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Approximation Algorithms for Multicoloring Planar Graphs and Powers of Square and Triangular Meshes

Mustapha Kchikech[1] and Olivier Togni[2]

LE2I, UMR CNRS 5158, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France
[1] *kchikech@u-bourgogne.fr*
[2] *olivier.togni@u-bourgogne.fr*

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A multicoloring of a weighted graph G is an assignment of sets of colors to the vertices of G so that two adjacent vertices receive two disjoint sets of colors. A multicoloring problem on G is to find a multicoloring of G. In particular, we are interested in a minimum multicoloring that uses the least total number of colors.

The main focus of this work is to obtain upper bounds on the weighted chromatic number of some classes of graphs in terms of the weighted clique number. We first propose an $\frac{11}{6}$-approximation algorithm for multicoloring any weighted planar graph. We then study the multicoloring problem on powers of square and triangular meshes. Among other results, we show that the infinite triangular mesh is an induced subgraph of the fourth power of the infinite square mesh and we present 2-approximation algorithms for multicoloring a power square mesh and the second power of a triangular mesh, 3-approximation algorithms for multicoloring powers of semi-toroidal meshes and of triangular meshes and 4-approximation algorithm for multicoloring the power of a toroidal mesh. We also give similar algorithms for the Cartesian product of powers of paths and of cycles.

Keywords: Graph theory; Coloring; Multicoloring; Planar graph; Power graph; Product graph; Approximation algorithm; Greedy algorithm.

1 Introduction

In this paper by *graph* we mean a simple graph. We denote by $G = (V, E)$ a graph G with vertex set V and edge set E. The *length* of a path between two vertices is the number of edges on that path. The *distance* between two vertices is the length of a shortest path between them. We define the *Cartesian product* $G_1 \square G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be the graph $G = (V, E)$ with $V = V_1 \times V_2$ and $((x, y), (x', y')) \in E$ whenever $x = x'$ and $(y, y') \in E_2$ or $(x, x') \in E_1$ and $y = y'$. Given a positive integer p, the p^{th} *power* G^p of a graph G is a graph with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most p between them in G. A *vertex coloring* of a graph G is an assignment of colors to the vertices of the graph such that no two adjacent vertices receive the same colors. The minimum number of colors for which a coloring of G exists

is called the *chromatic number* and is denoted by $\chi(G)$.
Let $G = (V, E)$ be a graph. A *weighted graph* of G is a pair $G_\omega = (G, \omega)$ where ω is a weight function that assigns non-negative integer $\omega_G(v)$ (or simply $\omega(v)$) to each vertex v of G, $\omega_G(v)$ is called the *weight* of v in G. A *vertex multicoloring* of the weighted graph G_ω consists of a set of colors $\mathcal{C}$ and a function Ψ that assigns to each $v \in V$ a subset of colors $\Psi(v) \subset \mathcal{C}$ such that:

i) $\forall\, v \in V$, $|\Psi(v)| = \omega_G(v)$, i-e. the vertex v gets $\omega_G(v)$ distinct colors.

ii) If $(u, v) \in E$ then $\Psi(u) \cap \Psi(v) = \emptyset$, i-e. two adjacent vertices get disjoint sets of colors.

The *weighted chromatic number* (or *multichromatic number*), denoted $\chi_\omega(G)$, of G is the minimum number of colors needed to multicolor all vertices of G_ω so that conditions i and ii above are satisfied.

A subgraph K of G is called a *clique* if every pair of vertices in K is connected by an edge. The weight of any clique in G_ω is defined as the sum of the weights of the vertices forming that clique. The *weighted clique number* of G, denoted $W(G)$ (for short, we will use W), is defined to be the maximum over the weights of all cliques in G_ω. Clearly, $\chi_\omega(G) \geq W$.

Note that a multicoloring of a weighted graph $G = (V, E)$ is the same as the usual vertex coloring of a graph G' that is obtained from G by replacing each vertex $u \in V$ by a clique $K(u)$ of size $\omega(u)$, and connect two vertices in distinct cliques $K(u)$ and $K(v)$ if and only if u and v are adjacent in G. Nevertheless, depending on the practical problem to be modeled, it is often more convenient to consider it as a multicoloring of a weighted graph since the structure of the graph remains the same even if the weights change. For instance, in the literature, almost all studies about frequency assignment in radio networks adopt this approach.

The multicoloring problem (also known as weighted coloring [10] or ω-coloring [12]) has been studied in several contexts. Halldórsson and Kortsarz in [4] present this problem as a model for scheduling of dependent jobs on multiple machines. In [8, 14], a technic based on graph multicoloring has been used to study the routing and wavelength assignment problem on WDM all-optical networks. Also the frequency (channel) assignment problem in a cellular network can be modeled as a multicoloring problem of graph. In the last two decades, this problem was largely studied. Thus, there is a vast literature on algorithms for the multicoloring problem on graphs (especially hexagon graphs), but generally there are no proven bounds on the approximation ratio of the proposed algorithms in terms of the number of colors used in relation to the weighted chromatic number.

