*Discrete Mathematics and Theoretical Computer Science* DMTCS vol. **19**:1, 2017, #19



# *Evaluations of series of the $q$-Watson, $q$-Dixon, and $q$-Whipple type*

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*received 16th Apr. 2016, revised 1st May 2017, accepted 6th June 2017.*

Using $q$-series identities and series rearrangement, we establish several extensions of $q$-Watson formulas with two extra integer parameters. Then they and Sears' transformation formula are utilized to derive some generalizations of $q$-Dixon formulas and $q$-Whipple formulas with two extra integer parameters. As special cases of these results, many interesting evaluations of series of $q$-Watson, $q$-Dixon, and $q$-Whipple type are displayed.

**Keywords:** $q$-Watson formula, $q$-Dixon formula, $q$-Whipple formula

## 1 Introduction

For two complex numbers $x$ and $q$, define the $q$-shifted factorial by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k) \quad \text{when} \quad n \in \mathbb{N}.$$

The fraction form of it reads as

$$\begin{bmatrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix} q \Bigg]_n = \frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}.$$

Following Gasper and Rahman [2004], define the basic hypergeometric series by

$${}_{1+r}\phi_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\Big|\, q; z\right] = \sum_{k=0}^{\infty}\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\Big|\, q\right]_k \left\{(-1)^k q^{\binom{k}{2}}\right\}^{s-r} \frac{z^k}{(q;q)_k},$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Sears' transformation formula (cf. Equation (2.10.4) of Gasper and Rahman [2004]) can be expressed as

$${}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, q^{1-n}abc/de \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} d/a, de/bc \\ d, de/abc \end{matrix} \,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, a, e/b, e/c \\ e, de/bc, q^{1-n}a/d \end{matrix} \,\Big|\, q; q\right]. \tag{1}$$

*I am supported by the National Natural Science Foundations of China (Nos. 11661032, 11301120).
†And she is supported by the National Natural Science Foundations of China (Nos. 11661032, 11201291).

ISSN 1365–8050 

There are many interesting formulas for basic hypergeometric series in the literature. Here we consider a number of ${}_4\phi_3$ summations. First, the $q$-Watson formula due to Andrews [1976] and the $q$-Watson formula that is Equation (3.17) of Jain [1981] read, respectively, as

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, c \end{matrix} \,\Big|\, q; q\right] = c^{n/2}\left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix} \,\Big|\, q^2\right]_{\frac{n}{2}} \chi(n) \tag{2} $$

where $\chi(n)=\{1+(-1)^n\}/2$,

$$ {}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{-2n} \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n. \tag{3} $$

Second, the $q$-Bailey-Dixon formula (cf. page 8 of Gasper and Rahman [2004]) can be stated as

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1-n}/ac \\ q^{1-n}/a, q^{1-n}/c, -ac \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} -a, c^2 \\ -ac, c \end{matrix} \,\Big|\, q\right]_n \left[\begin{matrix} q, q^na^2c^2 \\ qc^2, q^na^2 \end{matrix} \,\Big|\, q^2\right]_{\frac{n}{2}} \chi(n). \tag{4} $$

By specifying the parameters in (1), we have the relation

$$ {}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1+n}a/c \\ qa/c, q^{1+n}a, -q^{-n}c \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+n}, -qa/c \\ q^{1+n}a, -q/c \end{matrix} \,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} a, qa/c^2, q^{-n}, -q^{-n} \\ qa/c, -qa/c, q^{-2n} \end{matrix} \,\Big|\, q; q\right]. $$

The combination of the last formula and (3) creates another $q$-Dixon formula

$$ {}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1+n}a/c \\ qa/c, q^{1+n}a, -q^{-n}c \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+n}, -qa/c \\ q^{1+n}a, -q/c \end{matrix} \,\Big|\, q\right]_n \left[\begin{matrix} qa, q^2a/c^2 \\ q, q^2a^2/c^2 \end{matrix} \,\Big|\, q^2\right]_n. \tag{5} $$

Third, the $q$-Whipple formula due to Andrews [1976] and the $q$-Whipple formula that is Equation (3.19) of Jain [1981] read, respectively, as

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}, \sqrt{qac}, -\sqrt{qac} \\ -q, qa, qc \end{matrix} \,\Big|\, q; q\right] = \frac{(q^{1-n}a;q^2)_n(q^{1-n}c;q^2)_n}{(qa;q)_n(qc;q)_n}\, q^{\binom{n+1}{2}}, \tag{6} $$

$$ {}_4\phi_3\left[\begin{matrix} a, q/a, q^{-n}, -q^{-n} \\ c, q^{1-2n}/c, -q \end{matrix} \,\Big|\, q; q\right] = \frac{(ac;q^2)_n(qc/a;q^2)_n}{(c;q)_{2n}}. \tag{7} $$

By means of contiguous relations for ${}_3F_2$-series, Lavoie et al. [1992, 1996, 1994] gave a lot of summation formulas for Watson, Dixon, and Whipple type ${}_3F_2$-series. For some related works, the reader may refer to Lavoie [1987] and Rathie and Paris [2009]. In 2011, Chu [2012] established the generalizations of Watson's ${}_3F_2$-series identity with two extra integer parameters and derived several summation formulas for Dixon and Whipple type ${}_3F_2$-series according to hypergeometric series identities and series rearrangement. Let $u$ and $v$ both be integers throughout the paper. Inspired by Chu's method, we shall explore summation formulas for the following six series:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+u+n}a, \sqrt{c}, -q^v\sqrt{c} \\ q\sqrt{a}, -q^{1+u}\sqrt{a}, q^vc \end{matrix} \,\Big|\, q; q\right], \qquad {}_4\phi_3\left[\begin{matrix} q^ua, c, q^{-n}, -q^{v-n} \\ \sqrt{qac}, -q^u\sqrt{qac}, q^{v-2n} \end{matrix} \,\Big|\, q; q\right], $$

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1+u+v-n}/ac \\ q^{1+u-n}/a, q^{1+v-n}/c, -ac \end{matrix} \,\Big|\, q; q\right], \qquad {}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1+u+v+n}a/c \\ q^{1+u}a/c, q^{1+v+n}a, -q^{-n}c \end{matrix} \,\Big|\, q; q\right], $$

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+u+n}, \sqrt{qac}, -q^v\sqrt{qac} \\ -q, q^{1+u+v}a, qc \end{matrix} \,\Big|\, q; q\right], \qquad {}_4\phi_3\left[\begin{matrix} a, q^{1+u}/a, q^{-n}, -q^{v-n} \\ c, q^{1+u+v-2n}/c, -q \end{matrix} \,\Big|\, q; q\right], $$

which can be regarded as terminating $q$-analogues of the formulas that appear in Lavoie [1987], Chu [2012], and Lavoie et al. [1992, 1996, 1994]. Note that when $u = v = 0$ we recover the series in equations (2)-(7).

To give just one example of our results, we record our Theorem 2 as follows:

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^m c \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{m} q^{2i+(m+n-i)j-\binom{j}{2}} \frac{c^{(n-i)/2}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+2\ell+n-i}a, \sqrt{c}, -\sqrt{c} \\ q, q^{1+\ell}\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^m c, q^{j-1}c \end{matrix} \,\Big|\, q\right]_j \left[\begin{matrix} q, q^{2+2\ell}a/c \\ q^{2+2\ell+2j}a, q^{1+2j}c \end{matrix} \,\Big|\, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j),
\end{aligned}
$$

where $\ell$ and $m$ are both nonnegative integers.

# 2 Extensions of $q$-Watson formulas

## 2.1 *Extensions of Andrews' $q$-Watson formula*

In this subsection, we establish several two-parameter extensions of equation (2). We begin with the following one-parameter extension.

**Proposition 1** *For two complex numbers $\{a, c\}$ and a nonnegative integer $m$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^m c \end{matrix} \,\Big|\, q; q\right] \\
&= \sum_{j=0}^{m} q^{(m+n)j-\binom{j}{2}} c^{n/2} \left[\begin{matrix} q^{-m}, q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^m c, q^{j-1}c \end{matrix} \,\Big|\, q\right]_j \\
&\times \left[\begin{matrix} q, q^2 a/c \\ q^{2+2j}a, q^{1+2j}c \end{matrix} \,\Big|\, q^2\right]_{\frac{n-j}{2}} \chi(n-j).
\end{aligned}
$$

**Proof:** Letting $a \to c/q$, $b \to q^{-k}$, $c \to -\sqrt{c}$ in the ${}_6\phi_5$-series identity (cf. page 42 of Gasper and Rahman [2004]):

$$
{}_6\phi_5\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ q^{-m} \\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ q^{1+m}a \end{matrix} \,\Big|\, q; \frac{q^{1+m}a}{bc}\right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \,\Big|\, q\right]_m, \tag{8}
$$

we get the equation

$$
\begin{aligned}
&\sum_{j=0}^{m} q^{mj+\binom{j}{2}} c^{j/2} \frac{1-q^{2j-1}c}{1-q^{j+m-1}c} \left[\begin{matrix} q^{-m} \\ q \end{matrix} \,\Big|\, q\right]_j \left[\begin{matrix} -\sqrt{c} \\ q^{j-1}c \end{matrix} \,\Big|\, q\right]_m \\
&\times \frac{(q^{k-j+1};q)_j (q^{k+j}c;q)_{m-j}}{(-q^k\sqrt{c};q)_m} = 1.
\end{aligned}
$$

Then there is the following relation

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^m c\end{matrix}\,\Big|\, q;q\right] = \sum_{k=0}^{n}\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^m c\end{matrix}\,\Big|\, q\right]_k q^k \\
&= \sum_{k=0}^{n}\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^m c\end{matrix}\,\Big|\, q\right]_k q^k \sum_{j=0}^{m} q^{mj+\binom{j}{2}} c^{j/2}\frac{1-q^{2j-1}c}{1-q^{j+m-1}c} \\
&\times \left[\begin{matrix} q^{-m} \\ q\end{matrix}\,\Big|\, q\right]_j \left[\begin{matrix} -\sqrt{c} \\ q^{j-1}c\end{matrix}\,\Big|\, q\right]_m \frac{(q^{k-j+1};q)_j(q^{k+j}c;q)_{m-j}}{(-q^k\sqrt{c};q)_m}.
\end{aligned}
$$

Interchange the summation order for the last double sum to obtain

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^m c\end{matrix}\,\Big|\, q;q\right] = \sum_{j=0}^{m} q^{mj+\binom{j}{2}} c^{j/2}\frac{1-q^{2j-1}c}{1-q^{j+m-1}c} \\
&\times \left[\begin{matrix} q^{-m} \\ q\end{matrix}\,\Big|\, q\right]_j \left[\begin{matrix} -\sqrt{c} \\ q^{j-1}c\end{matrix}\,\Big|\, q\right]_m \\
&\times \sum_{k=j}^{n}\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^m c\end{matrix}\,\Big|\, q\right]_k q^k \frac{(q^{k-j+1};q)_j(q^{k+j}c;q)_{m-j}}{(-q^k\sqrt{c};q)_m}.
\end{aligned}
$$

