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Generalized connected domination in graphs

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As a generalization of connected domination in a graph $G$ we consider domination by sets having at most $k$ components. The order $\gamma_k^c(G)$ of such a smallest set we relate to $\gamma_c(G)$, the order of a smallest connected dominating set. For a tree $T$ we give bounds on $\gamma_k^c(T)$ in terms of minimum valency and diameter. For trees the inequality $\gamma_k^c(T) \leq n - k - 1$ is known to hold, we determine the class of trees, for which equality holds.

Keywords: connected domination, domination, tree
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1 Introduction

We consider simple non-oriented graphs. The largest valency in $G$ is denoted by $\Delta(G) = \Delta$, the smallest by $\delta(G) = \delta$. By $P_n$ we denote a path on $n$ vertices and $C_n$ denotes a circuit on $n$ vertices. In a graph a leaf or pendant vertex is a vertex of valency one and a stem is a vertex adjacent to at least one leaf. In $K_2$ each vertex is both a leaf and a stem. The set of leaves in a graph $G$ is denoted by $\Omega(G)$. The set of neighbours to a vertex $x$ is denoted $N(x)$. By $K_1,k$ we denote a star with one central vertex joined to $k$ other vertices. A subdivided star is a star with a subdivision vertex on each edge. By the corona graph on $H$ we understand the graph $G = H \circ K_1$ obtained from the graph $H$ by adding for each vertex $x$ in $H$ one new vertex $x'$ and one new edge $xx'$. In a corona graph each vertex is either a leaf or a stem adjacent to exactly one leaf. In particular, if $H$ is a tree, we obtain a corona tree $T = H \circ K_1$.

The eccentricity $e(x)$ of a vertex $x$ is defined by $e(x) = \max\{d(x,y) | y \in V(G)\}$. The diameter of $G$ is $\text{diam}(G) = \max\{e(x) | x \in V(G)\}$. Let $D \subseteq V(G)$, then $N(D)$ is the set of vertices which have a neighbour in $D$ and $N[D]$ is the set of vertices which are in $D$ or have a neighbour in $D$. $N[D] = D \cup N(D)$. A set $D \subseteq V(G)$ dominates $G$ if $V(G) \subseteq N[D]$, i.e. each vertex not in $D$ is adjacent to a vertex in $D$. The domination number $\gamma(G)$ is the cardinality of a smallest dominating set in $G$.

For a given graph $G$ it is NP-hard to determine its domination number $\gamma(G)$, but we can search for for upper bounds as O. Ore started doing about fifty years ago. Also it may be more tractable to restrict the
minimum dominating set problem to consider only such dominating sets which induce a connected subset of $G$, this problem is called the minimum connected dominating problem and it is still NP-complete; in network design theory it is called the maximum leaf spanning tree problem [4], the name will be clear from Section 2 below. We shall study a concept intermediate to the classical and the connected domination, namely by demanding the dominating set to induce at most a given number $k$ of components, we aim at presenting upper bounds for its order $\gamma_k^c$. Quite likely there is a corresponding problem in network design theory, although we are aware of no reference.

A comprehensive introduction to domination theory is given in [7, 14] and variations are discussed in [5, 13, 15].


**Proposition 1** Let $G$ be a connected graph with $n$ vertices, $n \geq 2$. Then $\gamma(G) \leq \frac{n}{2}$ and equality holds if and only if $G$ is either a corona graph or a 4-circuit.

If a tree $T$ has $\gamma(T) = \frac{n}{2}$, then $n$ is even and Proposition 1 implies that $T$ is a corona tree.

**Definition** For a positive integer $k$ and a graph $G$ with at most $k$ components we define

$$\gamma_k^c(G) = \min \{|D| : D \subseteq V(G), D \text{ has at most } k \text{ components and } D \text{ dominates } G\}.$$ 

A set $D$ attaining the minimum above is called a $\gamma_k^c$-set for $G$.

**Example**

$$\gamma_k^c(P_n) = \gamma_k^c(C_n) = \begin{cases} n - 2k & \text{for } n \geq 3k \\ \left\lceil \frac{n}{3} \right\rceil & \text{for } 1 \leq n \leq 3k \end{cases}$$

For $k = 1$ we have that $\gamma_1^c$ is the usual connected domination number, $\gamma_1^c(G) = \gamma_c(G)$.

There exists for every graph $G$ a $k$ such that $\gamma_k^c(G) = \gamma_c(G)$, e.g. $k = |G|$.

