Some new optimal and suboptimal infinite families of undirected double-loop networks

Bao-Xing Chen[1,2] †, Ji-Xiang Meng[2] and Wen-Jun Xiao[3]

[1] *Dept. of Computer Science, Zhangzhou Teacher's College, Zhangzhou, P.R. of China*
Email: cbaoxing@hotmail.com
[2] *College of Mathematics & System Science, Xinjiang University, Wulumuqi, P.R. of China*
[3] *Dept. of Computer Science, South China University of Technology, Guangzhou, P.R. of China*

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Let n, s be positive integers such that $2 \leq s < n$ and $s \neq \frac{n}{2}$. An undirected double-loop network $G(n; 1, s)$ is an undirected graph (V, E), where V=$\mathbb{Z}_n$=$\{0, 1, 2, \ldots, n-1\}$ and E=$\{(i, i+1 \pmod n), (i, i+s \pmod n) \mid i \in \mathbb{Z}\}$. It is a circulant graph with n nodes and degree 4. In this paper, the sufficient and necessary conditions for a class of undirected double-loop networks to be optimal are presented. By these conditions, 6 new optimal and 5 new suboptimal infinite families of undirected double-loop networks are given.

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1 Introduction

Double-loop networks are popular in the design and implementation of metropolitan networks and parallel processing computer systems. They have many attractive properties like vertex symmetry, incremental extensibility, low valency, ease of implementation, etc. Some researchers are interested in the study of double-loop networks [1-22]. They mainly focus on designs of optimal double-loop networks [1], [2], [4], [9], [12] – [15], [18], [19], diameters [1]–[4], [9]–[15], [18]–[22] and routing [5], [7], [8], [17]. For more details we refer readers to [3], [16] and the references therein.

Let n, s be positive integers such that $2 \leq s < n$ and $s \neq \frac{n}{2}$. The undirected double-loop network $G(n; 1, s)$ is an undirected graph (V, E), where V=$\mathbb{Z}_n$=$\{0, 1, 2, \ldots, n-1\}$ and E=$\{(i, i+1 \pmod n), (i, i+s \pmod n) \mid i \in \mathbb{Z}_n\}$. It is a circulant graph with n nodes and degree 4. Let $d(i, j)$ be the length of a shortest path from node i to node j. Let $d(n; 1, s)$ denote the diameter of $G(n; 1, s)$. Since $G(n; 1, s)$ is vertex symmetric, $d(n; 1, s) = \max\{d(i, j) \mid 0 \leq i, j < n\} = \max\{d(0, i) \mid 0 \leq i < n\}$. Let $D(n)$=min $\{d(n; 1, s) \mid 1 < s < n\}$. Wong and Coppersmith [20] gave the lower bound $\frac{1}{2}(\sqrt{2n} - 3)$

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for $D(n)$. Boesch and Wang [4] sharpened the bound by giving $lb(n)$=$\lceil \frac{\sqrt{2n-1}-1}{2} \rceil$, where $\lceil x \rceil$ denotes the minimum integer $\geq x$.

If s exists such that $D(n) = d(n; 1, s) = lb(n)$, then n, s, and $G(n; 1, s)$ will be called optimal. If s exists such that $D(n) = d(n; 1, s) = lb(n) + 1$, then n, s, and $G(n; 1, s)$ will be called suboptimal.

A set Θ of natural numbers will be called an optimal (suboptimal) family if each $n \in \Theta$ is optimal (suboptimal).

Some optimal infinite families of undirected double-loop networks were given in [2], [11], [12], [19]. And as we know, only one suboptimal infinite family of undirected double-loop networks $\{2t^2 + 2t \mid t \geq 2\}$ has been given so far [12]. In this paper, the sufficient and necessary conditions for a class of undirected double-loop networks to be optimal are presented. And by using these conditions, we obtain 6 new optimal and 5 new suboptimal infinite families of undirected double-loop networks.

2 Definitions and some lemmas

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and nonnegative integers respectively. Let $\lfloor x \rfloor$ denote the maximum integer $\leq x$.

Definition 2.1 (a_1, a_2) *is said to be a non-negative solution of the equation*

$$x + ys \equiv 0 \pmod n \tag{1}$$

if $a_1 + a_2 s \equiv 0 \pmod n$*,* $a_1 \geq 0, a_2 \geq 0$ *and* $(a_1, a_2) \neq (0, 0)$*.*
(u, v) *is said to be the smallest non-negative solution of the equation (1) if* (u, v) *is a non-negative solution of the equation (1) and the following conditions hold:*

(1) if (a_1, a_2) *is a non-negative solution of the equation (1), then* $u + v \leq a_1 + a_2$*.*

(2) if (a_1, a_2) *is a non-negative solution of the equation (1),where* $(a_1, a_2) \neq (u, v)$ *and* $u + v = a_1 + a_2$*, then* $u > a_1$*.*

Clearly, the smallest non-negative solution of equation (1)is unique. For example, it is easy to see that (4, 1), (2, 3), (0, 5), (8, 2), (4, 6), $\cdots$ are non-negative solutions of the equation $x + 6y \equiv 0 \pmod{10}$. Thus (4, 1) is the smallest non-negative solution of the equation $x + 6y \equiv 0 \pmod{10}$.