Shifting the index $k \to i+j$ for the sum on the last line, the result reads as

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^m c\end{matrix}\,\Big|\, q;q\right] \\
&= \sum_{j=0}^{m} q^{mj+\binom{j+1}{2}} c^{j/2}\left[\begin{matrix} q^{-m}, q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^m c, q^{j-1}c\end{matrix}\,\Big|\, q\right]_j \\
&\times\ {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^{1+n+j}a, q^j\sqrt{c}, -q^j\sqrt{c} \\ q^{1+j}\sqrt{a}, -q^{1+j}\sqrt{a}, q^{2j}c\end{matrix}\,\Big|\, q;q\right]. \qquad (9)
\end{aligned}
$$

Calculating the ${}_4\phi_3$-series on the last line by (2), we complete the proof Proposition 1. □

**Example 1 ($m=1$ in Proposition 1)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -q\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, qc\end{matrix}\,\Big|\, q;q\right] \\
&= \begin{cases} c^s\left[\begin{matrix} q, q^2a/c \\ q^2a, qc\end{matrix}\,\Big|\, q^2\right]_s, & n=2s; \\ c^{\frac{1}{2}+s}\left[\begin{matrix} q \\ qc\end{matrix}\,\Big|\, q^2\right]_{1+s}\left[\begin{matrix} q^2a/c \\ q^2a\end{matrix}\,\Big|\, q^2\right]_s, & n=1+2s. \end{cases}
\end{aligned}
$$

**Theorem 2** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^m c \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{m} q^{2i+(m+n-i)j-\binom{j}{2}} \frac{c^{(n-i)/2}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+2\ell+n-i}a, \sqrt{c}, -\sqrt{c} \\ q, q^{1+\ell}\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^m c, q^{j-1}c \end{matrix} \,\Big|\, q\right]_j \left[\begin{matrix} q, q^{2+2\ell}a/c \\ q^{2+2\ell+2j}a, q^{1+2j}c \end{matrix} \,\Big|\, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j).
\end{aligned}
$$

**Proof:** Performing the replacements $m \to \ell$, $a \to q^{-2\ell-2n-1}/a$, $b \to q^{k-n}$, $c \to q^{-\ell-n}/\sqrt{a}$ in (8), we get the equation

$$
\begin{aligned}
&\left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q^{1+k}\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \sum_{i=0}^{\ell} \frac{q^{2i}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \left[\begin{matrix} q^{-\ell}, q^{k-n}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{\ell+n+k+1}a \\ q^{\ell+n+1}a \end{matrix} \,\Big|\, q\right]_{\ell-i} = 1.
\end{aligned}
$$

Then there exists the following relation

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^v\sqrt{c} \\ q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^v c \end{matrix} \,\Big|\, q; q\right] = \sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^v\sqrt{c} \\ q, q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^v c \end{matrix} \,\Big|\, q\right]_k q^k \\
&= \sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^v\sqrt{c} \\ q, q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^v c \end{matrix} \,\Big|\, q\right]_k q^k \left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q^{1+k}\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \\
&\times \sum_{i=0}^{\ell} \frac{q^{2i}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \left[\begin{matrix} q^{-\ell}, q^{k-n}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \left[\begin{matrix} q^{\ell+n+k+1}a \\ q^{\ell+n+1}a \end{matrix} \,\Big|\, q\right]_{\ell-i} \\
&= \left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \sum_{i=0}^{\ell} \frac{q^{2i}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \left[\begin{matrix} q^{-\ell}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \\
&\times \sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^v\sqrt{c} \\ q, q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^v c \end{matrix} \,\Big|\, q\right]_k q^k \frac{(q\sqrt{a};q)_\ell (q^{k-n};q)_i}{(q^{1+k}\sqrt{a};q)_\ell} \left[\begin{matrix} q^{\ell+n+k+1}a \\ q^{\ell+n+1}a \end{matrix} \,\Big|\, q\right]_{\ell-i}.
\end{aligned}
$$

After some routine simplification, the result reads as

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^v\sqrt{c} \\ q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^v c \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \\
&\times \sum_{i=0}^{\ell} \frac{q^{2i}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \\
&\times {}_4\phi_3\left[\begin{matrix} q^{i-n}, q^{1+2\ell+n-i}a, \sqrt{c}, -q^v\sqrt{c} \\ q^{1+\ell}\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^v c \end{matrix} \,\Big|\, q; q\right]. \qquad (10)
\end{aligned}
$$

Setting $v=m$ in (10) and evaluating the ${}_4\phi_3$-series on the right hand side by Proposition 1, we finish the proof of Theorem 2. $\square$

**Example 2 ($\ell=1, m=0$ in Theorem 2)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{2+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q^2\sqrt{a}, c \end{matrix} \,\Big|\, q; q\right] \\
&= \begin{cases} \frac{(1+q\sqrt{a})c^s}{1+q^{1+2s}\sqrt{a}} \left[\begin{matrix} q, q^4a/c \\ q^2a, qc \end{matrix} \,\Big|\, q^2\right]_s, & n=2s; \\ \frac{(q^2-q)\sqrt{a}\,c^s}{(1-q\sqrt{a})(1+q^{2+2s}\sqrt{a})} \left[\begin{matrix} q^3, q^4a/c \\ q^4a, qc \end{matrix} \,\Big|\, q^2\right]_s, & n=1+2s. \end{cases}
\end{aligned}
$$

**Example 3 ($\ell=1, m=1$ in Theorem 2)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{2+n}a, \sqrt{c}, -q\sqrt{c} \\ q\sqrt{a}, -q^2\sqrt{a}, qc \end{matrix} \,\Big|\, q; q\right] \\
&= \begin{cases} \frac{(1+q\sqrt{a})(\sqrt{c}+q^{1+2s}\sqrt{a})c^s}{(\sqrt{c}+q\sqrt{a})(1+q^{1+2s}\sqrt{a})} \left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix} \,\Big|\, q^2\right]_s, & n=2s; \\ \frac{(1-q)(\sqrt{c}-q\sqrt{a})(1+q^{2+2s}\sqrt{ac})c^s}{(1-q\sqrt{a})(1-qc)(1+q^{2+2s}\sqrt{a})} \left[\begin{matrix} q^3, q^4a/c \\ q^4a, q^3c \end{matrix} \,\Big|\, q^2\right]_s, & n=1+2s. \end{cases}
\end{aligned}
$$

Replacing $\sqrt{a}$ by $-q^{-\ell}\sqrt{a}$ in Theorem 2, we obtain the equation

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1-\ell+n}a, \sqrt{c}, -q^m\sqrt{c} \\ q\sqrt{a}, -q^{1-\ell}\sqrt{a}, q^mc \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{-n}/a, -q^{-n}/\sqrt{a} \\ q^{-2n}/a, -1/\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^i q^{(\ell+2)i+(m+n-i)j-\binom{j}{2}} \frac{c^{(n-i)/2}}{a^{i/2}} \frac{1-q^{1+2n-2i}a}{1-q^{1+2n}a} \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2n-1}/a \\ q, q^{-n}/a, q^{\ell-2n}/a \end{matrix} \,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+n-i}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^mc, q^{j-1}c \end{matrix} \,\Big|\, q\right]_j \left[\begin{matrix} q, q^2a/c \\ q^{2+2j}a, q^{1+2j}c \end{matrix} \,\Big|\, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j). \qquad (11)
\end{aligned}
$$

Employing the substitution $\sqrt{c}\to -q^{-m}\sqrt{c}$ in Theorem 2, we get the formula

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+\ell+n}a, \sqrt{c}, -q^{-m}\sqrt{c} \\ q\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^{-m}c \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+\ell+n}a, q^{1+n}\sqrt{a} \\ q^{1+\ell+2n}a, q\sqrt{a} \end{matrix} \,\Big|\, q\right]_\ell \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^j q^{2i+(n-i-j)(j-m)+\binom{j+1}{2}} \frac{c^{(n-i)/2}}{a^{i/2}} \frac{1-q^{1+2\ell+2n-2i}a}{1-q^{1+2\ell+2n}a} \\
&\times \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-2n-1}/a \\ q, q^{-\ell-2n}/a, q^{-2\ell-n}/a \end{matrix} \,\Big|\, q\right]_i \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+2\ell+n-i}a, q^{-m}\sqrt{c}, -q^{-m}\sqrt{c} \\ q, q^{1+\ell}\sqrt{a}, -q^{1+\ell}\sqrt{a}, q^{-m}c, q^{j-2m-1}c \end{matrix} \,\Big|\, q\right]_j \\
&\times \left[\begin{matrix} q, q^{2+2\ell+2m}a/c \\ q^{2+2\ell+2j}a, q^{1-2m+2j}c \end{matrix} \,\Big|\, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j). \qquad (12)
\end{aligned}
$$

Letting $\sqrt{a} \to -q^{-\ell}\sqrt{a}$, $\sqrt{c} \to -q^{-m}\sqrt{c}$ in Theorem 2, we obtain the result

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1-\ell+n}a, \sqrt{c}, -q^{-m}\sqrt{c} \\ q\sqrt{a}, -q^{1-\ell}\sqrt{a}, q^{-m}c \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q^{-n}/a, -q^{-n}/\sqrt{a} \\ q^{-2n}/a, -1/\sqrt{a} \end{matrix} \Big| \, q\right]_{\ell} \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^{i+j}q^{(\ell+2)i+(n-i-j)(j-m)+\binom{j+1}{2}}\frac{c^{(n-i)/2}}{a^{i/2}}\frac{1-q^{1+2n-2i}a}{1-q^{1+2n}a} \\
&\times \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2n-1}/a \\ q, q^{-n}/a, q^{\ell-2n}/a \end{matrix} \Big| \, q\right]_i \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+n-i}a, q^{-m}\sqrt{c}, -q^{-m}\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^{-m}c, q^{j-2m-1}c \end{matrix} \Big| \, q\right]_j \\
&\times \left[\begin{matrix} q, q^{2+2m}a/c \\ q^{2+2j}a, q^{1-2m+2j}c \end{matrix} \Big| \, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j). \qquad (13)
\end{aligned}$$

**Remark**: Theorem 2, (11), (12) and (13) are equivalent to each other, but the corresponding hypergeometric series identities are essentially different.

## 2.2 *Extensions of Jain's $q$-Watson formula*

In this subsection, we prove some two-parameter extensions of equation (3). We start with the following one-parameter extension.