For $G$ connected and $k \geq 1$, obviously, $\gamma(G) \leq \gamma_k^c(G) \leq \gamma_c(G)$.

### 2 General graphs

Let $G$ be a connected graph with $n$ vertices and $k$ a positive integer. Let $\epsilon_F(G)$ be the maximum number of leaves among all spanning forests of $G$, and $\epsilon_T(G)$ be the maximum number of leaves among all spanning trees of $G$. With this notation Niemen [9] proved statement (i) below about $\gamma$ and Hedetniemi and Laskar [8] generalized it to statement (ii) about $\gamma_c$.

(i) $\gamma(G) = n - \epsilon_F(G)$,

(ii) $\gamma_c(G) = n - \epsilon_T(G)$.

In the next two theorems we extend these results to $\gamma_k^c$.

**Theorem 1** Let $G$ be a connected graph with $n$ vertices and $k$ a positive integer. Let $\epsilon_{F_k}(G)$ be the maximum number of leaves among all spanning forests of $G$ with at most $k$ trees. Then $\gamma_k^c(G) = n - \epsilon_{F_k}(G)$. 

**Proposition 2** Let \( \gamma_c^k(G) \) be a positive integer and \( G \) a connected graph. Then

\[
\gamma_c^k(G) = \min \{ \gamma_c^k(F_k) | F_k \text{ is a spanning forest of } G \text{ with at most } k\text{ trees} \}
\]

\[
= \min \{ \gamma_c^k(T) | T \text{ is a spanning tree of } G \}
\]

**Proof:** Let \( F_k \) be a spanning forest of \( G \) with at most \( k \) trees. Certainly \( \gamma_c^k(G) \leq \gamma_c^k(F_k) \) since a set which dominates \( F_k \) also dominates \( G \). Conversely, we can in \( G \) find a spanning forest \( F_k \) with at most \( k \) components such that \( \gamma_c^k(G) = \gamma_c^k(F_k) \): As was originally also done in the proofs for (i) and (ii) above we construct \( F_k \) from a \( \gamma_c^k \)-set \( D = D_1 \cup D_2 \cup \cdots \cup D_t \), \( 1 \leq t \leq k \), by choosing a spanning tree \( T_i \) in each connected subgraph \( D_i \) and joining each vertex in \( V(G) \setminus D \) to precisely one vertex in \( D \). Obviously, \( \gamma_c^k(F_k) \leq |D| = \gamma_c^k(G) \). This proves the first equality. For the second equality we observe that the first minimum is chosen among a larger set, so that \( \min \gamma_c^k(F_k) \leq \min \gamma_c^k(T) \), and also that any \( F_k \) by addition of edges can produce a tree \( T \) with \( \gamma_c^k(T) \leq \gamma_c^k(F_k) \).

Hartnell and Vestroigard [6] proved the following result.

**Proposition 2** For \( k \geq 1 \) and \( G \) connected

\[
\gamma_c(G) - 2(k - 1) \leq \gamma_c^k(G) \leq \gamma_c(G).
\]

From Proposition 2 we can easily derive the following corollary which is a classical result proven by Duchet and Meyniel. [2]

**Corollary 3** For any connected graph \( G \), \( \gamma_c(G) \leq 3\gamma(G) - 2 \).

**Proof:** Let \( G \) be a connected graph with domination number \( \gamma(G) \). Choose \( k = \gamma(G) \), then \( \gamma_c^k(G) = \gamma_c(G) \). Substituting into Proposition 2 above we obtain \( \gamma_c(G) - 2(k - 1) \leq \gamma(G) \) and that proves the corollary.

**2.1 Other bounds on \( \gamma_c^k \)**

**Theorem 4** For a positive integer \( k \) and a connected graph \( G \) with maximum valency \( \Delta \) we have

(A) \( \gamma_c(G) \leq n - \Delta \) and for trees \( T \) equality holds if and only if \( T \) has at most one vertex of valency \( \geq 3 \).

(B) \( \gamma_c^k(G) \leq n - \frac{(r - 1)(\delta - 2)}{3} - 2k \) if \( G \) has diameter \( r \geq 3k - 1 \) and the minimum valency \( \delta = \delta(G) \) is at least 3.
Proof:

(A) Let $T$ be a spanning tree of $G$ with $\Delta(T) = \Delta(G) = \Delta$, then $T$ has at least $\Delta$ leaves, and hence $\gamma_c(G) \leq \gamma_c(T) \leq n - \Delta$.