The multicoloring problem is NP-hard in general. Hence, it would be interesting to find algorithms that approximate the weighted chromatic number. Note that an algorithm is an f-approximation algorithm for the vertex multicoloring problem if the algorithm runs in polynomial time and it always produces a solution that is within a factor of f of the optimal solution.

Previous results: Exact and approximate algorithms for the multicoloring problem have been given for various classes of graphs, as indicated in Table 1. Note that the multicoloring problem on a triangular mesh has been extensively studied and proved to be NP-hard by McDiarmid and Reed [10]. This problem corresponds to the frequency (radio channel) assignment in cellular networks. Moreover, if the triangular mesh is of power $p \geq 2$ then the problem models channel allocation with interference constraints [2] also called reuse distance r with $r = p - 1$. Some authors independently gave approximation algorithms for this problem. In case where $r = 2$, a $\frac{4}{3}$-approximation algorithm has been described both in [10, 12]. For $r = 3$, [2] gives a simple algorithm that has a guaranteed approximation ratio of $\frac{7}{3}$. For $r \geq 4$, the best known upper bound on the number of colors needed is $4W$ [6], where W is the weighted clique number.

In contrast, the best known lower bound on the number of colors needed is $\frac{9}{8}W$ if $r = 2$ [12] and is $\frac{5}{4}W$ if $r \geq 3$ [13].

Our results: In Section 2, we give an algorithm for multicoloring any planar graph with at most $\frac{11}{6}W$ colors. In Section 3, we present two upper bounds on the weighted chromatic number of a graph, one in relation with the chromatic number and the other in relation with the minimum number of cliques necessary to cover the neighborhood of any vertex in the graph. Section 4 is devoted to the study of the multicoloring problem on powers of square and triangular meshes. In particular, we show that the infinite triangular mesh is an induced subgraph of the fourth power of the infinite square mesh. We present 2-approximation algorithms for multicoloring a power square mesh and the second power of a triangular mesh, 3-approximation algorithms for multicoloring powers of semi-toroidal meshes and of triangular meshes and a 4-approximation algorithm for multicoloring powers of toroidal meshes. In Section 5, we present similar results for graphs of the form $G^p \Box H^q$, where G and H are either a path or a cycle.

The best known upper bounds and our results for various classes of graphs are summarized in Table 1.

Graphs	**Approximation ratio**
k-colorable graph	$\frac{k}{2}$ [5];$(*)$
Bipartite graph	1 [10]
Outerplanar graph	1 [12]
Planar graph	$\frac{11}{6}$ $(*)$
Triangular mesh	$\frac{4}{3}$ [12, 10]
2^{th} power triangular mesh H^2	$\frac{7}{3}$ [2], 2 $(*)$
p^{th} power triangular mesh H^p $(p \geq 3)$	4 [6], 3 $(*)$
p^{th} power square mesh $(p \geq 2)$	2 $(*)$
$(G \Box H)^p$; $G^p \Box H^q$, G is a path; H is a cycle	3 $(*)$
p^{th} power toroidal mesh	4 $(*)$

Tab. 1: Best approximation ratios for multicoloring algorithms of some classes of graphs. $(*)$ our results in this paper.

2 Multicoloring planar graphs

In this section we consider the multicoloring problem on planar graphs. Before presenting our result, we remind a result of Narayanan and Shende [12] showing that there exists an efficient algorithm to optimally multicolor any outerplanar graph. A graph is *planar* if it can be drawn in a plane without edge crossings. A graph is said to be *outerplanar* if it is a planar graph so that all vertices may lie on the outer face.

Theorem 1 ([12]) *Let G be an arbitrary outerplanar graph, then its associated weighted graph G_ω can be multicolored optimally using $\chi_\omega(G)$ colors in linear time.*

Theorem 2 *Let G be a planar graph, then*

$$\chi_\omega(G) \leq \left\lceil \frac{11}{6} W(G) \right\rceil .$$

Proof: We may assume $G = (V_G, E_G)$ to be connected, since disconnected components of G can be multicolored independently. As G is a planar graph of order n, then using the $\mathcal{O}(n^2)$ algorithm described by Robertson et al. in [15], we color G with 4 colors from $\{1, 2, 3, 4\}$. We call these colors *base colors*. We denote by s_i any vertex $s \in V_G$ which has color $i \in \{1, 2, 3, 4\}$, and by $\{c_i^1, c_i^2, \cdots, c_i^\alpha\}$ a set of α (nonnegative integer) distinct *hues* associated with base color i so that if $i \neq j$ then for every integers $p, q \geq 1$, we have $c_i^p \neq c_j^q$.

Case 1: $W(G) \equiv 0 \pmod 3$.

