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Three matching intersection property for matching covered graphs∗

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In connection with Fulkerson's conjecture on cycle covers, Fan and Raspaud proposed a weaker conjecture: For every bridgeless cubic graph G, there are three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$. We call the property specified in this conjecture *the three matching intersection property* (and *3PM property* for short). We study this property on matching covered graphs. The main results are a necessary and sufficient condition and its applications to characterization of special graphs, such as the Halin graphs and 4-regular graphs.

Keywords: matching-covered graph, Fan-Raspaud's conjecture, 3PM-admissible graph

1 Introduction

Fulkerson's conjecture asserts that every bridgeless cubic graph has six perfect matchings such that each edge appears in exactly two of them (cf. [2, 4, 6]). If we take three of these six perfect matchings, then each edge appears in at most two of them. This motivates the following weaker conjecture proposed by Fan and Raspaud [5]: In every bridgeless cubic graph there exist three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$. For brevity, this conjecture is referred to as the *three matching intersection conjecture* or *3PM conjecture*.

A graph is said to be *matching covered* if it is connected and each edge is contained in a perfect matching. Note that every bridgeless cubic graph is matching covered (or 1-extendable in [7]). So we generally discuss the matching covered graphs below. In a viewpoint of generalization to the 3PM conjecture, we propose the following.

Definition 1.1. A matching covered graph G is called a *3PM-admissible graph* (or G admits the 3PM property) if there exist three perfect matchings M_1, M_2, and M_3 of G such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

Our goal is to characterize 3PM-admissible graphs. Within the realm of cubic graphs, this amounts to the 3PM conjecture. Many 3PM-admissible cubic graphs have been found to support this conjecture, such as the 3-edge-colourable cubic graphs (including bipartite graphs, hamiltonian graphs), the cubic graphs with independent perfect matching polytope $P(G)$ or with low dimension perfect matching polytope (see [8, 9]). Here, a cubic graph G is 3-edge-colorable if there are three perfect matchings of G which form a partition of $E(G)$. Some basic cubic graphs are shown in Figure 1, which are 3PM-admissible.

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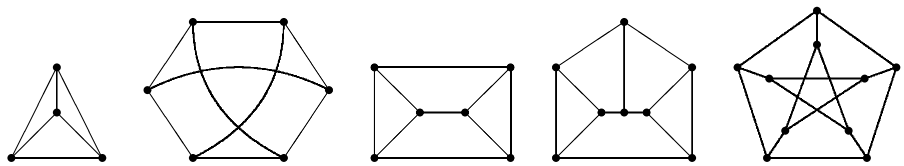
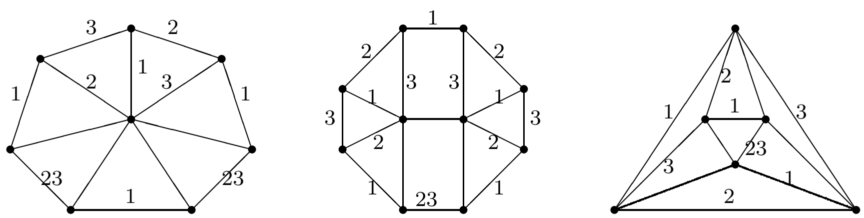


(a) K_4 (b) $K_{3,3}$ (c) 3-prism B_6 (d) B_8 (e) Petersen graph

Figure 1. Some important cubic graphs

Furthermore, apart from those cubic graphs, there are more 3PM-admissible matching covered graphs. For example, a *wheel* W_n is a cycle C_n with every vertex joining to a single vertex, the *hub*. When n is odd, W_n is called an odd wheel, which is matching covered (see Figure 2(a)). The wheels form a basic family of 3-connected graphs in the sense that every 3-connected graph can be constructed from a wheel via some kind of operations (see Tutte's theorem in [2]). When performing an 'expansion' at the hub of a wheel, we can obtain another matching covered graph, called the *double wheel*. An example is shown in Figure 2(b). Moreover, the tetrahedron K_4, the cube Q_3, the dodecahedron, the octahedron and the icosahedron, which are well-known *platonic graphs*, are matching covered and the last two are not cubic [2]. Here the octahedron is shown in Figure 2(c), and the icosahedron is shown in Figure 3. To see that these graphs are 3PM-admissible, we define the perfect matchings M_i for $1 \leq i \leq 3$ in Figures 2 and 3, where M_i is represented by the edges with label i at the edges.