Definition 2.2 *Let* (u, v) *be the smallest non-negative solution of the equation (1).* $(-a_1, a_2)$ *is said to be a cross solution of the equation (1) if* $-a_1 + a_2 s \equiv 0 \pmod n$*,* $a_1 \geq 0$*,* $a_2 \geq 0$*,* $(-a_1, a_2) \neq (0, 0)$*, and the three points* $(-a_1, a_2)$*,* $(0, 0)$*,* (u, v) *are not on the same line.* $(-a, b)$ *is said to be the smallest cross solution of the equation (1)if* $(-a, b)$ *is a cross solution of the equation (1)and the following conditions hold:*

(1) if $(-a_1, a_2)$ *is a cross solution of the equation (1),then* $a + b \leq a_1 + a_2$*.*

(2) if $(-a_1, a_2)$ *is a cross solution of the equation (1),where* $(-a_1, a_2) \neq (-a, b)$ *and* $a + b = a_1 + a_2$*, then* $b > a_2$*.*

Clearly, the smallest cross solution of equation (1)is unique. For example, it is easy to see that $(2, 2)$ is the smallest non-negative solution of the equation $x + 5y \equiv 0 \pmod{12}$, and (-5, 1), (-3, 3), (-1, 5), (-12,0), (-10, 2), (-8, 4), (-6, 6), (-4, 8), (-2, 10), $\cdots$ are cross solutions of the equation $x + 5y \equiv 0 \pmod{12}$. Thus (-1, 5) is the smallest cross solution of the equation $x + 5y \equiv 0 \pmod{12}$.

From [11] we have the following two lemmas.

Lemma 2.3 *Let (u, v) be the smallest non-negative solution of the equation (1) and $(-a, b)$ be the smallest cross solution of the equation (1) If $u < v$, then $a > u, a > b, b < v$ and n=$av + bu$.*

Lemma 2.4 *Let (u, v) be the smallest non-negative solution of the equation (1) and $(-a, b)$ be the smallest cross solution of the equation (1). If $u \geq v$, then $a < u, a \leq b, v < b$ and n=$av + bu$.*

Given $G(n; 1, s)$, we construct an infinite grid $G_{n,1,s}$ in $\mathbb{Z}^2$ as D. Tzvieli did in [19], labelling each lattice point (i, j) by $i + js \pmod{n}$. Every label $m,\ 0 \leq m < n$, is repeated in $G_{n,1,s}$ infinitely many times. We refer to a lattice point with label i as an i-point. If $i + js \equiv 0 \pmod{n}$, then we call (i, j) a 0-point.

The smallest non-negative solution (u, v) of the equation (1) can be seen as a 0-point in the first quadrant with $u + v$ is minimum (in case of tie, take the maximum u). The smallest cross solution $(-a, b)$ of the equation (1) can be seen as a 0-point in the second quadrant with $a + b$ is minimum (in case of tie, take the maximum b).

For $G(12; 1, 5)$, one can see that $(2, 2)$ is a 0-point in the first quadrant, and $(-1, 5)$ is a 0-point in the second quadrant(see Fig. 1).

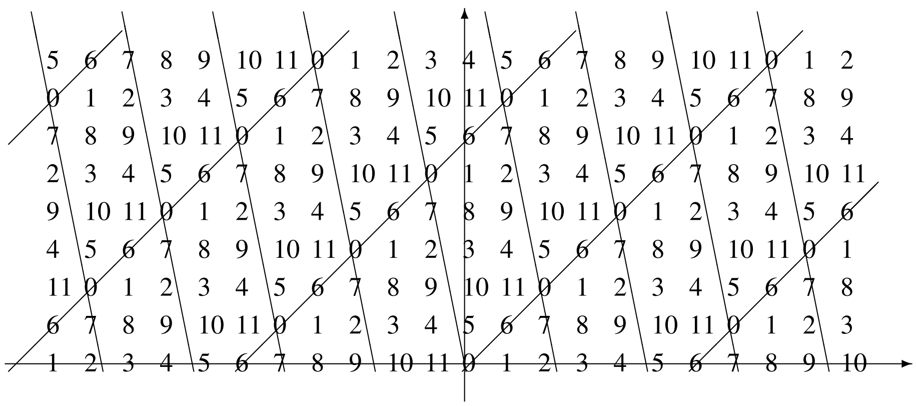

Fig. 1 0-points in the $G_{12;1,5}$.

From Lemma 4 [10] and Lemma 5 [10] we have the following lemma:

Lemma 2.5 *Given $G(n; 1, s)$ with $1 < s < n$ and $s \neq \frac{n}{2}$. Suppose that (u, v) and $(-a, b)$ are two 0-points and u, v, a, b are all non-negative integers. Let r_1=$\lfloor (u + v)/2 \rfloor$, r_2=$\lfloor (a + b)/2 \rfloor$, r_3=$\lfloor (|u - a| + v + b)/2 \rfloor$ and r_4=$\lfloor (u + a + |v - b|)/2 \rfloor$. If $n = av + bu$ and $b \geq a, u \geq v, u > a,\ v < b$ (or $a > b, v > u, a > u, b < v$), then*

$$d(n; 1, s) = \begin{cases} r_3 - 1, & if\ r_3 = r_4\ and\ (u + a)(v + b) \equiv 1 \pmod{2} \\ \max\{r_1, r_2, \min\{r_3, r_4\}\}, & otherwise. \end{cases}$$

By Lemma 2.5,it is easy to prove the following corollary.

