearly the only translation set is $C = 3\mathbb{Z} \times 3\mathbb{Z}$ and so that $A$ is a normal tile. The periods of $C$ are $(3x, 3y)$ where $x, y \in \mathbb{Z}$. See Figure 1.
We show that that \( A \) does not satisfy the condition of Theorem 1.3. Suppose not, then there are integers \( a, b \) such that \( as_0 + bt_0, as_1 + bt_1, \ldots, as_{n-1} + bt_{n-1} \) are distinct and \( \{as_0 + bt_0, as_1 + bt_1, \ldots, as_{n-1} + bt_{n-1}\} \) is a \( \mathbb{Z} \)-tile. We may assume that \( a \) and \( b \) are coprime. Let \( u, v \) be two integers such that \( au - bv = 1 \), and let

\[ \phi = \begin{pmatrix} a & b \\ v & u \end{pmatrix}. \]

Since

\[ \phi(A) = \{(as_0 + bt_0, vs_0 + ut_0), \ldots, (as_{n-1} + bt_{n-1}, vs_{n-1} + ut_{n-1})\} \]

and \( \{as_0 + bt_0, as_1 + bt_1, \ldots, as_{n-1} + bt_{n-1}\} \) is a \( \mathbb{Z} \)-tile, we infer that \( \phi(C) \), the unique translation set of \( \phi(A) \), has a period \((0, 1)\). So there exist \( x, y \in \mathbb{Z} \) such that \( \phi(3x, 3y) = (0, 1) \), which is impossible.

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References