We fix $\ell = \frac{1}{3}W(G)$, and we let $\mathcal{C} = \bigcup_{i=1}^{4} \{c_i^1, c_i^2, \cdots, c_i^\ell\}$ denote a set of available colors. Consider the multicoloring function f of G_ω defined as follows:

$$\begin{array}{llll} f: & V & \longrightarrow & \mathcal{P}(\mathcal{C}) \\ & s_i & \longmapsto & f(s_i) = \{c_i^1, c_i^2, \cdots, c_i^\alpha\}, \text{ with } \alpha = min(\omega_G(s_i), \ell) \end{array}$$

where each vertex s_i with weight $\omega_G(s_i)$ is assigned the hues of $\{c_i^1, c_i^2, \cdots, c_i^\alpha\}$. We call a vertex v *heavy* if $\omega(v) > \ell$ or *light* if $\omega(v) \leq \ell$. Hence, only the heavy vertices remain to be completely colored and their weights may be decreased by ℓ. All light vertices are colored completely and are deleted from G.

Let $H = (V_H, E_H)$ denote the remaining graph obtained after this process. Thus H is such that

- $u \in V_H \Leftrightarrow \omega_G(u) > \ell$,
- $\omega_H(u) = \omega_G(u) - \ell$,
- $(u, v) \in E_H \Rightarrow (u, v) \in E_G$.

It is easily seen that H has no clique of size 3. In fact, if (x, y, z) is a triangle in H, then these vertices must have been heavy in G. Hence, there exist positive integers ε_1, ε_2 and ε_3 such that $\omega_G(x) = \ell + \varepsilon_1$, $\omega_G(y) = \ell + \varepsilon_2$ and $\omega_G(z) = \ell + \varepsilon_3$. As $\omega_G(x) + \omega_G(y) + \omega_G(z) \leq W_G = 3\ell$, we obtain $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq 0$ a contradiction. Consequently, every clique K in H has size at most 2. Furthermore, if $(u, v) \in K$ then $\omega_H(u) + \omega_H(v) = \omega_G(u) - \ell + \omega_G(v) - \ell \leq W_G - 2\ell = \ell$. In addition, if u is an isolated vertex in H, then there exists a positive integer ε such that $\omega_G(u) = \ell + \varepsilon$, and all neighbors of u must be light vertices in G. Suppose that u has i for its base color and that it has a neighbor in G of base color $j \neq i$. Let N_j be the set of all its neighbors in G of base color j. Let $v_j \in N_j$ such that $\omega_G(v_j) = \max_{s \in N_j} \omega_G(s)$. Further, we have $\omega_G(v_j) = \ell - \varepsilon_{v_j}$ with $0 \leq \varepsilon_{v_j} \leq \ell$ since u is isolated in H. Then u can borrow ε_{v_j} available colors from $\{c_j^1, c_j^2, \cdots, c_j^\ell\}$, which are unused by all vertices of N_j. As $\omega_G(u) + \omega_G(v_j) \leq W_G = 3\ell$, we get $\varepsilon - \varepsilon_{v_j} \leq \ell$. For this reason, we consider that each isolated vertex $u \in H$ has $\omega_H(u) = \omega_G(u) - (\ell + \varepsilon_{v_j}) = \varepsilon - \varepsilon_{v_j}$. Thus, $W_H \leq \ell$ and we can therefore distinguish two cases:

(a) If H is a bipartite graph, then H can be multicolored optimally with exactly W_H colors (see [5, 10]). In this case, to avoid color conflicts, we use a new set $\mathcal{C}'$ of W_H distinct colors. Thus, multicoloring all vertices of G_ω requires at most $|\mathcal{C}| + |\mathcal{C}'| \leq 4\ell + \ell = \frac{5}{3}W_G$ colors.

(b) If H is not a bipartite graph, then as H is a planar graph without 3-cycles (triangle-free), it is well known that H is 3-colorable [3]. Using the $\mathcal{O}(n \log n)$ algorithm of Kowalik [9], we color H with 3 colors. Thus, there is an algorithm for multicoloring H that requires at most $\frac{3}{2}W_H$ colors (See Theorem 4 or [5]). Similarly, we use a new set $\mathcal{C}'$ of $\frac{3}{2}W_H$ distinct colors. Thus, multicoloring all vertices of G_ω requires at most $|\mathcal{C}|+|\mathcal{C}'| \leq 4\ell+\frac{3}{2}\ell = \frac{11}{6}W_G$ colors.

Case 2: $W(G) \equiv 1 \pmod{3}$.

In this case, the main idea remains the same as that of Case 1, except that we fix four integers $\ell_1, \ell_2, \ell_3, \ell_4$ such that $\ell_1 = \frac{W+2}{3}$ and $\ell_2 = \ell_3 = \ell_4 = \frac{W-1}{3}$. Note that ℓ_i represents the number of distinct hues associated with base color i. Then, each vertex s_i of base color i is assigned the hues of $\{c_i^1, c_i^2, \cdots, c_i^\alpha\}$, with $\alpha = min(\omega(s_i), \ell_i)$. A vertex v_i is called heavy if $\omega(v_i) > \ell_i$ and is called light if $\omega(v_i) \leq \ell_i$. All the light vertices thus get completely colored and are deleted from G. The weight of every remaining heavy vertex u_i is decreased by ℓ_i, resulting in the graph H.