Corollary 2.6 *Given $G(n;1,s)$ with $1 < s < n$ and $s \neq \frac{n}{2}$. Suppose that (u,v) and $(-a,b)$ are two 0-points and u,v,a,b are all non-negative integers. Let d denote the diameter of $G(n;1,s)$. If $n = av + bu$ and $b \geq a, u \geq v, u > a, v < b$ (or $a > b, v > u, a > u, b < v$), then $u+v \leq 2d+1$ and $a+b \leq 2d+1$.*

Lemma 2.7 *Suppose that u,v,a,b are all integers and there exist two integers λ, χ such that $\lambda b + \chi v = 1$. Let $s_0 \equiv \lambda a - \chi u$ and $s = s_0 \pmod{n}$. If $n = av + bu$, then $u + vs \equiv 0 \pmod{n}$ and $-a + bs \equiv 0 \pmod{n}$, i.e. , (u,v) and $(-a,b)$ are two 0-points.*

Proof: Since $\begin{pmatrix} b & v \\ a & -u \end{pmatrix} \binom{\lambda}{\chi} = \binom{1}{s_0}$, we have $n \binom{\lambda}{\chi} = \begin{pmatrix} u & v \\ a & -b \end{pmatrix} \binom{1}{s_0}$. That is, $u+vs \equiv u+vs_0 \pmod{n} = n\lambda \pmod{n} = 0$ and $-a+bs \equiv -a+bs_0 \pmod{n} = -n\chi \pmod{n} = 0$. □

We use $\gcd(a_1, a_2)$ to denote the greatest common divisor of two integers a_1, a_2. As the following lemma is well-known and easy to prove, its proof is omitted.

Lemma 2.8 *Suppose that a_1, b_1 and c_1 are integers.*

(1) $\gcd(a_1, b_1) = 1$ *if and only if there exist two integers λ, χ such that $\lambda a_1 + \chi b_1 = 1$.*

(2) If $\gcd(a_1, b_1) = 1$ *and* $\gcd(a_1, c_1) = 1$*, then* $\gcd(a_1, b_1 c_1) = 1$.

3 Main results

The sufficient and necessary conditions for a class of undirected double-loop networks to be optimal are given in the following theorem. By using this theorem, we can give some new optimal or suboptimal infinite families of undirected double-loop networks in this section.

Theorem 3.1 *Suppose that $n = 2t^2 + 2t - B$, where $t > B > 0$, and $t, B \in \mathbb{Z}^+$. Then there exists a positive integer s such that $d(n;1,s) = t = lb(n)$ if and only if there exist four non-negative integers a,b,u,v satisfying the following five conditions:*

1 $n = av + bu$;

2 $a + b \leq 2t + 1$ *and* $u + v \leq 2t + 1$;

3 $b \geq a, u \geq v, u > a, v < b$;

4 One of the following two conditions holds:

- *(a)* $x = 0$, $b - a = y$, y *is odd,* $y \mid 2B+1$ *and* $1 \leq y \leq \sqrt{2B+1}$.
- *(b)* $y = 0$, $u - v = x$, x *is odd,* $x \mid 2B+1$ *and* $1 \leq x \leq \sqrt{2B+1}$. *where* $x = 2t + 1 - (a+b)$, $y = 2t + 1 - (u+v)$.

5 $\gcd(b, v) = 1$.

Proof: Suppose that there exists a positive integer s such that $d(n;1,s) = t$. Let (u,v) be the smallest non-negative solution of the equation (1) and $(-a,b)$ be the smallest cross solution of the equation (1).

In the following we consider two cases: $u \geq v$ and $u < v$.

Case: $u \geq v$ Then by Lemma 2.4 and Corollary 2.6 , we see that (1), (2) and (3) hold.

Now we will prove that condition (4) holds.

Let r_1=$\lfloor (u+v)/2 \rfloor$, r_2=$\lfloor (a+b)/2 \rfloor$, r_3=$\lfloor (u-a+v+b)/2 \rfloor$ and r_4=$\lfloor (u+a+b-v)/2 \rfloor$.

If $r_3 = r_4$ and $(u+a)(v+b) \equiv 1 \pmod 2$, then as $u-a+v+b \equiv 0 \pmod 2$ and $u+a-v+b \equiv 0 \pmod 2$, we have $u-a+v+b = u+a-v+b$, i.e., $a = v$. Thus the diameter of $G(n;1,s)$ is $t = (u+b)/2 - 1$. So $b-a = b-v = (u+b)-(u+v) = 2t+2-(2t+1-y) = 1+y$ and $u-v = u-a = (u+b)-(a+b) = 2t+2-(2t+1-x) = 1+x$. As $(u+a)(v+b) \equiv 1 \pmod 2$ and $a = v$, we have $x = 2t+1-(a+b) = 2t+1-(v+b) \equiv 0 \pmod 2$ and $y = 2t+1-(u+v) = 2t+1-(u+a) \equiv 0 \pmod 2$.