**Proposition 3** *For two complex numbers $\{a, c\}$ and a nonnegative integer $m$ with $m \le n$, there holds*

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{m-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{m-2n} \end{matrix} \Big| \, q; q\right] \\
&= \sum_{j=0}^{m} q^{(m-n)j+\binom{j+1}{2}} \left[\begin{matrix} q^{-m}, q^{-n}, -q^{-n}, a, c \\ q, \sqrt{qac}, -\sqrt{qac}, q^{m-2n}, q^{j-2n-1} \end{matrix} \Big| \, q\right]_j \\
&\times \left[\begin{matrix} q^{1+j}a, q^{1+j}c \\ q, q^{1+2j}ac \end{matrix} \Big| \, q^2\right]_{n-j}.
\end{aligned}$$

**Proof:** Performing the replacement $a \to q^{-1-n}a$ in (9), we have

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, \sqrt{c}, -q^{m}\sqrt{c} \\ \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{m}c \end{matrix} \Big| \, q; q\right] \\
&= \sum_{j=0}^{m} q^{mj+\binom{j+1}{2}}c^{j/2} \left[\begin{matrix} q^{-m}, q^{-n}, a, \sqrt{c}, -\sqrt{c} \\ q, \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{m}c, q^{j-1}c \end{matrix} \Big| \, q\right]_j \\
&\times {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^{j}a, q^{j}\sqrt{c}, -q^{j}\sqrt{c} \\ q^{j}\sqrt{q^{1-n}a}, -q^{j}\sqrt{q^{1-n}a}, q^{2j}c \end{matrix} \Big| \, q; q\right]. \qquad (14)
\end{aligned}$$

Replacing $c$ and $q^{-n}$ by $q^{-2n}$ and $c$, respectively, in (14), the result reads as

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a, c, q^{-n}, -q^{m-n}\\ \sqrt{qac}, -\sqrt{qac}, q^{m-2n}\end{matrix}\Big|\, q; q\right]\\
&= \sum_{j=0}^{m} q^{(m-n)j+\binom{j+1}{2}}\left[\begin{matrix}q^{-m}, q^{-n}, -q^{-n}, a, c\\ q, \sqrt{qac}, -\sqrt{qac}, q^{m-2n}, q^{j-2n-1}\end{matrix}\Big|\, q\right]_j\\
&\times\ {}_4\phi_3\left[\begin{matrix}q^j a, q^j c, q^{j-n}, -q^{j-n}\\ q^j\sqrt{qac}, -q^j\sqrt{qac}, q^{2j-2n}\end{matrix}\Big|\, q; q\right].
\end{aligned}$$

Calculating the ${}_4\phi_3$-series on the last line by (3), we establish the proposition. $\square$

**Example 4** (**$m = 1$ in Proposition 3:** $n \geq 1$)

$$ {}_4\phi_3\left[\begin{matrix}a, c, q^{-n}, -q^{1-n}\\ \sqrt{qac}, -\sqrt{qac}, q^{1-2n}\end{matrix}\Big|\, q; q\right] = \left[\begin{matrix}qa, qc\\ q, qac\end{matrix}\Big|\, q^2\right]_n + q^n\left[\begin{matrix}a, c\\ q, qac\end{matrix}\Big|\, q^2\right]_n. $$

**Proposition 4** *For two complex numbers $\{a, c\}$ and a nonnegative integer $m$, there holds*

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a, c, q^{-n}, -q^{-m-n}\\ \sqrt{qac}, -\sqrt{qac}, q^{-m-2n}\end{matrix}\Big|\, q; q\right]\\
&= \sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}-nj}\left[\begin{matrix}q^{-m}, q^{-m-n}, -q^{-m-n}, a, c\\ q, \sqrt{qac}, -\sqrt{qac}, q^{-m-2n}, q^{j-2m-2n-1}\end{matrix}\Big|\, q\right]_j\\
&\times\left[\begin{matrix}q^{1+j}a, q^{1+j}c\\ q, q^{1+2j}ac\end{matrix}\Big|\, q^2\right]_{m+n-j}.
\end{aligned}$$

**Proof:** Employing the substitution $\sqrt{c} \to -q^{-m}\sqrt{c}$ in (14), we get

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}q^{-n}, a, \sqrt{c}, -q^{-m}\sqrt{c}\\ \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{-m}c\end{matrix}\Big|\, q; q\right]\\
&= \sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}c^{j/2}\left[\begin{matrix}q^{-m}, q^{-n}, a, q^{-m}\sqrt{c}, -q^{-m}\sqrt{c}\\ q, \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{-m}c, q^{j-2m-1}c\end{matrix}\Big|\, q\right]_j\\
&\times\ {}_4\phi_3\left[\begin{matrix}q^{j-n}, q^j a, q^{j-m}\sqrt{c}, -q^{j-m}\sqrt{c}\\ q^j\sqrt{q^{1-n}a}, -q^j\sqrt{q^{1-n}a}, q^{2j-2m}c\end{matrix}\Big|\, q; q\right].
\end{aligned}$$

Letting $c \to q^{-2n}$, $q^{-n} \to c$ in the last relation, the result reads as

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a, c, q^{-n}, -q^{-m-n}\\ \sqrt{qac}, -\sqrt{qac}, q^{-m-2n}\end{matrix}\Big|\, q; q\right]\\
&= \sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}-nj}\left[\begin{matrix}q^{-m}, q^{-m-n}, -q^{-m-n}, a, c\\ q, \sqrt{qac}, -\sqrt{qac}, q^{-m-2n}, q^{j-2m-2n-1}\end{matrix}\Big|\, q\right]_j\\
&\times\ {}_4\phi_3\left[\begin{matrix}q^j a, q^j c, q^{j-m-n}, -q^{j-m-n}\\ q^j\sqrt{qac}, -q^j\sqrt{qac}, q^{2j-2m-2n}\end{matrix}\Big|\, q; q\right].
\end{aligned}$$

Evaluating the ${}_4\phi_3$-series on the last line by (3), we establish the proposition. $\square$

**Example 5 ($m=1$ in Proposition 4)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-1-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{-1-2n} \end{matrix} \Big| \, q; q\right] \\
&= \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \Big| \, q^2\right]_{n+1} - q^{n+1}\left[\begin{matrix} a, c \\ q, qac \end{matrix} \Big| \, q^2\right]_{n+1}.
\end{aligned}
$$

**Theorem 5** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$ with $m \le n$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^\ell a, c, q^{-n}, -q^{m-n} \\ \sqrt{qac}, -q^\ell\sqrt{qac}, q^{m-2n} \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q^\ell a, \sqrt{qa/c} \\ q^\ell a/c, \sqrt{qac} \end{matrix} \Big| \, q\right]_\ell \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m} q^{\frac{5i}{2}+(m-n)j+\binom{j+1}{2}} \frac{1}{(ac)^{i/2}} \frac{1-q^{2\ell-2i}a/c}{1-q^{2\ell}a/c} \left[\begin{matrix} q^{-\ell}, c, q^{-2\ell}c/a \\ q, q^{1-2\ell}/a, q^{1-\ell}c/a \end{matrix} \Big| \, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-n}, -q^{-n}, q^{2\ell-i}a, q^i c \\ q, q^\ell\sqrt{qac}, -q^\ell\sqrt{qac}, q^{m-2n}, q^{j-2n-1} \end{matrix} \Big| \, q\right]_j \left[\begin{matrix} q^{1+2\ell-i+j}a, q^{1+i+j}c \\ q, q^{1+2\ell+2j}ac \end{matrix} \Big| \, q^2\right]_{n-j}.
\end{aligned}
$$

**Proof:** Performing the replacement $a \to q^{-n-1}a$ in (10), we obtain

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^\ell a, \sqrt{c}, -q^v\sqrt{c} \\ \sqrt{q^{1-n}a}, -q^\ell\sqrt{q^{1-n}a}, q^v c \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q^\ell a, \sqrt{q^{1+n}a} \\ q^{\ell+n}a, \sqrt{q^{1-n}a} \end{matrix} \Big| \, q\right]_\ell \\
&\times \sum_{i=0}^{\ell} \frac{q^{(5+n)i/2}}{a^{i/2}} \frac{1-q^{2\ell+n-2i}a}{1-q^{2\ell+n}a} \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-n}/a \\ q, q^{1-\ell-n}/a, q^{1-2\ell}/a \end{matrix} \Big| \, q\right]_i \\
&\times {}_4\phi_3\left[\begin{matrix} q^{i-n}, q^{2\ell-i}a, \sqrt{c}, -q^v\sqrt{c} \\ q^\ell\sqrt{q^{1-n}a}, -q^\ell\sqrt{q^{1-n}a}, q^v c \end{matrix} \Big| \, q; q\right].
\end{aligned}
$$

Replacing $q^{-n}$ and $c$ by $c$ and $q^{-2n}$, respectively, in the last relation, the result reads as

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^\ell a, c, q^{-n}, -q^{v-n} \\ \sqrt{qac}, -q^\ell\sqrt{qac}, q^{v-2n} \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q^\ell a, \sqrt{qa/c} \\ q^\ell a/c, \sqrt{qac} \end{matrix} \Big| \, q\right]_\ell \\
&\times \sum_{i=0}^{\ell} \frac{q^{5i/2}}{(ac)^{i/2}} \frac{1-q^{2\ell-2i}a/c}{1-q^{2\ell}a/c} \left[\begin{matrix} q^{-\ell}, c, q^{-2\ell}c/a \\ q, q^{1-2\ell}/a, q^{1-\ell}c/a \end{matrix} \Big| \, q\right]_i \\
&\times {}_4\phi_3\left[\begin{matrix} q^{2\ell-i}a, q^i c, q^{-n}, -q^{v-n} \\ q^\ell\sqrt{qac}, -q^\ell\sqrt{qac}, q^{v-2n} \end{matrix} \Big| \, q; q\right].
\end{aligned}
\tag{15}
$$

Taking $v = m$ in (15) and calculating the ${}_4\phi_3$-series on the right hand side by Proposition 3, we establish the theorem. $\square$

**Example 6 ($\ell = 1, m = 0$ in Theorem 5)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} qa, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{q^3ac}, q^{-2n} \end{matrix} \Big| \, q; q\right] \\
&= \frac{1-qa}{(1-\sqrt{qac})(1+\sqrt{qa/c})}\left[\begin{matrix} q^3a, qc \\ q, q^3ac \end{matrix} \Big| \, q^2\right]_n \\
&+ \frac{1-c}{(1-\sqrt{qac})(1+\sqrt{c/qa})}\left[\begin{matrix} q^2a, q^2c \\ q, q^3ac \end{matrix} \Big| \, q^2\right]_n.
\end{aligned}
$$

**Example 7 ($\ell = 1, m = 1$ in Theorem 5: $n \geq 1$)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} qa, c, q^{-n}, -q^{1-n} \\ \sqrt{qac}, -\sqrt{q^3ac}, q^{1-2n} \end{matrix} \Big| \, q; q\right] \\
&= \frac{(1-\sqrt{q^{1+2n}ac})(1+\sqrt{q^{1+2n}a/c})}{(1-\sqrt{qac})(1+\sqrt{qa/c})}\left[\begin{matrix} qa, qc \\ q, q^3ac \end{matrix} \Big| \, q^2\right]_n \\
&+ \frac{(1-\sqrt{q^{1+2n}ac})(q^n+\sqrt{qa/c})}{(1-\sqrt{qac})(1+\sqrt{qa/c})}\left[\begin{matrix} q^2a, c \\ q, q^3ac \end{matrix} \Big| \, q^2\right]_n.
\end{aligned}
$$

Employing the substitution $\sqrt{a} \to -q^{-\ell}\sqrt{a}$ in Theorem 5, we obtain the equation

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-\ell}a, c, q^{-n}, -q^{m-n} \\ \sqrt{qac}, -q^{-\ell}\sqrt{qac}, q^{m-2n} \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q/a, -\sqrt{qc/a} \\ qc/a, -\sqrt{q/ac} \end{matrix} \Big| \, q\right]_\ell \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^i q^{\frac{(1+2\ell)i}{2}+(m-n)j+\binom{j+1}{2}} \frac{1}{(ac)^{i/2}} \frac{1-q^{2i}c/a}{1-c/a}\left[\begin{matrix} q^{-\ell}, c, c/a \\ q, q/a, q^{1+\ell}c/a \end{matrix} \Big| \, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-n}, -q^{-n}, q^{-i}a, q^ic \\ q, \sqrt{qac}, -\sqrt{qac}, q^{m-2n}, q^{j-2n-1} \end{matrix} \Big| \, q\right]_j \left[\begin{matrix} q^{1-i+j}a, q^{1+i+j}c \\ q, q^{1+2j}ac \end{matrix} \Big| \, q^2\right]_{n-j}.
\end{aligned} \tag{16}
$$

Setting $v = -m$ in (15) and evaluating the ${}_4\phi_3$-series on the right hand side by Proposition 4, we get the following theorem.