Observe that every triangle in G must contain at least one light vertex. Otherwise, there exist three heavy vertices x, y, z of G such that (x, y, z) is a triangle in H. Hence, there exist positive integers $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 1$ such that $\omega_G(x_i) = \ell_i + \varepsilon_1$, $\omega_G(y_j) = \ell_j + \varepsilon_2$ and $\omega_G(z_k) = \ell_k + \varepsilon_3$. As $\omega_G(x_i) + \omega_G(y_j) + \omega_G(z_k) \leq W_G$, we obtain $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq 0$ or $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq 1$ (because $\ell_i + \ell_j + \ell_k = W_G - 1$ if $i, j, k \in \{2, 3, 4\}$ or $\ell_i + \ell_j + \ell_k = W_G$ otherwise) a contradiction. Consequently, H has no cliques of size greater than 2.

Let K be a clique of H. If $|K| = 2$ then K contains two vertices u_i and v_j such that $\omega_H(u_i) + \omega_H(v_j) = \omega_G(u_i) - \ell_i + \omega_G(v_j) - \ell_j \leq W_G - (\ell_i + \ell_j) \leq \ell_1$. If $|K| = 1$ then K contains an isolated vertex u_i such that $\omega_G(u_i) = \ell_i + \varepsilon$, with $\varepsilon \geq 1$, and all neighbors of u_i must be light vertices in G. Assume u_i has a neighbor in G of base color $j \neq i$ and let $N_j(u_i)$ be the set of all neighbors of u_i in G of base color j. Let $v_j \in N_j(u_i)$ such that $\omega_G(v_j) = \max_{s \in N_j(u_i)} \omega_G(s)$. Further, we have $\omega_G(v_j) = \ell_j - \varepsilon_{v_j}$ with $0 \leq \varepsilon_{v_j} \leq \ell_j$ since u_i is isolated in H. Then u_i can borrow ε_{v_j} available colors from $\{c_j^1, c_j^2, \cdots, c_j^{\ell_j}\}$, which are unused by all vertices of $N_j(u_i)$. As $\omega_G(u_i) + \omega_G(v_j) \leq W_G$, we get $\varepsilon - \varepsilon_{v_j} \leq W_G - (\ell_i + \ell_j) \leq \ell_1$ (because $\ell_i + \ell_j = \frac{2W_G-2}{3}$ if $i, j \in \{2, 3, 4\}$ or $\ell_i + \ell_j = \frac{2W_G+1}{3}$ otherwise). For this reason, we consider that each isolated vertex $u_i \in H$ has $\omega_H(u_i) = \omega_G(u_i) - (\ell_i + \varepsilon_{v_j}) = \varepsilon - \varepsilon_{v_j} \leq \ell_1$.

Consequently, for any clique K of H we have $\omega_H(K) \leq \ell_1$. Thus, $W_H \leq \ell_1$ and so H can be multicolored using a new set $\mathcal{C}'$ of colors, with $|\mathcal{C}'| = \ell_1$ if H is bipartite or $|\mathcal{C}'| = \frac{3}{2}\ell_1$ otherwise. Then the total number of colors used in this case is $|\mathcal{C}|+|\mathcal{C}'| \leq \ell_1+\ell_2+\ell_3+\ell_4+\frac{3}{2}\ell_1 = \frac{11}{6}W_G+\frac{2}{3}$.

Case 3: $W(G) \equiv 2 \pmod{3}$.

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four non-negative integers such that $\ell_1 = \frac{W-2}{3}$ and $\ell_2 = \ell_3 = \ell_4 = \frac{W+1}{3}$, and we can use the same method we used in Case 2.

□

3 Relationships between $W(G)$ and $\chi_\omega(G)$

The problem of precisely determining the weighted chromatic number is an intractable one in general. Nonetheless, it is possible to determine the exact value of the weighted chromatic number on specific families of graphs, such as bipartite graphs [10] and outerplanar graphs [12]. In this section, we provide two upper bounds for $\chi_\omega(G)$, one in relation with the chromatic number and the other in relation with the minimum number of cliques necessary to cover the neighborhood of any vertex in the graph. The second result plays a role in the rest of our work where it helps to determine upper bounds for weighted chromatic numbers of various classes of graphs of the form $(G\Box H)^p$ or $G^p\Box H^q$.

Given a k-colorable graph G (then G have chromatic number $\chi(G) \leq k$), Janssen, Kilakos and Marcotte [5] showed that there exists a $\frac{k}{2}$-approximation algorithm for multicoloring all vertices of G.

Theorem 3 ([5]) *If G is a k-colorable graph, then*

$$\chi_\omega(G) \leq \frac{k}{2} W(G).$$

However, finding a maximal clique has been proven to be an NP-complete problem. Thus, determining $W(G)$ becomes also a difficult problem. So, to calculate the upper bound on $\chi_\omega(G)$ by using Theorem 3 is complicated. In this section, we present an improvement of the above result using an approximation algorithm that can be implemented in a distributed manner.