As $0 \leq (b-a)(u-v) = (1+y)(u-v) \leq (1+y)(u+v)$ and $(b-a)(u-v) = 2(av+bu)-(a+b)(u+v) = 2n-(2t+1-x)(2t+1-y)$, we have $2n-(2t+1-x)(2t+1-y) \leq (1+y)(u+v) = (1+y)(2t+1-y)$, i.e., $(2t+2-x+y)(2t+1-y) \geq 2n$. Thus we have $(y+\frac{1-x}{2})^2+(x-2)(2t+1-\frac{x}{4}) \leq \frac{1}{4}+2B-2t$. As $\frac{1}{4}+2B-2t < 0$ and $2t+1-\frac{x}{4} > 0$, we have $x < 2$. Similarly we can prove that $y < 2$. Since $x \equiv 0 \pmod 2$, $y \equiv 0 \pmod 2$, $x \geq 0$ and $y \geq 0$, we have $x = y = 0$. As $b-a = 1+y = 1$, $u-v = 1+x = 1$, $a+b = 2t+1-x = 2t+1$ and $u+v = 2t+1-y = 2t+1$, we have $a = v = t$, $b = u = t+1$. Since $av+bu = t^2+(t+1)^2 = 2t^2+2t+1$ and $n = av+bu$, we get a contradiction. Thus $r_3 = r_4$ and $(u+a)(v+b) \equiv 1 \pmod 2$ can not hold simultaneously.

In other cases, the diameter of $G(n;1,s)$ is $\max\{r_1, r_2, \min\{r_3, r_4\}\} = t$. Thus $r_3 \leq t$ or $r_4 \leq t$.

If $r_3 \leq t$, we have $b-a \leq y$. As $0 \leq (b-a)(u-v) \leq y(u+v)$ and $(b-a)(u-v) = 2(av+bu)-(a+b)(u+v) = 2n-(2t+1-x)(2t+1-y)$, we have $(2t+1-x+y)(2t+1-y) \geq 2n$, i.e., $4t^2+(4-2x)t+(1-x+y)(1-y) \geq 2n$. Thus we have $(y-\frac{x}{2})^2+(x-1)(2t+1-\frac{x+1}{4}) \leq \frac{1}{4}+2B-2t$. As $\frac{1}{4}+2B-2t < 0$ and $2t+1-\frac{x+1}{4} > 0$, we have $x < 1$. Thus $x = 0$ and $y \leq \sqrt{2B+1}$. As $x = 0$ and $0 \leq (b-a)(u-v) = -2B-1-xy+(x+y)(2t+1) = -2B-1+y(2t+1)$, we have $y \geq 1$. As $u-v = \frac{-2B-1+y(2t+1)}{b-a} = \frac{2t-2B+(y-1)(2t+1)}{b-a} \leq u+v = 2t+1-y$, we have $b-a > y-1$. Since $b-a \leq y$, we have $b-a = y$. As $b+a = 2t+1$, $b-a = b+a-2a$, we see that $y = b-a$ is odd. As $b-a = y$ and $(b-a)(u-v) = -2B-1+y(2t+1)$, we have $2B+1 = y(2t+1-u+v)$. Thus y divides $2B+1$, i.e., $y \mid 2B+1$.

If $r_4 \leq t$, we have $u-v \leq x$. As in the case of $r_3 \leq t$, we can prove that $y = 0$, $u-v = x$, x is odd, $x \mid 2B+1$ and $1 \leq x \leq \sqrt{2B+1}$.

From above we see that condition (4) holds.

Since $u+vs \equiv 0 \pmod n$ and $-a+bs \equiv 0 \pmod n$, there exist two integers λ and χ such that $\begin{pmatrix} u & v \\ -a & b \end{pmatrix} \binom{1}{s}$=$n\binom{\lambda}{\chi}$. Thus $\binom{1}{s}$=$\begin{pmatrix} b & -v \\ a & u \end{pmatrix} \binom{\lambda}{\chi}$. So $\lambda b+(-\chi)v = 1$. By Lemma 2.8, we have $\gcd(v,b) = 1$. Thus condition (5) holds.

Case: $u < v$ By a similar argument we can deduce the following:

(1) $n = av+bu$;

(2) $a+b \leq 2t+1$ and $u+v \leq 2t+1$;

(3) $a > b, v > u, a > u, b < v$;

(4) One of the following two conditions holds:

(a) $x = 0$, $a-b = y$, y is odd, $y \mid 2B+1$ and $1 \leq y \leq \sqrt{2B+1}$.

(b) $y = 0, v - u = x, x$ is odd, $x \mid 2B + 1$ and $1 \le x \le \sqrt{2B+1}$. where $x = 2t + 1 - (a + b)$, $y = 2t + 1 - (u + v)$.

(5) $\gcd(b, v) = 1$.

By letting $a' = u, b' = v, u' = a, v' = b, x' = 2t + 1 - (a' + b') = y$ and $y' = 2t + 1 - (u' + v') = x$, it is routine to verify that a', b', u' and v' satisfy the five conditions of the theorem.