If $T$ has two vertices of valency $\geq 3$, the number of leaves in $T$ will be larger than $\Delta$, and we get strict inequality in (A). Clearly, a tree $T$ with exactly one vertex of valency $\Delta \geq 3$ has equality in (A) and for $\Delta = 2$, we obtain a path $P_n$ with $\gamma_c(P_n) = n - 2$.

(B) Let $P = v_1v_2v_3\ldots v_{2t+u}$, $k \leq t, 0 \leq u \leq 2$, be a diametrical path in $G$. The diameter of $T$ equals the length of $P$, which is $r = 3t + u - 1$. For $i = 1, \ldots, t$ let $v_{3i-1}$ have neighbours $v_{3i-2}, v_{3i}$ on $P$ and $a_{ij}$ off $P$, $j = 1, \ldots, s_i$, $s_i \geq \delta - 2 \geq 1$. In $G - \{v_{3i}v_{3i+1} | 1 \leq i \leq k - 1\}$ consider the $k - 1$ disjoint stars with center $v_{3i-1}$ and neighbours $N(v_{3i-1}), 1 \leq i \leq k - 1$, and the remaining tree to the right consisting of the path $v_{3k-2}v_{3k-1}v_{3k}\ldots v_{3t+u}$ and leaves $v_{3i-1}a_{3i-1}, \; j = 1, \ldots, s_i, \; s_i \geq \delta - 2 \geq 1$ adjacent to vertices $v_{3i-1}, \; k \leq i \leq t$.

Case 1, $t \geq k - 1$: In $G - \{v_{3i-1}v_{3k} | 1 \leq i \leq k - 2\}$ we extend the $k$ trees listed below to a spanning forest $F$ of $G$.

1. The star consisting of $v_1$ joined to all its neighbours,
2. the $k - 2$ paths of length two $v_{3i}v_{3i+1}v_{3i+2}, 1 \leq i \leq k - 2$,
3. the path $v_{3i-3}v_{3i-2}\ldots v_{3s}$ together with all $\Delta - 1$ neighbours of $v_s$ outside of $P$.

$F$ will have at least $2(\Delta - 1) + 2(k - 1)$ leaves.

Case 2, $t \leq k - 2$: $s = 3t + 1 + u, d = d(v_1, v_s) = s - 1 = 3t + u, t - 1 = \frac{d - u}{3} - 1 \geq \frac{d - 2}{3} - 1$. As before, we can find a spanning forest $F$ of $G$ whose number of leaves is at least $2\Delta + 2(t - 1) \geq 2(\Delta - 1) + 2\frac{d - 2}{3}$ and consequently $\gamma^k_c(G) \leq n - 2(\Delta - 1) - 2\frac{d - 2}{3}$.

The proof of D is similar.
3 Trees

For trees Hartnell and Vestergaard [6] found

**Proposition 3** Let $k$ be a positive integer and $T$ a tree with $|V(T)| = n, n \geq 2k + 1$. Then $\gamma^k_c(T) \leq n - k - 1$.

This inequality is best possible. For $k = 1$ the extremal trees are paths $P_n$, and for $k \geq 2$ extremal trees will be described in the following Theorem 5.

A tree $T$ is of type A if it contains a vertex $x_0$ such that $T - x_0$ is a forest of trees $T_1, T_2, \ldots, T_\alpha$, $\alpha \geq 1$, such that each tree $T_i$ is a corona tree and $x_0$ is joined to a stem in each of the trees $T_i$, $1 \leq i \leq \alpha$. We note that a subdivision of a star is a tree of type A.

A tree $T$ is of type B if it contains a path $uvw$ such that $T - \{u, v, w\}$ is a forest of corona trees $T_1, T_2, \ldots, T_s, T_{s+1}, \ldots, T_\alpha$, $\alpha \geq 2$, $1 \leq s < \alpha$ and $u$ is joined to a stem in each of the trees $T_1, T_2, \ldots, T_s$, while $w$ is joined to a stem in each of the trees $T_{s+1}, \ldots, T_\alpha$.

Proposition 4 below was proven by Randerath and Volkmann [12], Baogen, Cockayne, Haynes, Hedetniemi and Shangchao [1].

**Proposition 4** If $T$ is a tree with $n$ vertices, $n$ odd, and $\gamma(T) = \lfloor \frac{n}{2} \rfloor$ then $T$ is a tree of type A or B.

We shall now determine the trees extremal for Proposition 3.