In the following, we denote by $W_t(G) = \max\{W(K) \,|\, K \textit{ clique of } G \textit{ s.t. } |K| \leq t\}$ the maximum over the weights of all cliques of size at most t in G_ω.

Theorem 4 *If G is a k-colorable graph without isolated vertices, then*

$$\chi_\omega(G) \leq \frac{k}{2} W_2(G).$$

Proof: The proof is based on the same general principle as that of Theorem 2. We may assume G to be connected, since disconnected components of G can be multicolored independently. As G is a k-colorable graph, then we color G with at most k colors from $\{1, 2, ..., k\}$. We call these colors base colors. We denote by v_i any vertex v of G which has base color i, and by $\{c_i^1, c_i^2, \cdots, c_i^\alpha\}$ a set of α (nonnegative integer) distinct hues associated with base color i so that if $i \neq j$ then for every integers $p, q \geq 1$, we have $c_i^p \neq c_i^q$.

First, we assume $W_2(G) \equiv 0 \pmod{2}$ and we fix $\ell = \frac{1}{2} W_2(G)$ (if $W_2(G) \equiv 1 \pmod{2}$, we can use similar arguments as those presented in the proof of Theorem 2). Every vertex v_i is assigned the first $\omega(v_i)$ hues of the set $\{c_i^1, c_i^2, \cdots, c_i^\ell\}$. Hence, only the heavy vertices (of weight at least $\ell + 1$) remain to be completely colored after this step and the other vertices are completely colored and are deleted from G. Let H denote the remaining graph obtained after this process where the new weight of each vertex $v \in H$ is $\omega_H(v) = \omega(v) - \ell$. We observe that H contains only isolated vertices, because if we suppose that there exist two heavy adjacent vertices v and u in H then, we get $2\ell < \omega(v) + \omega(u) \leq W_2(G) = 2\ell$, a contradiction.

Assume v_j has a neighbor in G of base color $i \neq j$ and let $N_i(v_j)$ be the set of neighbors vertices of v_j having i as base color in G. Let $v_i \in N_i(v_j)$ such that $\omega(v_i) = \max_{v \in N_i(v_j)} \omega(v)$. Further, we have $\omega(v_j) = \ell + \epsilon_j$ and $\omega(v_i) = \ell - \epsilon_i$ where $\epsilon_j > 0$ and $0 \leq \epsilon_i \leq \ell$. As $\omega(v_j) + \omega(v_i) \leq W_2(G) = 2\ell$,

we get $\epsilon_j \leq \epsilon_i$. Then v_j can borrow from ϵ_i colors available in the set $\{c_i^1, c_i^2, \cdots, c_i^\ell\}$ for coloring the remaining weight on v_j.
Consequently, for multicoloring all vertices of G, we use at most $k\ell = \frac{k}{2}W_2(G)$ colors.
Thus, $\chi_\omega(G) \leq \frac{k}{2}W_2(G)$. □

Remark that, for a graph G on m edges, $W_2(G)$ can be computed in time $\mathcal{O}(m)$ and $W_2(G) \leq W(G)$, thus our result is an improvement of Theorem 3.

Now, we give a upper bound on the ratio $\frac{\chi_\omega(G)}{W(G)}$ in relation with the minimum number of cliques necessary to cover the neighborhood of any vertex.

For a vertex u of a graph $G = (V, E)$, let $N_G[u]$ be the closed neighborhood of u: $N_G[u] = \{v \in V | u = v \text{ or } uv \in E\}$.

Definition 1 *A connected graph G has Property P_q if and only if for any vertex u, $N_G[u]$ can be covered with q distinct cliques each containing u.*

Lemma 1 *Let $G = (V, E)$ be a simple graph and ω be a weighting.*

1. *If $q \in \mathbb{N}$ is such that G has Property P_q, then the vertices of G can be multicolored in a greedy fashion using at most $q(W-1)+1$ colors in any given order. In particular $\chi_\omega(G) \leq q(W-1)+1$.*
2. *If further there is a $q' \leq q$ such that for a specific ordering $V = \{u_1, u_2, \cdots,\}$ (not necessarily finite) of the vertices of G, we have that $N_G[u_i] \cap \{u_1, \cdots, u_{i-1}\}$ can be covered by at most q' cliques, then the vertices can be multicolored in a greedy fashion using at most $q'W$ colors in this order. In particular $\chi_\omega(G) \leq q'(W-1)+1$.*

Proof:

1. Assume G is k-colorable and we associate with each base color ℓ distinct hues, where ℓ is some fraction of W to be determined later. Thus, we obtain $k\ell$ available colors.

   Now, we assign to each vertex u_i of G of color i $\alpha = min(\omega_G(u_i), \ell)$ colors in the set $\{c_i^1, c_i^2, \cdots, c_i^\ell\}$, associated to the base color i of u. Then, we note that the heavy vertices (of weight at least $\ell + 1$) are not completely colored. In order to complete the coloring of the heavy vertices, we proceed in the following manner.