On the other hand, if there exist four non-negative integers a, b, u, v satisfying the five conditions of the theorem, then by condition 5 there exist two integers λ and χ such that $\lambda b + \chi v = 1$. Let $s \equiv \lambda a - \chi u \pmod{n}$. By Lemma 2.7 , we see that (u, v) and $(-a, b)$ are two 0-points.

As $u \ge v, b \ge a, u > a$, and $v < b$, by Lemma 2.5 we have $d(n; 1, s) \le t$. Since $d(n; 1, s) \ge lb(n) = t$, we have $d(n; 1, s) = t$. □

Corollary 3.2 *Suppose that* $n = 2t^2 + 2t - B$, *where* $t > B > 0$, $t, B \in \mathbb{Z}^+$ *and* $2B + 1$ *is a prime number. Then* n *is optimal if and only if* $\gcd(t, B) = 1$ *or* $\gcd(t + 1, B) = 1$.

Proof: Suppose that n is optimal, by Theorem 3.1 there exist four non-negative integers a, b, u, v satisfying the condition 4 of Theorem 3.1. Let $x = 2t + 1 - (a + b)$ and $y = 2t + 1 - (u + v)$. Since $2B + 1$ is a prime number, we have (a) $x = 0, b - a = y = 1$ or (b) $y = 0, u - v = x = 1$.

When $x = 0, b-a = y = 1$, as $(b-a)(u-v) = 2(av+bu)-(a+b)(u+v) = 2n-(2t+1)*2t = 2t-2B$, we have $a = t, b = t + 1, u = 2t - B$ and $v = B$. Since a, b, u, v satisfy the condition 5 of Theorem 3.1, we have $1 = \gcd(b, v) = \gcd(t + 1, B)$.

When $y = 0, u - v = x = 1$, as $(b - a)(u - v) = 2(av + bu) - (a + b)(u + v) = 2n - 2t * (2t + 1) = 2t - 2B$, we have $a = B, b = 2t - B, u = t + 1$ and $v = t$. Since a, b, u, v satisfy the condition 5 of Theorem 3.1, we have $1 = \gcd(b, v) = \gcd(2t - B, t) = \gcd(-B, t) = \gcd(t, B)$.

Thus we have proved that if n is optimal, then $\gcd(t, B) = 1$ or $\gcd(t + 1, B) = 1$.

If $\gcd(t, B) = 1$ or $\gcd(t + 1, B) = 1$, we will prove that n is optimal.

When $\gcd(t, B) = 1$, let $a = B, b = 2t - B, u = t + 1$ and $v = t$. It is routine to verify that four non-negative integers a, b, u, v satisfy the five conditions of Theorem 3.1.

When $\gcd(t + 1, B) = 1$, let $a = t, b = t + 1, u = 2t - B$ and $v = B$. It is easy to see that four non-negative integers a, b, u, v satisfy the five conditions of Theorem 3.1.

From above we see that Corollary 3.2 holds. □

By Theorem 3.1 and Corollary 3.2, we can give some new optimal and suboptimal infinite families of undirected double-loop networks in the following theorems.

Theorem 3.3 *(1) Let* $\Theta = \{2t^2 + 2t - 6 \mid\ t > 6,\ t \ne 6e + 2$ *and* $t \ne 6e + 3,\ e \in \mathbb{Z}^+\}$. *Then* Θ *is an optimal infinite family, and when* $t = 6e, 6e + 1, 6e + 4, 6e + 5$ *respectively, the optimal step s is* $12e^2, 12e^2 + 4e - 2, 60e^2 + 88e + 28, 60e^2 + 108e + 42$ *correspondingly.*

(2) $\{G(2t^2 + 2t - 6; 1, 2t^2 - 3) \mid\ t = 6e + 2$ *or* $t = 6e + 3,\ 1 \le e \in \mathbb{Z}^+\}$ *is a suboptimal infinite family.*

Proof:

(1) Let n be $2t^2 + 2t - 6$. In the following we consider two cases: $t = 6e + 1$ or $t = 6e + 5$ and $t = 6e$ or $t = 6e + 4$.

$t = 6e + 1$ **or** $t = 6e + 5$ When $t = 6e + 1$ or $t = 6e + 5$, we have $\gcd(t, 6) = 1$.

$t = 6e$ **or** $t = 6e + 4$ When $t = 6e$ or $t = 6e + 4$, we have $\gcd(t + 1, 6) = 1$.

As $2 \times 6 + 1 = 13$ and 13 is a prime number, by Corollary 3.2, we see that Θ is an optimal infinite family.

When $t = 6e + 1$, let $a = 6, b = 2t - 6, u = t + 1$, and $v = t$. As $e \times b + (-2e + 1) \times v = 1$, the optimal step s is $e \times a - (-2e + 1) \times u \pmod n = 12e^2 + 4e - 2$.

When $t = 6e + 5$, let $a = 6, b = 2t - 6, u = t + 1$, and $v = t$. As $-(e + 1) \times b + (2e + 1) \times v = 1$, $s = -(e + 1) \times a - (2e + 1) \times u \pmod n = 60e^2 + 108e + 42$.