**Theorem 6** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{\ell}a, c, q^{-n}, -q^{-m-n} \\ \sqrt{qac}, -q^{\ell}\sqrt{qac}, q^{-m-2n} \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q^{\ell}a, \sqrt{qa/c} \\ q^{\ell}a/c, \sqrt{qac} \end{matrix} \Big| \, q\right]_\ell \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^j q^{\frac{5i}{2}-nj+\binom{j+1}{2}} \frac{1}{(ac)^{i/2}} \frac{1-q^{2\ell-2i}a/c}{1-q^{2\ell}a/c}\left[\begin{matrix} q^{-\ell}, c, q^{-2\ell}c/a \\ q, q^{1-2\ell}/a, q^{1-\ell}c/a \end{matrix} \Big| \, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-m-n}, -q^{-m-n}, q^{2\ell-i}a, q^ic \\ q, q^{\ell}\sqrt{qac}, -q^{\ell}\sqrt{qac}, q^{-m-2n}, q^{j-2m-2n-1} \end{matrix} \Big| \, q\right]_j \left[\begin{matrix} q^{1+2\ell-i+j}a, q^{1+i+j}c \\ q, q^{1+2\ell+2j}ac \end{matrix} \Big| \, q^2\right]_{m+n-j}.
\end{aligned}
$$

**Example 8 ($\ell = 1, m = 1$ in Theorem 6)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} qa, c, q^{-n}, -q^{-1-n} \\ \sqrt{qac}, -\sqrt{q^3ac}, q^{-1-2n} \end{matrix} \,\Big|\, q; q\right] \\
&= \frac{(1+\sqrt{q^{3+2n}ac})(1-\sqrt{q^{3+2n}a/c})}{(1-\sqrt{qac})(1+\sqrt{qa/c})}\left[\begin{matrix} qa, qc \\ q, q^3ac \end{matrix} \,\Big|\, q^2\right]_{1+n} \\
&- \frac{(1+\sqrt{q^{3+2n}ac})(q^{1+n}-\sqrt{qa/c})}{(1-\sqrt{qac})(1+\sqrt{qa/c})}\left[\begin{matrix} q^2a, c \\ q, q^3ac \end{matrix} \,\Big|\, q^2\right]_{1+n}.
\end{aligned}
$$

Letting $\sqrt{a} \to -q^{-\ell}\sqrt{a}$ in Theorem 6, we obtain the formula

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-\ell}a, c, q^{-n}, -q^{-m-n} \\ \sqrt{qac}, -q^{-\ell}\sqrt{qac}, q^{-m-2n} \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q/a, -\sqrt{qc/a} \\ qc/a, -\sqrt{q/ac} \end{matrix} \,\Big|\, q\right]_{\ell} \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^{i+j}q^{\frac{(1+2\ell)i}{2}-nj+\binom{j+1}{2}}\frac{1}{(ac)^{i/2}}\frac{1-q^{2i}c/a}{1-c/a}\left[\begin{matrix} q^{-\ell}, c, c/a \\ q, q/a, q^{1+\ell}c/a \end{matrix} \,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-m-n}, -q^{-m-n}, q^{-i}a, q^ic \\ q, \sqrt{qac}, -\sqrt{qac}, q^{-m-2n}, q^{j-2m-2n-1} \end{matrix} \,\Big|\, q\right]_j \left[\begin{matrix} q^{1-i+j}a, q^{1+i+j}c \\ q, q^{1+2j}ac \end{matrix} \,\Big|\, q^2\right]_{m+n-j}. \qquad (17)
\end{aligned}
$$

It should be pointed out that the corresponding hypergeometric series identities of Theorem 5 and (16) are different. This similarly applies to Theorem 6 and (17).

# 3 Extensions of $q$-Dixon formulas

## *3.1 Extensions of Bailey's $q$-Dixon formula*

**Theorem 7** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1+\ell+m-n}/ac \\ q^{1+\ell-n}/a, q^{1+m-n}/c, -ac \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+\ell-n}/a^2, -q/a \\ q^{1+\ell}/a^2, -q^{1-n}/a \end{matrix} \,\Big|\, q\right]_{\ell}\left[\begin{matrix} -a, q^{-m}c^2 \\ -ac, q^{-m}c \end{matrix} \,\Big|\, q\right]_n \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^{i+j}q^{ni+(n-i-j)(j-m)+\binom{j+1}{2}}a^ic^{n-i}\frac{1-q^{2i-2\ell-1}a^2}{1-q^{-2\ell-1}a^2} \\
&\times \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-1}a^2 \\ q, q^{n-2\ell}a^2, q^{-\ell}a^2 \end{matrix} \,\Big|\, q\right]_i \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+2\ell-n-i}/a^2, q^{-m}c, -q^{-m}c \\ q, q^{1+\ell-n}/a, -q^{1+\ell-n}/a, q^{-m}c^2, q^{j-2m-1}c^2 \end{matrix} \,\Big|\, q\right]_j \\
&\times \left[\begin{matrix} q, q^{2+2\ell+2m-2n}/a^2c^2 \\ q^{2+2\ell-2n+2j}/a^2, q^{1-2m+2j}c^2 \end{matrix} \,\Big|\, q^2\right]_{\frac{n-i-j}{2}}\chi(n-i-j).
\end{aligned}
$$

**Proof:** Using (1) with $a = c,\ b = a,\ c = -q^{1+u+v-n}/ac,\ d = q^{1+v-n}/c,\ e = q^{1+u-n}/a$, we get the relation

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1+u+v-n}/ac \\ q^{1+u-n}/a, q^{1+v-n}/c, -ac \end{matrix} \,\Big|\, q; q\right] &= \left[\begin{matrix} -a, q^{-v}c^2 \\ -ac, q^{-v}c \end{matrix} \,\Big|\, q\right]_n \\
&\times {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+u-n}/a^2, c, -q^{-v}c \\ q^{1+u-n}/a, -q^{1-n}/a, q^{-v}c^2 \end{matrix} \,\Big|\, q; q\right]. \qquad (18)
\end{aligned}
$$

Taking $u=\ell$, $v=m$ in (18) and calculating the ${}_4\phi_3$-series on the right hand side by (12), we establish the theorem. $\square$

**Example 9 ($\ell=0, m=1$ in Theorem 7)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{2-n}/ac \\ q^{1-n}/a, q^{2-n}/c, -ac \end{matrix}\,\Big|\, q; q\right]\\
&= \begin{cases} \left[\begin{matrix} -a, c^2/q \\ -ac, c/q \end{matrix}\,\Big|\, q\right]_{2s} \left[\begin{matrix} q, q^{2s-2}a^2c^2 \\ c^2/q, q^{2s}a^2 \end{matrix}\,\Big|\, q^2\right]_s, & n=2s; \\ \frac{(q-1)c}{q-c^2}\left[\begin{matrix} -a, c^2/q \\ -ac, c/q \end{matrix}\,\Big|\, q\right]_{1+2s} \left[\begin{matrix} q^3, q^{2s}a^2c^2 \\ qc^2, q^{2+2s}a^2 \end{matrix}\,\Big|\, q^2\right]_s, & n=1+2s. \end{cases}
\end{aligned}
$$

**Example 10 ($\ell=1, m=1$ in Theorem 7)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{3-n}/ac \\ q^{2-n}/a, q^{2-n}/c, -ac \end{matrix}\,\Big|\, q; q\right]\\
&= \begin{cases} \frac{(q^2+ac)(q^2-q^{2s}a^2)}{(q^2-a^2)(q^2+q^{2s}ac)}\left[\begin{matrix} -a/q, c^2/q \\ -ac, c/q \end{matrix}\,\Big|\, q\right]_{2s} \left[\begin{matrix} q, q^{2s-2}a^2c^2 \\ c^2/q, q^{2s-2}a^2 \end{matrix}\,\Big|\, q^2\right]_s, & n=2s; \\ \frac{(a+c)(q-1)(q-q^{2s}ac)}{(q^2-a^2)(1-c^2/q^2)}\left[\begin{matrix} -a/q, c^2/q^2 \\ -ac, c/q \end{matrix}\,\Big|\, q\right]_{1+2s} \left[\begin{matrix} q^3, q^{2s-2}a^2c^2 \\ c^2/q, q^{2s}a^2 \end{matrix}\,\Big|\, q^2\right]_s, & n=1+2s. \end{cases}
\end{aligned}
$$

Setting $u=\ell$, $v=-m$ in (18) and evaluating the ${}_4\phi_3$-series on the right hand side by Theorem 2, we obtain the following theorem.