(a) Wheel (b) Double wheel (c) Octahedron

Figure 2. Examples of 3PM-admissible graphs

We can see from the above examples that in addition to the cubic graphs, there would be many 3PM-admissible matching covered graphs. In this paper, we consider the characterization of 3PM-admissibility for matching covered graphs. Especially, we are concerned with several special classes of matching covered graphs, such as the platonic graphs, wheels, Halin graphs, outerplanar graphs, 4-regular graphs on small size.

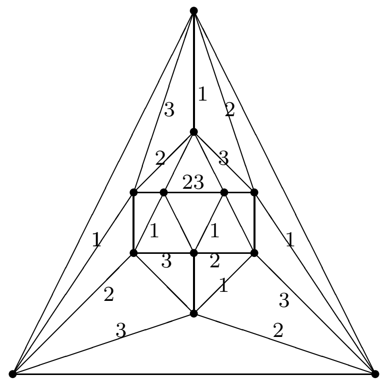


Figure 3. The icosahedron

The organization of the paper is as follows. In Section 2, we present a necessary and sufficient condition and its consequences. Section 3 is dedicated to the 4-regular graphs. We give a short summary in Section 4. We shall follow the graph-theoretic terminology and notation of [2].

2 Basic theorems

Throughout the paper, we consider G as a matching covered graph. So G has a perfect matching and has even number of vertices.

Matching covered graphs have a basic property (see [7]): If G' results from G by subdividing an edge with two vertices, then G' is matching covered if and only if G is matching covered. For this, the subdivision from G to G' is called a *bisubdivision*. A graph results from G by performing several times of this kind of operations is also called a bisubdivision of G. On the other hand, the inverse operation, namely, replacing a path of G' whose length is three and whose internal vertices have degree two in G' by an edge, is called a *bicontraction*. The resulting graph obtained from G' by performing several times of this kind of operations is also called a bicontraction of G'.

A spanning subgraph G' of G is called a *2-factor* if every vertex of G' has degree two. We have the following basic criterion.

Theorem 2.1 *A graph G is 3PM-admissible if and only if (1) G has a 2-factor G' with even components, or (2) G has a spanning subgraph G' which is a bisubdivision of a 3-edge-colorable cubic graph.*

Proof: If G is 3PM-admissible, then there exist three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$. Consider the spanning subgraph $G' = G[M_1 \cup M_2 \cup M_3]$. Note that the maximum degree of G' is at most three. If every vertex of G' has degree two, then G' is a 2-factor and so each of its components is a cycle. Since each M_i ($1 \leq i \leq 3$) is a perfect matching, these cycle components must be (M_i, M_j)-alternating cycles, where $1 \leq i, j \leq 3$ and $i \neq j$. Thus they have even number of edges. Hence (1) holds. Otherwise, G' has vertices of degree three. If every vertex of G' has degree three, then G' is a cubic graph with edge set $M_1 \cup M_2 \cup M_3$ and so is 3-edge-colorable. If this is not the case, then G' has vertices of degree two. Suppose that a vertex u has degree two and it is incident with two edges xu and uv. Without loss of generality, assume that $xu \in M_1$ and $uv \in M_2 \cap M_3$. Then v must be incident with an edge $vy \in M_1$. Thus $xuvy$ is a path of G' whose length is three and whose internal vertices have degree two in G'. Replacing this path by an edge xy, we get a bicontraction H' of G'. Moreover, $(M_1 \setminus \{ux, vy\}) \cup \{xy\}$, $M_2 \setminus \{uv\}$ and $M_3 \setminus \{uv\}$ are three perfect matchings of H' with empty intersection. If there are more vertices of degree two, then we can repeatedly perform this kind of bicontractions. As a result, we finally obtain a cubic graph H, and G' is a bisubdivision of H. Furthermore, H is 3-edge colorable. Hence (2) holds.