   Consider a heavy vertex u not yet completely colored. As G satisfies Property P_q, $N_G[u]$ can be covered with q cliques, each of them containing u. Note that each clique has weight at most W, since W is the weighted clique number. Thus, the total weight of $N_G[u]$ is at most $\omega(u) + q(W - \omega(u)) = qW - (q-1)\omega(u) \leq qW - (q-1)(\ell+1)$ because $\omega(u) \geq \ell + 1$.

   Consequently, $k\ell$ colors will be sufficient to color all vertices of $N_G[u]$ if $qW-(q-1)(\ell+1)\ell \leq k\ell$. This gives $q(W-1) \leq (q-1+k)\ell - 1$. So, we can take $\ell = \frac{q(W-1)+1}{q-1+k}$.

   Thus, we are able to color all vertices of G using at most $k\ell = \frac{k(q(W-1)+1)}{q-1+k} \leq q(W-1)+1$ colors.

2. We consider the vertices of G in a specific order $V(G) = \{u_1, u_2, \cdots\}$ such that $N_G[u_i] \cap \{u_1, \cdots, u_{i-1}\}$ can be covered by at most q' cliques. The idea of the proof is to multicolor in a greedy fashion each vertex u_i of G using the same method we used in the first assertion.

□

4 Multicoloring powers of square and triangular meshes

In this section we present greedy algorithms for multicoloring square and triangular meshes and we compute their approximation ratios.

4.1 Powers of square meshes

We begin with the following easy to observe lemma.

Lemma 2 *Consider $(G\Box H)^p$, where both G and H are either a simple path (finite or infinite) or a simple cycle.*

1. *Any closed neighborhood of a vertex can be covered by four cliques of $(G\Box H)^p$.*

2. *If exactly one of the graphs G or H is a cycle, then there is an ordering of the vertices, such that the previously listed neighbors of each vertex can be covered by three cliques of $(G\Box H)^p$.*

3. *If neither of the graphs G nor H are cycles, then there is an ordering of the vertices, such that the previously listed neighbors of each vertex can be covered by two cliques of $(G\Box H)^p$.*

Proof: Without loss of generality, we denote by (i, j), with $(i, j) \in \mathbb{N}^2$, the vertices of $(G\Box H)^p$.

1. Let $s = (i, j)$ be a vertex of $(G\Box H)^p$, and let $N_{(G\Box H)^p}[s]$ be the closed neighborhood of s. Consider the following four subsets of $V((G\Box H)^p)$:

   - $K^1_{ij} = \{(i \pm r, j - l) \in V((G\Box H)^p) \,|\, 0 \leq r + l \leq p;\ 0 \leq r \leq l\}$,
   - $K^2_{ij} = \{(i + r, j \pm l) \in V((G\Box H)^p) \,|\, 0 \leq r + l \leq p;\ 0 \leq l \leq r\}$,
   - $K^3_{ij} = \{(i \pm r, j + l) \in V((G\Box H)^p) \,|\, 0 \leq r + l \leq p;\ 0 \leq r \leq l\}$,
   - $K^4_{ij} = \{(i - r, j \pm l) \in V((G\Box H)^p) \,|\, 0 \leq r + l \leq p;\ 0 \leq l \leq r\}$.

   It is easy to see that each subset K^l_{ij} with $l \in \{1, 2, 3, 4\}$ is a clique in $V((G\Box H)^p)$ and that $N_{(G\Box H)^p}[s]$ is covered by these four cliques. An illustration is given in Fig. 1 for the case $p = 3$ (only the edges of $(G\Box H)^1$ are drawn for clarity).

2. If exactly one of the graphs G or H is a cycle, then the vertices of $(G\Box H)^p$ are ordered using the *degree lexicographical order (deglex for short)* [1] of the points of $\mathbb{Z}^2$: $V((G\Box H)^p) = \{u_1, u_2, \cdots\}$, with
$$u_k = (i_k, j_k) < u_\ell = (i_\ell, j_\ell) \Leftrightarrow \begin{cases} i_k + j_k < i_\ell + j_\ell \text{ or} \\ i_k + j_k = i_\ell + j_\ell \text{ and } i_k < i_\ell. \end{cases}$$

   It is then easy to observe that for each vertex u_k, $N_{(G\Box H)^p}[u_k] \cap \{u_1, u_2, \cdots u_{k-1}\}$ can be covered by three cliques. As above, an illustration is given in Fig. 2 for the case $p = 3$ (only the edges of $(G\Box H)^1$) are drawn for clarity).

3. If neither of the graphs G nor H are cycles, then, as previous, listing the vertices of $(G\Box H)^p$ in deglex order yelds the desired condition that for each vertex u_k, $N_{(G\Box H)^p}[u_k] \cap \{u_1, u_2, \cdots, u_{k-1}\}$ can be covered by two cliques (See Fig. 3).