When $t = 6e$, let $a = t, b = t + 1, u = 2t - 6$, and $v = 6$. As $1 \times b - e \times v = 1$, $s = 1 \times a - (-e) \times u \pmod n = 12e^2$.

When $t = 6e + 4$, let $a = t, b = t + 1, u = 2t - 6$, and $v = 6$. As $-1 \times b + (e + 1) \times v = 1$, $s = -1 \times a - (e + 1) \times u \pmod n = 60e^2 + 88e + 28$.

(2) When $t = 6e + 2$, we have $\gcd(t + 1, 6) = \gcd(6e + 3, 6) = \gcd(3, 6) = 3$ and $\gcd(t, 6) = \gcd(6e + 2, 6) = \gcd(2, 6) = 2$.

When $t = 6e + 3$, we have $\gcd(t + 1, 6) = \gcd(6e + 4, 6) = \gcd(4, 6) = 2$ and $\gcd(t, 6) = \gcd(6e + 3, 6) = \gcd(3, 6) = 3$.

As $2 \times 6 + 1 = 13$ and 13 is a prime number, by Corollary 3.2, we see that when $t = 6e + 2$ or $t = 6e + 3$, $1 \le e \in \mathbb{Z}^+$, $n = 2t^2 + 2t - 6$ can not be optimal.

Let $a = t, b = t + 2, u = 2t - 3$ and $v = 1$. As $0 \times b + 1 \times v = 1$, let $s = 0 \times a - 1 \times u \equiv 2t^2 - 3 \pmod n$. By Lemma 2.7, we see that $(2t - 3, 1)$ and $(-t, t + 2)$ are two 0-points. By Lemma 2.5 we have $d(n; 1, s) = t + 1$.

From above we see that when $t = 6e + 2$ or $t = 6e + 3$, $1 \le e \in \mathbb{Z}^+$, $G(2t^2 + 2t - 6; 1, 2t^2 - 3)$ is suboptimal. □

Theorem 3.4 *1. Let* $n = 2t^2 + 2t - 12$ *and* $\Theta = \{2t^2 + 2t - 12 \mid t > 12, t \neq 6e + 2$ *and* $t \neq 6e + 3, e \in \mathbb{Z}^+\}$. *Then* Θ *is an optimal infinite family, and when* $t = 6e, 6e + 1, 6e + 4, 6e + 5$ *respectively, the optimal step s is* $6e - 36e^3 \pmod n, 36e^3 + 24e^2 - 2e - 2 \pmod n, -36e^3 - 96e^2 - 78e - 16 \pmod n, 36e^3 + 96e^2 + 78e + 18 \pmod n$ *correspondingly.*

2. Let $\Psi = \{2t^2 + 2t - 12 \mid t = 6e + 2$ *or* $t = 6e + 3$*, where* $e \in \mathbb{Z}^+$ *and* $e \ge 2\}$. *Then* Ψ *is a suboptimal infinite family, and when* $t = 6e + 2$ *or* $6e + 3$*, the suboptimal step s is* $t^2 + 2t - 3, t^2 - 4$ *correspondingly.*

Proof:

(1) By Theorem 3.1, we only need to find four non-negative integers a, b, u, v which satisfy the five conditions of Theorem 3.1. In the following we consider two cases: $t = 6e$ or $t = 6e + 4$, and $t = 6e + 1$ or $t = 6e + 5$.

$t = 6e$ **or** $t = 6e + 4$ When $t = 6e$ or $t = 6e + 4$, let $a = t, b = t + 1, u = 2t - 12$ and $v = 12$.

When $t = 6e$ or $t = 6e + 4$, As $\gcd(2, t + 1) = 1$ and $\gcd(3, t + 1) = 1$, by Lemma 2.8, we have that $\gcd(4, t + 1) = 1$ and $\gcd(12, t + 1) = 1$. That is, $\gcd(v, b) = 1$. Thus it is easy to verify that four integers a, b, u, v satisfy the five conditions of Theorem 3.1.

$t = 6e+1$ **or** $t = 6e+5$ When $t = 6e+1$ or $t = 6e+5$, let $a = 12, b = 2t-12, u = t+1$ and $v = t$.

When $t = 6e+1$ or $t = 6e+5$, As $\gcd(2,t) = 1$ and $\gcd(3,t) = 1$, by Lemma 2.8, we have that $\gcd(4,t) = 1$ and $\gcd(12,t) = 1$. Thus $\gcd(v,b) = \gcd(t, 2t-12) = \gcd(t, -12) = \gcd(t, 12) = 1$. So it is easy to verify that four integers a, b, u, v satisfy the five conditions of Theorem 3.1.

From above we see that Θ is an optimal infinite family.

When $t = 6e$, let $a = t, b = t+1, u = 2t-12$, and $v = 12$. As $(1-6e) \times b + 3e^2 \times v = 1$, $s = (1-6e) \times a - 3e^2 \times u \pmod{n} = 6e - 36e^3 \pmod{n}$.

When $t = 6e+1$ or $6e+4$ or $6e+5$, the corresponding optimal step s can be computed similarly. They are $36e^3 + 24e^2 - 2e - 2 \pmod{n}, -36e^3 - 96e^2 - 78e - 16 \pmod{n}, 36e^3 + 96e^2 + 78e + 18 \pmod{n}$ respectively.