**Theorem 5** Let $k \geq 2$ be a positive integer and $T$ a tree with $n$ vertices, $n \geq 2k + 1$. Then $\gamma^k_c(T) = n - k - 1$ if and only if one of cases (i)-(iii) below occur:

(i) $k = \frac{n - 1}{2}$, $\gamma^k_c(T) = \gamma(T) = \frac{n - 1}{2}$ and $T$ is of type A or B.

(ii) $k = \frac{n - 2}{2}$, $\gamma^k_c(T) = \gamma(T) = \frac{n}{2}$ and $T$ is a corona tree.

(iii) $k = \frac{n - 3}{2}$, $\gamma^k_c(T) = \frac{n + 1}{2}$, $\gamma(T) = \frac{n - 1}{2}$ and $T$ is a star $K_{1,k+1}$ with a subdvision vertex on each edge.

**Proof:** First, let $k \geq 2$ and a tree $T$ of order $n$ be given such that $n \geq 2k + 1$ and $\gamma^k_c(T) = n - k - 1$. We shall prove that $T$ is as described in one of the three cases (i)-(iii).

We note in passing that

**Remark 1** $\gamma(T) \leq k$ implies $\gamma^k_c(T) = \gamma(T)$, and that likewise $\gamma^k_c(T) \leq k$ implies $\gamma^k_c(T) = \gamma(T)$.

If $n = 2k + 1$, or equivalently $k = \frac{n - 1}{2}$, we have by assumption $\gamma^k_c(T) = n - k - 1 = k$ and, as just observed above, that implies that also $\gamma(T) = k$. Since $k = \lfloor \frac{n}{2} \rfloor$ we obtain from Proposition 4 that $T$ is a tree of type A or B, so Case (i) occurs.

If $n = 2k + 2$, or equivalently $k = \frac{n - 2}{2}$ we have by assumption $\gamma^k_c(T) = n - k - 1 = k + 1$. Certainly $\gamma(T) \leq \gamma^k_c(T)$, but if $\gamma(T) \leq k$ then we should have that $\gamma^k_c(T) = \gamma(T) \leq k$ in contradiction
to $\gamma^k_c(T) = k + 1$, therefore $\gamma(T) = k + 1 = \frac{n}{2}$. From Proposition 1 we obtain that $T$ is a corona tree, i.e. Case (ii) occurs.

We may now assume $n \geq 2k + 3$, and we shall prove that, in fact, $n$ equals $2k + 3$ and that Case (iii) occurs.

Let $v_1, v_2, \ldots, v_n$ be a longest path in $T$. Since $\gamma^k_c(T) = n - k - 1 \geq k + 2 \geq 4$, $T$ is neither a star nor a bistar and therefore $\alpha \geq 5$. We must have $d_T(v_1) = 2$, because otherwise $d_T(v_2) \geq 3$ and we could from $T$ delete three leaves adjacent to $v_2$, if $d_T(v_2) \geq 4$, and in case $d_T(v_2) = 3$ we could delete $v_2$ and its two adjacent leaves. In both cases we would obtain a tree $T'$ of order $n - 3 \geq 2(k - 1) + 1$ which by Proposition 3 has $\gamma^{k-1}_c(T') \leq (n - 3) - (k - 1) - 1 \leq n - k - 3$. Adding $v_2$ to a $\gamma^{k-1}_c(T')$-set we would obtain $\gamma^k_c(T) \leq n - k - 2$, a contradiction. Therefore $d_T(v_2) = 2$.

The vertex $v_3$ cannot be adjacent to two leaves $c$ and $d$, say, because, then the tree $T' = T - \{v_1, v_2, c, d\}$ would have order $n - 4 \geq 2(k - 1) + 1$. Thus Proposition 3 gives that $\gamma^{k-1}_c(T') \leq (n - 4) - (k - 1) - 1 \leq n - k - 4$ and adding $v_2, v_3$ to a $\gamma^{k-1}_c(T')$-set we would obtain $\gamma^k_c(T) \leq n - k - 2$, a contradiction. So $v_3$ can be adjacent to at most one leaf. The case $d_T(v_3) = 3$ and $v_3$ adjacent to one leaf $c$ can similarly be seen to be impossible by considering $T'' = T - \{v_1, v_2, v_3\}$.

On the other hand $d_T(v_3) \geq 3$, for assume $d_T(v_3) = 2$, then $T' = T - \{v_1, v_2, v_3\}$ has $\gamma^{k-1}_c(T') \leq n - k - 3$ and addition of $v_3$ to a $\gamma^{k-1}_c(T')$-set would give $\gamma^k_c(T) \leq n - k - 2$, a contradiction.