Conversely, if (1) holds, then G has a 2-factor G' with even components. Here, each component of G' is an even cycle. So we can define perfect matchings M_1 and M_2 of G' by making each even cycle in G' to be an (M_1, M_2)-alternating cycle. Further, let $M_3 := M_2$. In this way, we obtain three perfect matchings M_1, M_2, and M_3 with $M_1 \cap M_2 \cap M_3 = \emptyset$.

On the other hand, if (2) holds, then G has a spanning subgraph G' which is a bisubdivision of a 3-edge-colorable cubic graph, say H. So H has three perfect matchings which cover $E(H)$ and whose intersection is empty. We can extend these three perfect matchings to G' as follows. Suppose that H' is a graph whose edge set is covered by three perfect matchings M_1, M_2, and M_3 with $M_1 \cap M_2 \cap M_3 = \emptyset$. Initially, $H' := H$. Suppose that we have made a bisubdivision of H' on xy by subdividing it with two vertices u and v. The resulting graph is also denoted by H'. Since $M_1 \cap M_2 \cap M_3 = \emptyset$, suppose, without loss of generality, that $xy \in M_1$ and $xy \notin M_3$. If $xy \in M_1 \setminus M_2$, then we delete xy from M_1, add xu, vy into M_1, and add uv into $M_2 \cap M_3$. If $xy \in M_1 \cap M_2$, then we delete xy from $M_1 \cap M_2$, add xu, vy into $M_1 \cap M_2$, and add uv into M_3. Then $M_1 \cup M_2 \cup M_3 = E(H')$ and $M_1 \cap M_2 \cap M_3 = \emptyset$. By this procedure, we construct three perfect matchings M_1, M_2, and M_3 in G' (and thus in G) such that $M_1 \cup M_2 \cup M_3 = E(G')$ and $M_1 \cap M_2 \cap M_3 = \emptyset$. This completes the proof. □

In condition (2) of this theorem, the cubic graph H is called the *cubic skeleton* of G. As we know, a graph is a *minor* of G if it can be obtained from G by a sequence of deleting vertices or edges, and contracting edges. So the cubic skeleton H is in fact a minor of G, a cubic minor.

Corollary 2.2 *If G is an odd wheel, a double wheel with even number of vertices, or the octahedron, then G is 3PM-admissible.*

Proof: First, an odd wheel W_n has K_4 as its cubic skeleton, that is, it has a spanning subgraph G' which is a bisubdivision of K_4. Second, a double wheel G has the 3-prism $B_6 = K_3 \times K_2$ as its cubic skeleton. Moreover, the octahedron contains a 3-prism B_6 as spanning subgraphs (see Figure 2(c)). And it is known that K_4 and B_6 in Figure 1 are 3-edge-colorable. The result follows from Theorem 2.1. □

Theorem 2.1 also implies the following.

Corollary 2.3 *A hamiltonian graph is 3PM-admissible.*

The well-known Tutte's theorem says that every 4-connected planar graph is hamiltonian (see [1]). So we have the following.

Corollary 2.4 *Every 4-connected planar graph is 3PM-admissible.*

A graph G is called a *Halin graph* if it can be drawn in the plane as a tree T, with all non-end-vertices having minimum degree 3, together with a cycle C passing through the end-vertices of T. Since Halin graphs are hamiltonian (see Exercise 10.2.4 of [2]), we have the following.

Corollary 2.5 *Every Halin graph is 3PM-admissible.*

As we know, the dodecahedral is hamiltonian. Moreover, the icosahedron is hamiltonian. In fact, the edges with labels 1 and 2 in Figure 3 constitute a Hamilton cycle. An *outerplanar graph* (it has a planar embedding in which all vertices lie on the boundary of its outer face) is also hamiltonian. So they are 3PM-admissible.