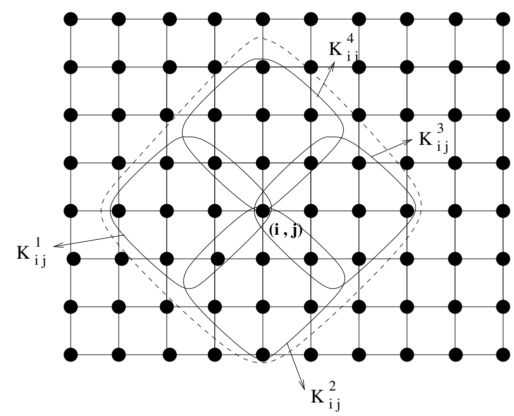

Fig. 1: The graph $(P_8 \square P_{10})^3$ with the neighbors of (i, j) being covered by 4 cliques K^1_{ij}, K^2_{ij}, K^3_{ij}, K^4_{ij} (only edges of $(P_8 \square P_{10})^1$ are represented).

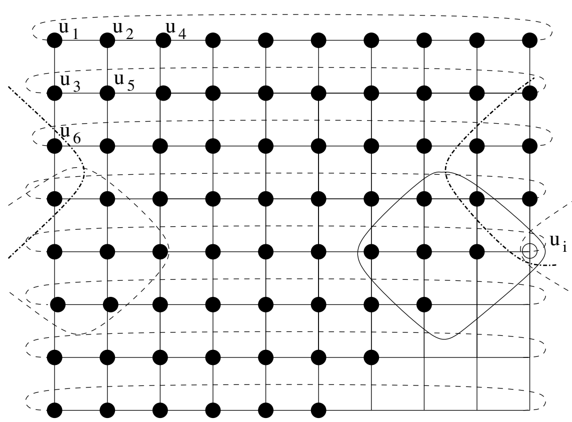

Fig. 2: The graph $(P_8 \square C_{10})^3$ with the previously listed neighbors of u_i in deglex order being covered by 3 cliques (only edges of $(P_8 \square C_{10})^1$ are represented).

□

Let M be the infinite square mesh with vertex set $\mathbb{Z} \square \mathbb{Z}$, where two vertices (x_1, y_1) and (x_2, y_2) of M are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$.

Consider the infinite weighted p^{th} power mesh M^p with weighted clique number W. If $p = 1$ then M^1 is a bipartite graph. McDiarmid and Reed in [10] proposed an algorithm to optimally multicolor this family of graphs. In addition, when $p = 2$, the multicoloring of a subgraph of M^2 called *lattice graph with diagonals*(i) has been studied by Miyamato and Matsui in [11]. They proposed an $\mathcal{O}(|M|)$ time

(i) Graph obtained from the standard square mesh by adding the two diagonals edges on each face.

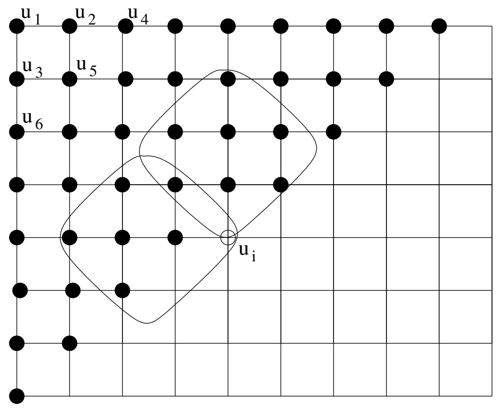

Fig. 3: The graph $(P_8\Box P_{10})^3$ with the previously listed neighbors of u_i in deglex order being covered by 2 cliques (only edges of $(P_8\Box P_{10})^1$ are represented).

approximation algorithm for multicoloring a finite square mesh with $\frac{4}{3}W + 4$ colors.

In the following three theorems, we present algorithms for multicoloring powers of square meshes, semi-toroidal meshes and toroidal meshes, with approximation ratios of 2, 3 and 4 respectively. Their proofs are direct consequences of Lemma 1 and Lemma 2.

Theorem 5 *For any $p \geq 2$, there exists a polynomial time greedy algorithm which multicolors all vertices every finite induced portion of a weighted p^{th} power of the infinite square mesh M^p using at most $(2W - 1)$ colors.*

Theorem 6 *Consider $(G\Box H)^p$, where one of the graphs G or H is a cycle. There exists a polynomial time greedy algorithm which multicolor all vertices of every finite induced portion of $(G\Box H)^p$ using at most $(3W - 2)$ colors.*

Theorem 7 *Consider the p^{th} power toroidal mesh $TM^p = (G\Box H)^p$ where both G and H are cycles. There exists a polynomial time algorithm which multicolors all vertices of TM^p with using at most $(4W - 3)$ colors.*

4.2 Powers of triangular meshes

Now, we consider the multicoloring problem on the p^{th} power of a triangular mesh. We define the infinite triangular mesh H as a mesh formed by tiling the plane regularly with equilateral triangles.

In the following, we present an approximation algorithm that multicolor all vertices of any finite induced portion of H^p using at most $(2W - 1)$ colors if $p = 2$ and at most $(3W - 2)$ colors if $p \geq 3$. The method used is based on the multicoloring of the p^{th} power of a square mesh.