(2) For $n = 2t^2 + 2t - 12$, where $t = 6e+2$ or $t = 6e+3$, $2 \le e \in \mathbb{Z}^+$, in the following we will prove that n can not be optimal.

If n is optimal, then there exist four non-negative integers a, b, u, v which satisfy the five conditions of Theorem 3.1. Let $x = 2t+1-(a+b)$ and $y = 2t+1-(u+v)$.

In the following we consider two cases:

(A) $x = 0, b-a = y, y$ is odd, $y \mid 2B+1$ and $1 \le y \le \sqrt{2B+1}$.

(B) $y = 0, u-v = x, x$ is odd, $x \mid 2B+1$ and $1 \le x \le \sqrt{2B+1}$, where $B = 12$.

Case A If $x = 0, b-a = y, y$ is odd, $y \mid 25$ and $1 \le y \le \sqrt{25}$, then $y = 1$ or $y = 5$.

Subcase: $x = 0, y = 1$**:** As $b-a = y$ and $(b-a)(u-v) = 2n-(a+b)(u+v) = 2n-(2t+1-x)(2t+1-y) = 2t-24$, we have that $b-a = 1$ and $u-v = 2t-24$. Thus $a = t, b = t+1, u = 2t-12$ and $v = 12$. When $t = 6e+2$ or $t = 6e+3$, $\gcd(b,v) \neq 1$.

Subcase: $x = 0, y = 5$**:** As $b-a = y$ and $(b-a)(u-v) = 2n-(a+b)(u+v) = 2n-(2t+1-x)(2t+1-y) = 10t-20$, we have that $b-a = 5$ and $u-v = 2t-4$. Thus $a = t-2, b = t+3, u = 2t-4$ and $v = 0$. So $\gcd(b,v) = t+3 \neq 1$.

Case B If $y = 0, u-v = x, x$ is odd, $x \mid 25$ and $1 \le x \le \sqrt{25}$, then $x = 1$ or $x = 5$.

Subcase: $y = 0, x = 1$**:** As $u-v = x$ and $(b-a)(u-v) = 2n-(a+b)(u+v) = 2n-(2t+1-x)(2t+1-y) = 2t-24$, we have $u-v = 1$ and $b-a = 2t-24$. Thus $a = 12, b = 2t-12, u = t+1$ and $v = t$. When $t = 6e+2$ or $t = 6e+3$, $\gcd(b,v) \neq 1$.

Subcase: $y = 0, x = 5$**:** As $u-v = x$ and $(b-a)(u-v) = 2n-(a+b)(u+v) = 2n-(2t+1-x)(2t+1-y) = 10t-20$, we have that $u-v = 5$ and $b-a = 2t-4$. Thus $a = 0, b = 2t-4, u = t+3$ and $v = t-2$. So $\gcd(b,v) = \gcd(2t-4, t-2) = t-2 \neq 1$.

From above we see that in any cases, two non-negative integers b, v can not satisfy condition 5 of Theorem 3.1. So n can not be optimal.

When $t = 6e+2$, let $a = 0, b = 2t-4, u = t+3$ and $v = t-3$. As $\frac{4-t}{2} \times b + (t-3) \times v = 1$, let $s = \frac{4-t}{2} \times a - (t-3) \times u \equiv 9 - t^2 \pmod{n} = t^2 + 2t - 3$. By Lemma 2.7, we see that $(t+3, t-3)$ and $(0, 2t-4)$ are two 0-points. By Lemma 2.5 we have $d(n; 1, s) = t+1$.

When $t = 6e + 3$, let $a = t - 2, b = t + 2, u = 2t - 4$ and $v = 2$. As $1 \times b - \frac{t+1}{2} \times v = 1$, let $s = 1 \times a + \frac{t+1}{2} \times u \pmod n = t^2 - 4$. By Lemma 2.7, we see that $(2t-4, 2)$ and $(-t+2, t+2)$ are two 0-points. By Lemma 2.5 we have $d(n; 1, s) = t + 1$.

Thus when $t = 6e + 2$ or $t = 6e + 3$, $2 \le e \in \mathbb{Z}^+$, $2t^2 + 2t - 12$ is suboptimal. □

Theorem 3.5 *Let $\Theta = \{2t^2 + 2t - (2A^2 + 2A - 2) \mid A > 0, A \in \mathbb{Z}^+, t \ge \frac{A^2+A}{2}\}$. Then Θ is an optimal infinite family. When $t + A$ is odd, the optimal step s is $t^2 - A^2 + 1$. When $t + A$ is even, the optimal step s is $t^2 - A^2 - 2A$.*

Proof: Let $n = 2t^2 + 2t - (2A^2 + 2A - 2)$. When $A > 0, A \in \mathbb{Z}^+, t \ge \frac{A^2+A}{2}$, $lb(n) = t$.