Let us see one more example taken from [8] whose perfect matching polytope is independent, as shown in Figure 4. It is hamiltonian (the edges with labels 1 and 2 in Figure 4 constitute a Hamilton cycle). Also, it contains a 3-prism B_6 as its cubic skeleton.

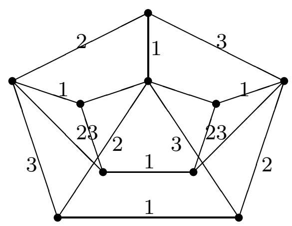


Figure 4. A 3-connected graph with independent polytope

3 4-regular graphs

Corollary 3.4.3 in [7] says that if a graph is $(k-1)$-edge-connected, k-regular, and has even number of vertices, then it is matching covered. A 3-connected 4-regular graph is 3-edge-connected and so is matching covered. Recall Jackson's theorem: Every 2-connected k-regular graph on at most $3k$ vertices is hamiltonian (see [3]). From this, we have an observation as follows.

Proposition 3.1 *Every 3-connected 4-regular graph G on at most 12 even number of vertices is 3PM-admissible.*

For example, the octahedron in Figure 2(c) is 4-regular and has 6 vertices. So it is 3PM-admissible. The following is a stronger result.

Theorem 3.2 *Every 3-connected 4-regular simple graph G on at most 18 even number of vertices is 3PM-admissible.*

Proof: Let M_1 be a perfect matching of G and let $G' = G - M_1$. Then G' is a cubic subgraph of G. If G' has a perfect matching M_2, then G has two disjoint perfect matchings M_1 and M_2. Thus G is 3PM-admissible. In the following, assume that G' has no perfect matchings.

We shall apply Gallai-Edmonds structure theorem (see [7]) to G'. Denote by D the set of all vertices not covered by at least one maximum matching of G', by A the set of neighbours of D in $V(G') \setminus D$, and by C the set of all other vertices of G'. Then

(a) each component of $G'[D]$ is factor critical;

(b) $G'[C]$ has a perfect matching;

(c) any maximum matching in G' contains a perfect matching in $G'[C]$ and near-perfect matchings of components of $G'[D]$, and matches all vertices of A to distinct components of $G'[D]$.

Here, a graph H is *factor critical* if $H - v$ has a perfect matching for each $v \in V(H)$, and a matching of H is *near perfect* if it covers all but one vertex in H.

Let D' be the vertex set of a component of $G'[D]$, and let t be the number of edges in G' connecting A and D'. Then $G'[D']$ is factor critical, and so $|D'|$ is odd. Recall that G' is a cubic graph. We have $3|D'| = 2|E(G'[D'])| + t$. This implies that t is odd. Since G is simple, if $t = 1$, then $|D'| \geq 5$. Let ω_1 denote the number of components of $G'[D]$ each of which is connected by only one edge to A. Let ω denote the number of components of $G'[D]$. Since G' has no perfect matchings, by Gallai-Edmonds structure theorem, we have $\omega > |A|$. Since the number of vertices of G' is even, ω and $|A|$ have the same parity, and so $\omega \geq |A| + 2$.

When $|A| = 1$, we have $\omega \geq 3$. Since G' is cubic, we have $\omega = \omega_1 = 3$. Let u be the vertex in A, G_1, G_2, G_3 the three components in $G'[D]$, and $x \in V(G_1), y \in V(G_2), z \in V(G_3)$ three neighbours of u. Then $|V(G_i)| \geq 5$, $i = 1, 2, 3$. Moreover, by the definition of C, in this case G' is the disjoint union of $G'[C]$ and $G'[A \cup D]$, both of which are cubic.