Theorem 8 *The infinite triangular mesh H is an induced subgraph of $(\mathbb{Z}\Box\mathbb{Z})^4$, the fourth power of the infinite square mesh.*

Proof: The infinite triangular mesh can be presented by $H = \{x(1,3) + y(3,1) : x, y \in \mathbb{Z}\} = \{(x + 3y, 3x + y) : x, y \in \mathbb{Z}\} \subseteq \mathbb{Z}^2$, where the edges are between all pairs of points of distance four apart (See Fig. 4). $\Box$

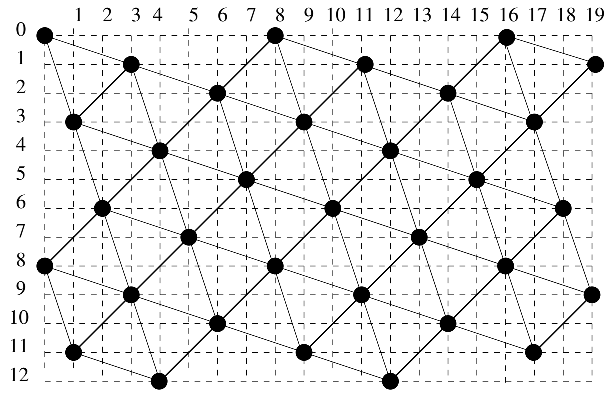

Fig. 4: The infinite triangular mesh induced in the 4^{th} power of the infinite square mesh.

As a consequence of Theorem 8, for every finite induced portion Γ of H, there are m and n such that Γ embeds into $(P_m \Box P_n)^4$.

Theorem 9 *There exists a polynomial time greedy algorithm which multicolor all vertices of every finite induced portion of the 2^{th} power of a triangular mesh H^2 using at most $(2W-1)$ colors.*

Proof: If we let H^2 be the subgraph of M^8 defined previously (see proof of Theorem 8), then we can use the same multicoloring process used in the proof of Theorem 5. In fact, keeping the same order on the vertices, when we multicolor vertex u_k of H^2, the previously listed neighbors of u_k can be covered by two cliques. Thus the same algorithm used in proof of Theorem 5 gives the result. $\Box$

Theorem 10 *For any $p \geq 3$, there exists a polynomial time greedy algorithm which multicolor all vertices of every finite induced portion of the p^{th} power of a triangular mesh H^p using at most $(3W-2)$ colors.*

Proof: The proof is similar to that of Theorem 5. However, when we multicolor the vertex u_k of H^p, one can see that the previously listed neighbors of u_k can be covered by three cliques. $\Box$

5 Multicoloring the product of powers of paths and of cycles

Consider $G^p \Box H^q$ where both G and H are either a simple path (finite or infinite) or a simple cycle. The multicoloring problem on this type of graphs is of interest since it can model the problem of coloring a set of line-column paths in a mesh [7, 8].

Note that $G^p \Box H^q$ is a subgraph of $(G \Box H)^r$, with $r = \max(p,q)$. But, as it is not an induced subgraph, the results for powers of square meshes cannot be extended directly to graphs of the form $G^p \Box H^q$. Nevertheless, the analogous of Lemma 2 for graphs $G^p \Box H^q$ is easy to state.

Lemma 3 *Consider $G^p \Box H^q$ where both G and H are either a simple path (finite or infinite) or a simple cycle.*

1. *Any closed neighborhood of a vertex can be covered by four cliques of $G^p \Box H^q$.*

2. *If exactly one of the graphs G or H is a cycle, then there is an ordering of the vertices, such that the previously listed neighbors of each vertex can be covered by three cliques of $G^p \square H^q$.*

3. *If neither of the graphs G nor H are cycles, then there is an ordering of the vertices, such that the previously listed neighbors of each vertex can be covered by two cliques of $G^p \square H^q$.*

Notice that, in the case where both G and H are a simple (finite or infinite) path, it is known that $\chi(G^p \square H^q) = max(\chi(G^p), \chi(H^p)) = max(p+1, q+1)$. Thus, for the vertex multicoloring problem on $G^p \square H^q$, if we apply Theorem 3, we obtain $\chi_\omega(G^p \square H^q) \leq \frac{max(p+1,q+1)}{2} W$. Consequently, we have $\chi_\omega(G^p \square H^q) \leq \frac{3}{2} W$ for the case where $max(p,q) \leq 2$. General upper bounds on $\chi_\omega(G^p \square H^q)$ are given in the following theorem whose proof is a direct consequence of Lemma 1 and Lemma 3.

Theorem 11 *Let Γ be any finite induced portion of $G^p \square H^q$, where both G and H are either a simple path (finite or infinite) or a simple cycle. There exists a polynomial time greedy algorithm which multicolor all vertices of Γ, using a total of at most*

1. $(2W-1)$ *colors if neither of the graphs G nor H are cycles,*

2. $(3W-2)$ *colors if exactly one of the graphs G or H is a cycle,*

3. $(4W-3)$ *colors if both G and H are cycles.*

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