**Theorem 8** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1+\ell-m-n}/ac \\ q^{1+\ell-n}/a, q^{1-m-n}/c, -ac \end{matrix}\,\Big|\, q; q\right] = \left[\begin{matrix} q^{1+\ell-n}/a^2, -q/a \\ q^{1+\ell}/a^2, -q^{1-n}/a \end{matrix}\,\Big|\, q\right]_\ell \left[\begin{matrix} -a, q^m c^2 \\ -ac, q^m c \end{matrix}\,\Big|\, q\right]_n\\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^i q^{ni+(m+n-i)j-\binom{j}{2}} a^i c^{n-i}\frac{1-q^{2i-2\ell-1}a^2}{1-q^{-2\ell-1}a^2}\\
&\times \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-2\ell-1}a^2 \\ q, q^{n-2\ell}a^2, q^{-\ell}a^2 \end{matrix}\,\Big|\, q\right]_i \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1+2\ell-n-i}/a^2, c, -c \\ q, q^{1+\ell-n}/a, -q^{1+\ell-n}/a, q^m c^2, q^{j-1}c^2 \end{matrix}\,\Big|\, q\right]_j\\
&\times \left[\begin{matrix} q, q^{2+2\ell-2n}/a^2c^2 \\ q^{2+2\ell-2n+2j}/a^2, q^{1+2j}c^2 \end{matrix}\,\Big|\, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j).
\end{aligned}
$$

**Example 11** (**$\ell = 0, m = 1$ in Theorem 8**)

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{-n}/ac \\ q^{1-n}/a, q^{-n}/c, -ac \end{matrix} \,\Big|\, q; q\right]
$$
$$
= \begin{cases} \left[\begin{matrix} -a, qc^2 \\ -ac, qc \end{matrix} \,\Big|\, q\right]_{2s} \left[\begin{matrix} q, q^{2s}a^2c^2 \\ qc^2, q^{2s}a^2 \end{matrix} \,\Big|\, q^2\right]_s, & n = 2s; \\ \frac{(1-q)c}{1-qc^2} \left[\begin{matrix} -a, qc^2 \\ -ac, qc \end{matrix} \,\Big|\, q\right]_{1+2s} \left[\begin{matrix} q^3, q^{2+2s}a^2c^2 \\ q^3c^2, q^{2+2s}a^2 \end{matrix} \,\Big|\, q^2\right]_s, & n = 1+2s. \end{cases}
$$

**Example 12** (**$\ell = 1, m = 1$ in Theorem 8**)

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1-n}/ac \\ q^{2-n}/a, q^{-n}/c, -ac \end{matrix} \,\Big|\, q; q\right]
$$
$$
= \begin{cases} \frac{(q^2-a^2c^2)(1-q^{2s-2}a^2)}{(q^2-a^2)(1-q^{2s-2}a^2c^2)} \left[\begin{matrix} -a/q, qc^2 \\ -ac/q, qc \end{matrix} \,\Big|\, q\right]_{2s} \left[\begin{matrix} q, q^{2s-2}a^2c^2 \\ qc^2, q^{2s-2}a^2 \end{matrix} \,\Big|\, q^2\right]_s, & n = 2s; \\ \frac{(1-q)(qc-a)}{(q-a)(1-qc)} \left[\begin{matrix} -a, qc^2 \\ -ac, q^2c \end{matrix} \,\Big|\, q\right]_{2s} \left[\begin{matrix} q^3, q^{2s}a^2c^2 \\ qc^2, q^{2s}a^2 \end{matrix} \,\Big|\, q^2\right]_s, & n = 1+2s. \end{cases}
$$

Taking $u = -\ell$, $v = -m$ in (18) and calculating the ${}_4\phi_3$-series on the right hand side by (11), we get the following theorem.

**Theorem 9** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$, there holds*

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{1-\ell-m-n}/ac \\ q^{1-\ell-n}/a, q^{1-m-n}/c, -ac \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} a, q^na^2 \\ a^2, q^na \end{matrix} \,\Big|\, q\right]_\ell \left[\begin{matrix} -a, q^mc^2 \\ -ac, q^mc \end{matrix} \,\Big|\, q\right]_n
$$
$$
\times \sum_{i=0}^{\ell}\sum_{j=0}^{m} q^{(\ell+n)i+(m+n-i)j-\binom{j}{2}} a^i c^{n-i} \frac{1-q^{2i-1}a^2}{1-q^{-1}a^2}
$$
$$
\times \left[\begin{matrix} q^{-\ell}, q^{-n}, q^{-1}a^2 \\ q, q^na^2, q^\ell a^2 \end{matrix} \,\Big|\, q\right]_i \left[\begin{matrix} q^{-m}, q^{i-n}, q^{1-n-i}/a^2, c, -c \\ q, q^{1-n}/a, -q^{1-n}/a, q^mc^2, q^{j-1}c^2 \end{matrix} \,\Big|\, q\right]_j
$$
$$
\times \left[\begin{matrix} q, q^{2-2n}/a^2c^2 \\ q^{2-2n+2j}/a^2, q^{1+2j}c^2 \end{matrix} \,\Big|\, q^2\right]_{\frac{n-i-j}{2}} \chi(n-i-j).
$$

**Example 13** (**$\ell = 1, m = 1$ in Theorem 9**)

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c, -q^{-1-n}/ac \\ q^{-n}/a, q^{-n}/c, -ac \end{matrix} \,\Big|\, q; q\right]
$$
$$
= \begin{cases} \left[\begin{matrix} -qa, qc^2 \\ -qac, qc \end{matrix} \,\Big|\, q\right]_{2s} \left[\begin{matrix} q, q^{2s+2}a^2c^2 \\ qc^2, q^{2s+2}a^2 \end{matrix} \,\Big|\, q^2\right]_s, & n = 2s; \\ \frac{a+c}{1+ac} \left[\begin{matrix} -qa, qc^2 \\ -qac, qc \end{matrix} \,\Big|\, q\right]_{1+2s} \left[\begin{matrix} q, q^{2s+2}a^2c^2 \\ qc^2, q^{2s+2}a^2 \end{matrix} \,\Big|\, q^2\right]_{1+s}, & n = 1+2s. \end{cases}
$$

Letting $u=-\ell,\ v=m$ in (18) and evaluating the ${}_4\phi_3$-series on the right hand side by (13), we can derive summation formula for the following series:

$$ {}_4\phi_3\left[\begin{matrix}q^{-n},a,c,-q^{1-\ell+m-n}/ac\\ q^{1-\ell-n}/a,q^{1+m-n}/c,-ac\end{matrix}\Big|\,q;q\right], $$

which is equivalent to Theorem 8. The corresponding concrete result has been omitted.

## 3.2 Extensions of another $q$-Dixon formula

**Theorem 10** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$, there holds*

$$ \begin{aligned} {}_4\phi_3&\left[\begin{matrix}a,c,q^{-n},-q^{1+\ell+m+n}a/c\\ q^{1+\ell}a/c,q^{1+m+n}a,-q^{-n}c\end{matrix}\Big|\,q;q\right]=\left[\begin{matrix}a,-q^{-\ell}c\\ q^{-\ell-1}c^2,-qa/c\end{matrix}\Big|\,q\right]_\ell\left[\begin{matrix}q^{1+m+n},-qa/c\\ q^{1+m+n}a,-q/c\end{matrix}\Big|\,q\right]_n\\ &\times\sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^{i+j}q^{\binom{j+1}{2}-nj}\left(\frac{c}{a}\right)^i\frac{1-q^{1+2i}/c^2}{1-q/c^2}\left[\begin{matrix}q^{-\ell},q^{1+\ell}a/c^2,q/c^2\\ q,q^{1-\ell}/a,q^{2+\ell}/c^2\end{matrix}\Big|\,q\right]_i\\ &\times\left[\begin{matrix}q^{-m},q^{-m-n},-q^{-m-n},q^{\ell-i}a,q^{1+\ell+i}a/c^2\\ q,q^{1+\ell}a/c,-q^{1+\ell}a/c,q^{-m-2n},q^{j-2m-2n-1}\end{matrix}\Big|\,q\right]_j\\ &\times\left[\begin{matrix}q^{1+\ell-i+j}a,q^{2+\ell+i+j}a/c^2\\ q,q^{2+2\ell+2j}a^2/c^2\end{matrix}\Big|\,q^2\right]_{m+n-j}. \end{aligned} $$

**Proof:** Utilizing (1) with $a=a,\ b=c,\ c=-q^{1+u+v+n}a/c,\ d=q^{1+v+n}a,\ e=q^{1+u}a/c$, we obtain the relation

$$ \begin{aligned} {}_4\phi_3\left[\begin{matrix}a,c,q^{-n},-q^{1+u+v+n}a/c\\ q^{1+u}a/c,q^{1+v+n}a,-q^{-n}c\end{matrix}\Big|\,q;q\right]&=\left[\begin{matrix}q^{1+v+n},-qa/c\\ q^{1+v+n}a,-q/c\end{matrix}\Big|\,q\right]_n\\ &\quad\times{}_4\phi_3\left[\begin{matrix}a,q^{1+u}a/c^2,q^{-n},-q^{-v-n}\\ q^{1+u}a/c,-qa/c,q^{-v-2n}\end{matrix}\Big|\,q;q\right]. \end{aligned} \tag{19} $$

Setting $u=\ell,\ v=m$ in (19) and evaluating the ${}_4\phi_3$-series on the right hand side by Theorem 6, we establish the theorem. □

**Example 14 ($\ell=0,m=1$ in Theorem 10)**

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix}a,c,q^{-n},-q^{2+n}a/c\\ qa/c,q^{2+n}a,-q^{-n}c\end{matrix}\Big|\,q;q\right]\\ &=\left[\begin{matrix}q^{2+n},-qa/c\\ q^{2+n}a,-q/c\end{matrix}\Big|\,q\right]_n\left[\begin{matrix}qa,q^2a/c^2\\ q,q^2a^2/c^2\end{matrix}\Big|\,q^2\right]_{1+n}\\ &-\ q^{1+n}\left[\begin{matrix}q^{2+n},-qa/c\\ q^{2+n}a,-q/c\end{matrix}\Big|\,q\right]_n\left[\begin{matrix}a,qa/c^2\\ q,q^2a^2/c^2\end{matrix}\Big|\,q^2\right]_{1+n}. \end{aligned} $$

**Example 15 ($\ell = 1, m = 0$ in Theorem 10)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{2+n}a/c \\ q^2a/c, q^{1+n}a, -q^{-n}c \end{matrix}\middle|\, q; q\right] \\
&= \frac{qc(1-a)}{(qa+c)(q-c)}\left[\begin{matrix} q^{1+n}, -qa/c \\ q^{1+n}a, -q/c \end{matrix}\middle|\, q\right]_n\left[\begin{matrix} q^2a, q^3a/c^2 \\ q, q^4a^2/c^2 \end{matrix}\middle|\, q^2\right]_n \\
&+ \frac{q^2a-c^2}{(qa+c)(q-c)}\left[\begin{matrix} q^{1+n}, -qa/c \\ q^{1+n}a, -q/c \end{matrix}\middle|\, q\right]_n\left[\begin{matrix} qa, q^4a/c^2 \\ q, q^4a^2/c^2 \end{matrix}\middle|\, q^2\right]_n.
\end{aligned}
$$

**Example 16 ($\ell = 1, m = 1$ in Theorem 10)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{3+n}a/c \\ q^2a/c, q^{2+n}a, -q^{-n}c \end{matrix}\middle|\, q; q\right] \\
&= \frac{q(1+q^nc)(c-q^{2+n}a)}{(qa+c)(q-c)}\left[\begin{matrix} q^{2+n}, -qa/c \\ q^{2+n}a, -q/c \end{matrix}\middle|\, q\right]_n\left[\begin{matrix} a, q^3a/c^2 \\ q, q^4a^2/c^2 \end{matrix}\middle|\, q^2\right]_{1+n} \\
&- \frac{(c+q^{2+n})(c-q^{2+n}a)}{(qa+c)(q-c)}\left[\begin{matrix} q^{2+n}, -qa/c \\ q^{2+n}a, -q/c \end{matrix}\middle|\, q\right]_n\left[\begin{matrix} qa, q^2a/c^2 \\ q, q^4a^2/c^2 \end{matrix}\middle|\, q^2\right]_{1+n}.
\end{aligned}
$$

Taking $u = \ell$, $v = -m$ in (19) and calculating the ${}_4\phi_3$-series on the right hand side by Theorem 5, we get the following theorem.