If $C = \emptyset$, then $G' - u = G'[D]$ (see an example in Figure 5). Since G is 3-connected, $G - u$ is connected. Thus there exist at least two edges of M_1, say e and f, which connect the components G_1, G_2, G_3. Suppose, without loss of generality, that e connects G_1 and G_2 and f connects G_1 and G_3. Let $G^* = G' + e + f$. Since G_1, G_2, G_3 are factor-critical, there exists a perfect matching M_2 of G^* containing $\{e, uz\}$, and a perfect matching M_3 of G^* containing $\{f, uy\}$. Then M_1, M_2, and M_3 are three perfect matchings of G such that $M_1 \cap M_2 = \{e\}$, $M_1 \cap M_3 = \{f\}$, and $M_2 \cap M_3$ may be nonempty. However, $M_1 \cap M_2 \cap M_3 = \emptyset$. Therefore, G is 3PM-admissible.

If G is not 3PM-admissible, then either $|A| = 1$ and $C \neq \emptyset$ or $|A| \geq 2$. For the former case, noting that $G'[C]$ is cubic and G is simple, there are at least four vertices in C. Thus $|V(G)| = |V(G')| = |C| + |A| + \sum_{i=1}^{3} |V(G_i)| \geq 20$. For the latter case, when $|A| = 2$, we have $\omega \geq 4$. Combining the fact that the number of edges in G' connecting A and a component of $G'[D]$ is odd and G' is a cubic graph, we have $\omega_1 \geq 3$. If $\omega_1 = 3$, then $\omega = 4$ and there is a component D'' of $G'[D]$ such that there are three edges in G' connecting A and D''. Since G' is simple and $|D''|$ is odd, we have $|D''| \geq 3$. So $|V(G)| \geq |A| + |D''| + 5\omega_1 \geq 20$. If $\omega_1 \geq 4$, then $|V(G)| \geq |A| + 5\omega_1 \geq 22$. When $|A| \geq 3$, we have $\omega \geq 5$. By counting the number of edges which connect A and D in two ways, we have $\omega_1 + 3(\omega - \omega_1) \leq 3|A|$. Thus $\omega_1 \geq \frac{3}{2}(\omega - |A|) \geq 3$, and so $|V(G)| \geq |A| + (\omega - \omega_1) + 5\omega_1 = |A| + \omega + 4\omega_1 \geq 20$. Therefore, a graph with at most 18 vertices admits the 3PM property. $\square$

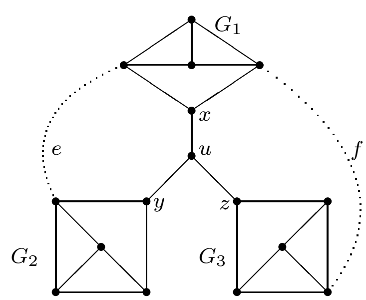


Figure 5. Cubic graph without perfect matching

4 Concluding remarks

To look for 3PM-admissible graphs, traversing from cubic graphs to matching covered graphs, we can see some connections and some new features. Many problems remain to be investigated.

• The concept of 3PM-admissible graphs is a generalization (relaxation) of that of the hamiltonian graphs. At the beginning we introduce five polyhedral graphs, the platonic graphs. They are all hamiltonian. In general, a graph is polyhedral if and only if it is planar and 3-connected (see [1]). Tutte presented a counterexample to show that a polyhedral graph is not necessarily hamiltonian. However, this counterexample is cubic and is 3PM-admissible. So it is not a counterexample for the statement that every polyhedral graph is 3PM-admissible. We can ask if this statement holds true.

• Jackson's theorem asserts that every 2-connected 4-regular graph on at most 12 vertices is hamiltonian. Further, Jackson conjectured that every 3-connected 4-regular graph on at most 16 vertices is hamiltonian (see [3]). Now, we obtain an easier assertion that every 3-connected 4-regular graph on at most 18 vertices is 3PM-admissible. Can we further improve this upper bound?

• For a cubic graph G, we have proved that if the perfect matching polytope is independent, then G is 3PM-admissible. In Figure 4, we show a 3-connected graph with independent polytope to be 3PM-admissible. Can we prove this for every 3-connected graph?

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