Assume therefore that $v_3$ besides $v_2$ and $v_4$ is adjacent to precisely one leaf $c$ and to at least one further vertex $a$, where $a$ has valency two and is adjacent to the leaf $b$. Then $T' = T - \{v_1, v_2, a, b\}$ has order $n - 4 \geq 2(k - 1) + 1$ and Proposition 3 gives that $2(c)$ $\gamma^{k-1}_c(T') \leq (n - 4) - (k - 1) - 1 \leq n - k - 4$. In $T'\{c\}$ the vertex $c$ is a leaf and as any $\gamma^{k-1}_c(T')$-set for $T'$ must contain one of $\{v_3, c\}$, we may assume it contains $v_3$. Addition of $\{v_2, a\}$ to a $\gamma^{k-1}_c(T')$-set now gives the contradiction $\gamma^k_c(T) \leq n - k - 2$.

Assume finally that $v_3$ has no leaf but besides $v_2$ and $v_4$ is adjacent to $a_1, a_2, \ldots, a_t$, $t \geq 1$, where each $a_i$ has valency two and is adjacent to the leaf $b_i$, $1 \leq i \leq t$.

We have $k - t \geq 1$ because $V(T) - \{v_1, b_1, b_2, \ldots, b_t, v_n\}$ is a connected subgraph with $n - t - 2$ vertices which dominate $T$, so that $n - k - 1 = \gamma^k_c(T) \leq n - t - 2$ giving $k - t \geq 1$. Consider the tree $T'' = T - \{v_1, v_2, a_1, a_2, \ldots, b_{t-1}, b_t, v_3\}$ of order $n - 2t - 3$.

If $n - 2t - 3 \geq 2(k - t) + 1$ we obtain by Proposition 3 that $\gamma^{k-t}_c(T'') \leq (n - 2t - 3) - (k - t) - 1 \leq n - k - t - 4$, and by addition of the $t + 2$ vertices $\{v_2, v_3, a_2, a_3, \ldots, a_t\}$, (which span a connected subgraph of $T$), to a $\gamma^{k-t}_c(T'')$-set we obtain $\gamma^k_c(T) \leq n - k - 2$, a contradiction. So we have $n - 2t - 3 \leq 2(k - t)$ and now $|V(T')| = n - 2t - 3 \leq 2(k - t)$ implies $\gamma(T') \leq \frac{|V(T')|}{2} \leq k - t$ which by remark 1 gives that $\gamma^{k-t}_c(T') = \gamma(T')$ and hence addition of the $t + 2$ vertices $\{v_2, v_3, a_1, a_2, \ldots, a_t\}$ to a $\gamma^{k-t}_c(T')$-set (having at most $k - t$ vertices) gives $\gamma^{k-t+1}_c(T) \leq k + 2$. We now have $n - k - 1 = \gamma^k_c(T) \leq \gamma^{k-t+1}_c(T) \leq k + 2$ giving $n \leq 2k + 3$, so the assumption $n \geq 2k + 3$ implies $n = 2k + 3$. By hypothesis $\gamma^k_c(T) = k + 2$ and we have $\gamma(T) \leq k + 1$ by Proposition 1.

Thus $\gamma(T) = k + 1$, (because otherwise $\gamma^k_c(T) = \gamma(T) < k + 2$), and any $\gamma(T)$-set must consist of $k + 1$ isolated vertices. As $\gamma(T) = \lfloor \frac{n}{2} \rfloor$ the tree $T$ by Proposition 4 is of type A or B. But $T$ cannot be of type B, for assume $T$ is of type B. Then $T$ consists of a 3-path, $uvw$, with each of its ends joined to stems of corona trees, and since we have just seen that $v_3, v_{n-2}$ are neither stems nor leaves, they must play the role of $u, w$, so $\alpha = 7$ and $T$ consists of two subdivided stars centered respectively at $u = v_3$ and $w = v_5$ and a vertex $v = v_4$ joined to $u$ and $w$. Among its $\gamma$-sets this tree $T$ has one with two adjacent vertices, namely $v_2$ and $v_3$, a contradiction, so $T$ is of type A.
Using, in analogy to $v_2, v_3$, that $d_T(v_{\alpha - 1}) = 2$ and that $v_{\alpha - 2}$ is not a stem, we get that $\alpha = 5$ and $T$ is a subdivided star so that (iii) occurs.

Conversely, it is easy to see that if (i), (ii) or (iii) holds then $\gamma^c_k(T) = \gamma(T) = n - k + 1$. This proves Theorem 5.

References