**Theorem 11** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$ with $m \leq n$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1+\ell-m+n}a/c \\ q^{1+\ell}a/c, q^{1-m+n}a, -q^{-n}c \end{matrix}\middle|\, q; q\right] = \left[\begin{matrix} a, -q^{-\ell}c \\ q^{-\ell-1}c^2, -qa/c \end{matrix}\middle|\, q\right]_\ell\left[\begin{matrix} q^{1-m+n}, -qa/c \\ q^{1-m+n}a, -q/c \end{matrix}\middle|\, q\right]_n \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^i q^{(m-n)j+\binom{j+1}{2}}\left(\frac{c}{a}\right)^i\frac{1-q^{1+2i}/c^2}{1-q/c^2}\left[\begin{matrix} q^{-\ell}, q^{1+\ell}a/c^2, q/c^2 \\ q, q^{1-\ell}/a, q^{2+\ell}/c^2 \end{matrix}\middle|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-n}, -q^{-n}, q^{\ell-i}a, q^{1+\ell+i}a/c^2 \\ q, q^{1+\ell}a/c, -q^{1+\ell}a/c, q^{m-2n}, q^{j-2n-1} \end{matrix}\middle|\, q\right]_j\left[\begin{matrix} q^{1+\ell-i+j}a, q^{2+\ell+i+j}a/c^2 \\ q, q^{2+2\ell+2j}a^2/c^2 \end{matrix}\middle|\, q^2\right]_{n-j}.
\end{aligned}
$$

**Example 17 ($\ell = 0, m = 1$ in Theorem 11: $n \geq 1$)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{n}a/c \\ qa/c, q^{n}a, -q^{-n}c \end{matrix}\middle|\, q; q\right] \\
&= \left[\begin{matrix} q^{n}, -qa/c \\ q^{n}a, -q/c \end{matrix}\middle|\, q\right]_n\left[\begin{matrix} qa, q^2a/c^2 \\ q, q^2a^2/c^2 \end{matrix}\middle|\, q^2\right]_n \\
&+ q^n\left[\begin{matrix} q^{n}, -qa/c \\ q^{n}a, -q/c \end{matrix}\middle|\, q\right]_n\left[\begin{matrix} a, qa/c^2 \\ q, q^2a^2/c^2 \end{matrix}\middle|\, q^2\right]_n.
\end{aligned}
$$

**Example 18** ($\ell=1, m=1$ **in Theorem 11:** $n\geq 1$)

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1+n}a/c \\ q^2a/c, q^na, -q^{-n}c \end{matrix}\,\Big|\, q; q\right] \\
&= \frac{q-q^nc}{q-c}\left[\begin{matrix} q^n, -q^2a/c \\ q^na, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} a, q^3a/c^2 \\ q, q^4a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_n \\
&+ \frac{q^{1+n}-c}{q-c}\left[\begin{matrix} q^n, -q^2a/c \\ q^na, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} qa, q^2a/c^2 \\ q, q^4a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_n.
\end{aligned}$$

Setting $u=-\ell$, $v=m$ in (19) and evaluating the ${}_4\phi_3$-series on the right hand side by (17), we obtain the following theorem.

**Theorem 12** *For two complex numbers* $\{a, c\}$ *and two nonnegative integers* $\{\ell, m\}$*, there holds*

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1-\ell+m+n}a/c \\ q^{1-\ell}a/c, q^{1+m+n}a, -q^{-n}c \end{matrix}\,\Big|\, q; q\right] = \left[\begin{matrix} a, c \\ q^{\ell-1}c^2, q^{1-\ell}a/c \end{matrix}\,\Big|\, q\right]_\ell \left[\begin{matrix} q^{1+m+n}, -qa/c \\ q^{1+m+n}a, -q/c \end{matrix}\,\Big|\, q\right]_n \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^j q^{\ell i-nj+\binom{j+1}{2}}\left(\frac{c}{a}\right)^i \frac{1-q^{1-2\ell+2i}/c^2}{1-q^{1-2\ell}/c^2}\left[\begin{matrix} q^{-\ell}, q^{1-\ell}a/c^2, q^{1-2\ell}/c^2 \\ q, q^{1-\ell}/a, q^{2-\ell}/c^2 \end{matrix}\,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-m-n}, -q^{-m-n}, q^{\ell-i}a, q^{1-\ell+i}a/c^2 \\ q, qa/c, -qa/c, q^{-m-2n}, q^{j-2m-2n-1} \end{matrix}\,\Big|\, q\right]_j \left[\begin{matrix} q^{1+\ell-i+j}a, q^{2-\ell+i+j}a/c^2 \\ q, q^{2+2j}a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_{m+n-j}.
\end{aligned}$$

**Example 19** ($\ell=1, m=0$ **in Theorem 12**)

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^na/c \\ a/c, q^{1+n}a, -q^{-n}c \end{matrix}\,\Big|\, q; q\right] \\
&= \frac{(1-a)c}{(1+c)(c-a)}\left[\begin{matrix} q^{1+n}, -qa/c \\ q^{1+n}a, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} q^2a, qa/c^2 \\ q, q^2a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_n \\
&+ \frac{c^2-a}{(1+c)(c-a)}\left[\begin{matrix} q^{1+n}, -qa/c \\ q^{1+n}a, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} qa, q^2a/c^2 \\ q, q^2a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_n.
\end{aligned}$$

**Example 20** ($\ell=1, m=1$ **in Theorem 12**)

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1+n}a/c \\ a/c, q^{2+n}a, -q^{-n}c \end{matrix}\,\Big|\, q; q\right] \\
&= \frac{(qa+c)(1-q^{1+n}c)}{(1+c)(c-a)}\left[\begin{matrix} q^{2+n}, -q^2a/c \\ q^{2+n}a, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} a, qa/c^2 \\ q, q^2a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_{1+n} \\
&+ \frac{(qa+c)(c-q^{1+n})}{(1+c)(c-a)}\left[\begin{matrix} q^{2+n}, -q^2a/c \\ q^{2+n}a, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} qa, a/c^2 \\ q, q^2a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_{1+n}.
\end{aligned}$$

Taking $u=-\ell$, $v=-m$ in (19) and calculating the ${}_4\phi_3$-series on the right hand side by (16), we get the following theorem.

**Theorem 13** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$ with $m \leq n$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{1-\ell-m+n}a/c \\ q^{1-\ell}a/c, q^{1-m+n}a, -q^{-n}c \end{matrix}\,\Big|\, q;q\right] = \left[\begin{matrix} a, c \\ q^{\ell-1}c^2, q^{1-\ell}a/c \end{matrix}\,\Big|\, q\right]_\ell \left[\begin{matrix} q^{1-m+n}, -qa/c \\ q^{1-m+n}a, -q/c \end{matrix}\,\Big|\, q\right]_n \\
&\times \sum_{i=0}^{\ell}\sum_{j=0}^{m} q^{\ell i+(m-n)j+\binom{j+1}{2}}\Big(\frac{c}{a}\Big)^i \frac{1-q^{1-2\ell+2i}/c^2}{1-q^{1-2\ell}/c^2} \left[\begin{matrix} q^{-\ell}, q^{1-\ell}a/c^2, q^{1-2\ell}/c^2 \\ q, q^{1-\ell}/a, q^{2-\ell}/c^2 \end{matrix}\,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-n}, -q^{-n}, q^{\ell-i}a, q^{1-\ell+i}a/c^2 \\ q, qa/c, -qa/c, q^{m-2n}, q^{j-2n-1} \end{matrix}\,\Big|\, q\right]_j \left[\begin{matrix} q^{1+\ell-i+j}a, q^{2-\ell+i+j}a/c^2 \\ q, q^{2+2j}a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_{n-j}.
\end{aligned}
$$

**Example 21 ($\ell = 1, m = 1$ in Theorem 13: $n \geq 1$)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{n-1}a/c \\ a/c, q^n a, -q^{-n}c \end{matrix}\,\Big|\, q;q\right] \\
&= \left[\begin{matrix} q^n, -a/c \\ q^n a, -1/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} qa, a/c^2 \\ q, a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_n \\
&+ \frac{1+q^n c}{1+c}\left[\begin{matrix} q^n, -a/c \\ q^n a, -q/c \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} a, qa/c^2 \\ q, a^2/c^2 \end{matrix}\,\Big|\, q^2\right]_n.
\end{aligned}
$$

# 4 Extensions of $q$-Whipple formulas

## 4.1 Extensions of Andrews' $q$-Whipple formula

**Theorem 14** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+\ell+n}, \sqrt{qac}, -q^m\sqrt{qac} \\ -q, q^{1+\ell+m}a, qc \end{matrix}\,\Big|\, q;q\right] = \left[\begin{matrix} q^{-n}\sqrt{qa/c}, -q^{m-n}\sqrt{qa/c}, q^{1+\ell+n} \\ q^{1+\ell+m}a, q^{-n}/c, -q \end{matrix}\,\Big|\, q\right]_n \\
&\times \left[\begin{matrix} q^{m-n}a, \sqrt{qac} \\ q^m ac, q^{-n}\sqrt{qa/c} \end{matrix}\,\Big|\, q\right]_m \sum_{i=0}^{\ell}\sum_{j=0}^{m}(-1)^i q^{(n+\frac{5}{2})j-ni+\binom{i+1}{2}}\Big(\frac{c}{a}\Big)^{\frac{j}{2}}\frac{1-q^{2m-2j}ac}{1-q^{2m}ac} \\
&\times \left[\begin{matrix} q^{-\ell}, q^{-\ell-n}, -q^{-\ell-n}, q^{2m-n-j}a, q^{j-n}/c \\ q, q^{m-n}\sqrt{qa/c}, -q^{m-n}\sqrt{qa/c}, q^{-\ell-2n}, q^{i-2\ell-2n-1} \end{matrix}\,\Big|\, q\right]_i \left[\begin{matrix} q^{-m}, q^{-n}/c, q^{-2m}/ac \\ q, q^{1-2m+n}/a, q^{1-m}/ac \end{matrix}\,\Big|\, q\right]_j \\
&\times \left[\begin{matrix} q^{1+2m-n+i-j}a, q^{1-n+i+j}/c \\ q, q^{1+2m-2n+2i}a/c \end{matrix}\,\Big|\, q^2\right]_{\ell+n-i}.
\end{aligned}
$$

**Proof:** The iteration of (1) produces the transformation formula:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, q^{1-n}abc/de \end{matrix}\,\Big|\, q;q\right] = \left[\begin{matrix} c, de/ac, de/bc \\ d, e, de/abc \end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, d/c, e/c, de/abc \\ de/ac, de/bc, q^{1-n}/c \end{matrix}\,\Big|\, q;q\right].
$$

Using the last equation with $a = \sqrt{qac},\ b = -q^v\sqrt{qac},\ c = q^{1+n+u},\ d = q^{1+u+v}a,\ e = -q$, we obtain the relation

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n+u}, \sqrt{qac}, -q^{v}\sqrt{qac} \\ -q, q^{1+u+v}a, qc \end{matrix} \Big| \, q; q\right] \\
&= \left[\begin{matrix} q^{-n}\sqrt{qa/c}, -q^{v-n}\sqrt{qa/c}, q^{1+n+u} \\ q^{1+u+v}a, q^{-n}/c, -q \end{matrix} \Big| \, q\right]_n \\
&\times {}_4\phi_3\left[\begin{matrix} q^{v-n}a, q^{-n}/c, q^{-n}, -q^{-u-n} \\ q^{-n}\sqrt{qa/c}, -q^{v-n}\sqrt{qa/c}, q^{-u-2n} \end{matrix} \Big| \, q; q\right]. \qquad (20)
\end{aligned}
$$

Setting $u = \ell$, $v = m$ in (20) and evaluating the ${}_4\phi_3$-series on the right hand side by Theorem 6, we establish the theorem. □

**Example 22** (**$\ell = 0, m = 1$ in Theorem 14)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}, \sqrt{qac}, -\sqrt{q^3ac} \\ -q, q^2a, qc \end{matrix} \Big| \, q; q\right] \\
&= \frac{q^{\binom{1+n}{2}}}{(1+\sqrt{qac})(1-\sqrt{qa/c})} \frac{(q^{1-n}a; q^2)_{1+n}(q^{1-n}c; q^2)_n}{(q^2a; q)_n(qc; q)_n} \\
&+ \frac{q^{\binom{1+n}{2}}}{(1+\sqrt{qac})(1-\sqrt{c/qa})} \frac{(q^{2-n}a; q^2)_n(q^{-n}c; q^2)_{1+n}}{(q^2a; q)_n(qc; q)_n}.
\end{aligned}
$$

**Example 23** (**$\ell = 1, m = 0$ in Theorem 14)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{2+n}, \sqrt{qac}, -\sqrt{qac} \\ -q, q^2a, qc \end{matrix} \Big| \, q; q\right] \\
&= \frac{q^{\binom{2+n}{2}}}{(qa-c)(1-q^{1+n})} \frac{(q^{1-n}a; q^2)_{1+n}(q^{-1-n}c; q^2)_{1+n}}{(q^2a; q)_n(qc; q)_n} \\
&- \frac{q^{\binom{2+n}{2}}}{(qa-c)(1-q^{1+n})} \frac{(q^{-n}a; q^2)_{1+n}(q^{-n}c; q^2)_{1+n}}{(q^2a; q)_n(qc; q)_n}.
\end{aligned}
$$

**Example 24** (**$\ell = 1, m = 1$ in Theorem 14)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{2+n}, \sqrt{qac}, -\sqrt{q^3ac} \\ -q, q^3a, qc \end{matrix} \Big| \, q; q\right] \\
&= \frac{q^{\binom{2+n}{2}}(1-\sqrt{q^{-1-2n}ac})}{(\sqrt{qa}-\sqrt{c})(\sqrt{q^3a}-\sqrt{c})(1+\sqrt{qac})(1-q^{1+n})} \frac{(q^{2-n}a; q^2)_{1+n}(q^{-n}c; q^2)_{1+n}}{(q^3a; q)_n(qc; q)_n} \\
&- \frac{q^{\binom{2+n}{2}}(1-\sqrt{q^{3+2n}ac})}{(\sqrt{qa}-\sqrt{c})(\sqrt{q^3a}-\sqrt{c})(1+\sqrt{qac})(1-q^{1+n})} \frac{(q^{1-n}a; q^2)_{1+n}(q^{-1-n}c; q^2)_{1+n}}{(q^3a; q)_n(qc; q)_n}.
\end{aligned}
$$

Taking $u = -\ell$, $v = m$ in (20) and calculating the ${}_4\phi_3$-series on the right hand side by Theorem 5, we get the following theorem.

**Theorem 15** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$ with $\ell\le n$, there holds*

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix}q^{-n},q^{1-\ell+n},\sqrt{qac},-q^m\sqrt{qac}\\ -q,q^{1-\ell+m}a,qc\end{matrix}\Big|\,q;q\right]&=\left[\begin{matrix}q^{-n}\sqrt{qa/c},-q^{m-n}\sqrt{qa/c},q^{1-\ell+n}\\ q^{1-\ell+m}a,q^{-n}/c,-q\end{matrix}\Big|\,q\right]_n\\
\times\left[\begin{matrix}q^{m-n}a,\sqrt{qac}\\ q^m ac,q^{-n}\sqrt{qa/c}\end{matrix}\Big|\,q\right]_m&\sum_{i=0}^{\ell}\sum_{j=0}^{m}q^{(\ell-n)i+\binom{i+1}{2}+(n+\frac{5}{2})j}\Big(\frac{c}{a}\Big)^{\frac{j}{2}}\frac{1-q^{2m-2j}ac}{1-q^{2m}ac}\\
\times\left[\begin{matrix}q^{-\ell},q^{-n},-q^{-n},q^{2m-n-j}a,q^{j-n}/c\\ q,q^{m-n}\sqrt{qa/c},-q^{m-n}\sqrt{qa/c},q^{\ell-2n},q^{i-2n-1}\end{matrix}\Big|\,q\right]_i&\left[\begin{matrix}q^{-m},q^{-n}/c,q^{-2m}/ac\\ q,q^{1-2m+n}/a,q^{1-m}/ac\end{matrix}\Big|\,q\right]_j\\
\times\left[\begin{matrix}q^{1+2m-n+i-j}a,q^{1-n+i+j}/c\\ q,q^{1+2m-2n+2i}a/c\end{matrix}\Big|\,q^2\right]_{n-i}&.
\end{aligned}
$$

**Example 25 ($\ell=1,m=0$ in Theorem 15)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}q^{-n},q^{n},\sqrt{qac},-\sqrt{qac}\\ -q,a,qc\end{matrix}\Big|\,q;q\right]\\
&=\frac{q^{\binom{1+n}{2}}}{1+q^n}\frac{(q^{1-n}a;q^2)_n(q^{1-n}c;q^2)_n}{(a;q)_n(qc;q)_n}\\
&+\frac{q^{\binom{1+n}{2}}}{1+q^n}\frac{(q^{-n}a;q^2)_n(q^{2-n}c;q^2)_n}{(a;q)_n(qc;q)_n}.
\end{aligned}
$$

**Example 26 ($\ell=1,m=1$ in Theorem 15)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}q^{-n},q^{n},\sqrt{qac},-\sqrt{q^3ac}\\ -q,qa,qc\end{matrix}\Big|\,q;q\right]\\
&=\frac{q^{\binom{1+n}{2}}(1+\sqrt{q^{1+2n}ac})}{(1+q^n)(1+\sqrt{qac})}\frac{(q^{1-n}a;q^2)_n(q^{1-n}c;q^2)_n}{(qa;q)_n(qc;q)_n}\\
&+\frac{q^{\binom{1+n}{2}}(1+\sqrt{q^{1-2n}ac})}{(1+q^n)(1+\sqrt{qac})}\frac{(q^{2-n}a;q^2)_n(q^{2-n}c;q^2)_n}{(qa;q)_n(qc;q)_n}.
\end{aligned}
$$

Performing the replacement $\sqrt{a}\to -q^{-m}\sqrt{a}$ in Theorems 14 and 15, we can derive summation formulas for the following two series:

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}q^{-n},q^{1+\ell+n},\sqrt{qac},-q^{-m}\sqrt{qac}\\ -q,q^{1+\ell-m}a,qc\end{matrix}\Big|\,q;q\right],\\
&{}_4\phi_3\left[\begin{matrix}q^{-n},q^{1-\ell+n},\sqrt{qac},-q^{-m}\sqrt{qac}\\ -q,q^{1-\ell-m}a,qc\end{matrix}\Big|\,q;q\right].
\end{aligned}
$$

The corresponding concrete results will not be displayed here.

## 4.2 Extensions of Jain's $q$-Whipple formula

**Theorem 16** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$ with $m\le n$, there holds*

$$\begin{aligned}
{}_4\phi_3\left[\begin{matrix}a,q^{1+\ell}/a,q^{-n},-q^{m-n}\\ c,q^{1+\ell+m-2n}/c,-q\end{matrix}\Big|\,q;q\right] &= \left[\begin{matrix}q^2/ac,-q/a\\ q^2/a^2,-q/c\end{matrix}\Big|\,q\right]_\ell\left[\begin{matrix}q^{1-m+n},-q^{-\ell}c\\ q^{n-m-\ell}c,-q\end{matrix}\Big|\,q\right]_n\\
&\times\sum_{i=0}^{\ell}\sum_{j=0}^{m}q^{(\ell+1)i+(m-n)j+\binom{j+1}{2}}\frac{(-1)^i}{c^i}\frac{1-q^{1+2i}/a^2}{1-q/a^2}\left[\begin{matrix}q^{-\ell},c/a,q/a^2\\ q,q^2/ac,q^{2+\ell}/a^2\end{matrix}\Big|\,q\right]_i\\
&\times\left[\begin{matrix}q^{-m},q^{-n},-q^{-n},q^{-i-1}ac,q^ic/a\\ q,c,-c,q^{m-2n},q^{j-2n-1}\end{matrix}\Big|\,q\right]_j\left[\begin{matrix}q^{j-i}ac,q^{1+i+j}c/a\\ q,q^{2j}c^2\end{matrix}\Big|\,q^2\right]_{n-j}.
\end{aligned}$$

**Proof:** Utilizing (1) with $a=-q^{v-n},\ b=a,\ c=q^{1+u}/a,\ d=-q,\ e=c$, we obtain the relation

$$\begin{aligned}
{}_4\phi_3\left[\begin{matrix}a,q^{1+u}/a,q^{-n},-q^{v-n}\\ c,q^{1+u+v-2n}/c,-q\end{matrix}\Big|\,q;q\right] &= \left[\begin{matrix}q^{1-v+n},-q^{-u}c\\ q^{n-u-v}c,-q\end{matrix}\Big|\,q\right]_n\\
&\times{}_4\phi_3\left[\begin{matrix}q^{-u-1}ac,c/a,q^{-n},-q^{v-n}\\ c,-q^{-u}c,q^{v-2n}\end{matrix}\Big|\,q;q\right]. \qquad (21)
\end{aligned}$$

Setting $u=\ell,\ v=m$ in (21) and evaluating the ${}_4\phi_3$-series on the right hand side by (16), we establish the theorem. □

**Example 27 ($\ell=0,m=1$ in Theorem 16)**

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a,q/a,q^{-n},-q^{1-n}\\ c,q^{2-2n}/c,-q\end{matrix}\Big|\,q;q\right]\\
&=\frac{1}{1+q^n}\frac{(ac;q^2)_n(qc/a;q^2)_n}{(c;q)_n(q^{n-1}c;q)_n}\\
&+\frac{q^n}{1+q^n}\frac{(ac/q;q^2)_n(c/a;q^2)_n}{(c;q)_n(q^{n-1}c;q)_n}.
\end{aligned}$$

**Example 28 ($\ell=1,m=0$ in Theorem 16)**

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a,q^2/a,q^{-n},-q^{-n}\\ c,q^{2-2n}/c,-q\end{matrix}\Big|\,q;q\right]\\
&=\frac{(q-c)(q^2-ac)}{(q-a)(q^2-q^{2n}c^2)}\frac{(ac;q^2)_n(qc/a;q^2)_n}{(c/q;q)_{2n}}\\
&+\frac{q(q-c)(c-a)}{(q-a)(q^2-q^{2n}c^2)}\frac{(ac/q;q^2)_n(q^2c/a;q^2)_n}{(c/q;q)_{2n}}.
\end{aligned}$$

**Example 29** ($\ell = 1, m = 1$ **in Theorem 16)**

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, q^2/a, q^{-n}, -q^{1-n} \\ c, q^{3-2n}/c, -q \end{matrix} \Big| \, q; q\right] \\
&= \frac{q - q^n a}{(q-a)(1+q^n)} \frac{(ac/q^2;q^2)_n (qc/a;q^2)_n}{(c;q)_n (q^{n-2}c;q)_n} \\
&+ \frac{q^{1+n} - a}{(q-a)(1+q^n)} \frac{(ac/q;q^2)_n (c/a;q^2)_n}{(c;q)_n (q^{n-2}c;q)_n}.
\end{aligned}$$

Taking $u = \ell$, $v = -m$ in (21) and calculating the ${}_4\phi_3$-series on the right hand side by (17), we get the following theorem.

**Theorem 17** *For two complex numbers $\{a, c\}$ and two nonnegative integers $\{\ell, m\}$, there holds*

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, q^{1+\ell}/a, q^{-n}, -q^{-m-n} \\ c, q^{1+\ell-m-2n}/c, -q \end{matrix} \Big| \, q; q\right] = \left[\begin{matrix} q^2/ac, -q/a \\ q^2/a^2, -q/c \end{matrix} \Big| \, q\right]_\ell \left[\begin{matrix} q^{1+m+n}, -q^{-\ell}c \\ q^{n+m-\ell}c, -q \end{matrix} \Big| \, q\right]_n \\
&\times \sum_{i=0}^{\ell} \sum_{j=0}^{m} q^{(\ell+1)i - nj + \binom{j+1}{2}} \frac{(-1)^{i+j}}{c^i} \frac{1-q^{1+2i}/a^2}{1-q/a^2} \left[\begin{matrix} q^{-\ell}, c/a, q/a^2 \\ q, q^2/ac, q^{2+\ell}/a^2 \end{matrix} \Big| \, q\right]_i \\
&\times \left[\begin{matrix} q^{-m}, q^{-m-n}, -q^{-m-n}, q^{-i-1}ac, q^i c/a \\ q, c, -c, q^{-m-2n}, q^{j-2m-2n-1} \end{matrix} \Big| \, q\right]_j \left[\begin{matrix} q^{j-i}ac, q^{1+i+j}c/a \\ q, q^{2j}c^2 \end{matrix} \Big| \, q^2\right]_{m+n-j}.
\end{aligned}$$

**Example 30** ($\ell = 0, m = 1$ **in Theorem 17)**

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, q/a, q^{-n}, -q^{-1-n} \\ c, q^{-2n}/c, -q \end{matrix} \Big| \, q; q\right] \\
&= \frac{1}{(1-q^{1+n})(1+q^n c)} \frac{(ac;q^2)_{1+n} (qc/a;q^2)_{1+n}}{(c;q)_{1+2n}} \\
&- \frac{q^{1+n}}{(1-q^{1+n})(1+q^n c)} \frac{(ac/q;q^2)_{1+n} (c/a;q^2)_{1+n}}{(c;q)_{1+2n}}.
\end{aligned}$$

**Example 31** ($\ell = 1, m = 1$ **in Theorem 17)**

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, q^2/a, q^{-n}, -q^{-1-n} \\ c, q^{1-2n}/c, -q \end{matrix} \Big| \, q; q\right] \\
&= \frac{q^2(1+q^n a)}{(q-a)(1+q^n c)(q+q^n c)(1-q^{1+n})} \frac{(ac/q^2;q^2)_{1+n} (qc/a;q^2)_{1+n}}{(c;q)_{2n}} \\
&- \frac{q(a+q^{2+n})}{(q-a)(1+q^n c)(q+q^n c)(1-q^{1+n})} \frac{(ac/q;q^2)_{1+n} (c/a;q^2)_{1+n}}{(c;q)_{2n}}.
\end{aligned}$$

Setting $u = -\ell$, $v = m$ in (21) and evaluating the ${}_4\phi_3$-series on the right hand side by Theorem 5, we obtain the following theorem.

**Theorem 18** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$ with $m\le n$, there holds*

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix}a,q^{1-\ell}/a,q^{-n},-q^{m-n}\\ c,q^{1-\ell+m-2n}/c,-q\end{matrix}\Big|\,q;q\right]&=\left[\begin{matrix}a,q^{\ell-1}ac\\ c,q^{\ell-1}a^2\end{matrix}\Big|\,q\right]_\ell\left[\begin{matrix}q^{1-m+n},-q^\ell c\\ q^{n-m+\ell}c,-q\end{matrix}\Big|\,q\right]_n\\
&\times\sum_{i=0}^{\ell}\sum_{j=0}^{m}q^{i+(m-n)j+\binom{j+1}{2}}\frac{1}{c^i}\frac{1-q^{1-2\ell+2i}/a^2}{1-q^{1-2\ell}/a^2}\left[\begin{matrix}q^{-\ell},c/a,q^{1-2\ell}/a^2\\ q,q^{2-2\ell}/ac,q^{2-\ell}/a^2\end{matrix}\Big|\,q\right]_i\\
&\times\left[\begin{matrix}q^{-m},q^{-n},-q^{-n},q^{2\ell-i-1}ac,q^ic/a\\ q,q^\ell c,-q^\ell c,q^{m-2n},q^{j-2n-1}\end{matrix}\Big|\,q\right]_j\left[\begin{matrix}q^{2\ell-i+j}ac,q^{1+i+j}c/a\\ q,q^{2\ell+2j}c^2\end{matrix}\Big|\,q^2\right]_{n-j}.
\end{aligned}
$$

**Example 32 ($\ell=1,m=0$ in Theorem 18)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a,1/a,q^{-n},-q^{-n}\\ c,q^{-2n}/c,-q\end{matrix}\Big|\,q;q\right]\\
&=\frac{1-ac}{1+a}\frac{(q^2ac;q^2)_n(qc/a;q^2)_n}{(c;q)_{1+2n}}\\
&+\frac{a-c}{1+a}\frac{(qac;q^2)_n(q^2c/a;q^2)_n}{(c;q)_{1+2n}}.
\end{aligned}
$$

**Example 33 ($\ell=1,m=1$ in Theorem 18)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}a,1/a,q^{-n},-q^{1-n}\\ c,q^{1-2n}/c,-q\end{matrix}\Big|\,q;q\right]\\
&=\frac{1+aq^n}{(1+a)(1+q^n)}\frac{(ac;q^2)_n(qc/a;q^2)_n}{(c;q)_{2n}}\\
&+\frac{a+q^n}{(1+a)(1+q^n)}\frac{(qac;q^2)_n(c/a;q^2)_n}{(c;q)_{2n}}.
\end{aligned}
$$

Taking $u=-\ell$, $v=-m$ in (21) and calculating the ${}_4\phi_3$-series on the right hand side by Theorem 6, we get the following theorem.

**Theorem 19** *For two complex numbers $\{a,c\}$ and two nonnegative integers $\{\ell,m\}$, there holds*

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix}a,q^{1-\ell}/a,q^{-n},-q^{-m-n}\\ c,q^{1-\ell-m-2n}/c,-q\end{matrix}\Big|\,q;q\right]&=\left[\begin{matrix}a,q^{\ell-1}ac\\ c,q^{\ell-1}a^2\end{matrix}\Big|\,q\right]_\ell\left[\begin{matrix}q^{1+m+n},-q^\ell c\\ q^{\ell+m+n}c,-q\end{matrix}\Big|\,q\right]_n\\
&\times\sum_{i=0}^{\ell}\sum_{j=0}^{m}q^{i-nj+\binom{j+1}{2}}\frac{(-1)^j}{c^i}\frac{1-q^{1-2\ell+2i}/a^2}{1-q^{1-2\ell}/a^2}\left[\begin{matrix}q^{-\ell},c/a,q^{1-2\ell}/a^2\\ q,q^{2-2\ell}/ac,q^{2-\ell}/a^2\end{matrix}\Big|\,q\right]_i\\
&\times\left[\begin{matrix}q^{-m},q^{-m-n},-q^{-m-n},q^{2\ell-i-1}ac,q^ic/a\\ q,q^\ell c,-q^\ell c,q^{-m-2n},q^{j-2m-2n-1}\end{matrix}\Big|\,q\right]_j\left[\begin{matrix}q^{2\ell-i+j}ac,q^{1+i+j}c/a\\ q,q^{2\ell+2j}c^2\end{matrix}\Big|\,q^2\right]_{m+n-j}.
\end{aligned}
$$

**Example 34** ($\ell = 1, m = 1$ **in Theorem 19)**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, 1/a, q^{-n}, -q^{-1-n} \\ c, q^{-1-2n}/c, -q \end{matrix} \;\middle|\; q; q\right] \\
&= \frac{1 - aq^{1+n}}{(1+a)(1-q^{1+n})} \frac{(ac; q^2)_{1+n}(qc/a; q^2)_{1+n}}{(c; q)_{2+2n}} \\
&+ \frac{a - q^{1+n}}{(1+a)(1-q^{1+n})} \frac{(qac; q^2)_{1+n}(c/a; q^2)_{1+n}}{(c; q)_{2+2n}}.
\end{aligned}
$$

With the change of the parameters $\ell$ and $m$, Theorems 2 and 5-19 can produce more concrete formulas. Due to the limit of space, the corresponding results will not be laid out in the paper.

## Acknowledgements

The authors are grateful to the reviewers for helpful comments.

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