In the following we consider two cases:

$t+A$ **is an odd number** When $t+A$ is odd, suppose that $t+A = 2j+1$. Let $a = t-A+1, b = t+A, u = 2t - 2A$ and $v = 2$. As $1 \times b - jv = 1$, let $s \equiv 1 \times (t - A + 1) + j(2t - 2A) \pmod n = t^2 - A^2 + 1$. By Lemma 2.7, we see that (u, v) and $(-a, b)$ are two 0-points. By Lemma 2.5 we have $d(n; 1, s) = t$.

$t + A$ **is an even number** When $t + A$ is even, let $a = 2, b = 2t - 2A, u = t + A$ and $v = t - A + 1$. Since $\frac{t-A}{2}b + (1-t+A)v = 1$, let $s \equiv \frac{t-A}{2}a - (1-t+A) * u \pmod n = t^2 - A^2 - 2A$. By Lemma 2.7, we see that (u, v) and $(-a, b)$ are two 0-points. By Lemma 2.5 we have $d(n; 1, s) = t$.

Thus when $t \ge \frac{A^2+A}{2}$, $0 < A \in \mathbb{Z}^+$, $2t^2 + 2t - (2A^2 + 2A - 2)$ is optimal. □

By Theorem 3.5, it is easy to prove the following corollary.

Corollary 3.6 *Let $\Gamma = \{2t^2 - 2 \mid t = A^2 + A - 2, A \ge 2, A \in \mathbb{Z}^+\}$. Then Γ is an optimal infinite family. When A is odd, the optimal step s is $t^2 - A^2 + 1$. When A is even, the optimal step s is $t^2 - A^2 - 2A$.*

By APPENDIX B [19], we see that $390 = 2 \times 14^2 - 2$ is suboptimal. In other words, there are some suboptimal integers in the set $\Sigma = \{2t^2 - 2 \mid t > 2, t \in \mathbb{Z}^+\}$. From Corollary 3.6, we see that the set Γ, which is a subset of the set Σ, is optimal. It would be interesting to find out a subset of the set Σ which is a suboptimal infinite family.

Theorem 3.7 *(1) Let $n = 2t^2 + 2t - 14$ and $\Theta = \{2t^2 + 2t - 14 \mid\ t > 14, t \ne 14e + 6$ and $t \ne 14e + 7, e \in \mathbb{Z}^+\}$. Then Θ is an optimal infinite family, and when $t = 14e, 14e + 1, 14e + 2, 14e + 3, 14e + 4, 14e + 5, 14e + 8, 14e + 9, 14e + 10, 14e + 11, 14e + 12, 14e + 13$ respectively, the optimal step s is $28e^2, 28e^2 + 4e - 2, 140e^2 + 48e, 140e^2 + 68e + 2, 84e^2 + 52e + 6, 84e^2 + 64e + 8, -84e^2 - 104e - 28 \pmod n, -84e^2 - 116e - 38 \pmod n, -140e^2 - 212e - 74 \pmod n, -140e^2 - 232e - 92 \pmod n, -28e^2 - 52e - 22 \pmod n, -28e^2 - 56e - 28 \pmod n$ correspondingly.*

(2) Let $\Psi = \{2t^2 + 2t - 14 \mid\ t = 14e + 6$ or $t = 14e + 7$, where $e \in \mathbb{Z}^+$ and $e \ge 1\}$. Then Ψ is a suboptimal infinite family, and when $t = 14e + 6$ or $14e + 7$, the suboptimal step s is $t^2 + 2t - 4, t^2 - 5$ correspondingly.

Proof:

(1) In the following we consider two cases:

(A) $t = 14e$, or $t = 14e + 2$, or $t = 14e + 4$, or $t = 14e + 8$, or $t = 14e + 10$, or $t = 14e + 12$;

(B) $t = 14e + 1$, or $t = 14e + 3$, or $t = 14e + 5$, or $t = 14e + 9$, or $t = 14e + 11$, or $t = 14e + 13$.

Case A When $t = 14e$, or $t = 14e+2$, or $t = 14e+4$, or $t = 14e+8$, or $t = 14e+10$, or $t = 14e+12$, we have $\gcd(t+1, 14) = 1$.

Case B When $t = 14e + 1$, or $t = 14e + 3$, or $t = 14e + 5$, or $t = 14e + 9$, or $t = 14e + 11$, or $t = 14e + 13$, we have $\gcd(t, 14) = 1$.

As $2 \times 14 + 1 = 29$ and 29 is a prime number, by Corollary 3.2, we see that Θ is an optimal infinite family.

When $t = 14e$, let $a = t, b = t+1, u = 2t-14$, and $v = 14$. As $1 \times b - e \times v = 1$, $s = 1 \times a + e \times u \pmod{n} = 28e^2$.

When $t = 14e + 1$ or $14e + 2$ or $14e + 3$ or $14e + 4$ or $14e + 5$ or $14e + 8$ or $14e + 9$ or $14e + 10$ or $14e + 11$ or $14e + 12$ or $14e + 13$, the corresponding optimal step s can be computed similarly. They are $28e^2 + 4e - 2, 140e^2 + 48e, 140e^2 + 68e + 2, 84e^2 + 52e + 6, 84e^2 + 64e + 8, -84e^2 - 104e - 28 \pmod{n}, -84e^2 - 116e - 38 \pmod{n}, -140e^2 - 212e - 74 \pmod{n}, -140e^2 - 232e - 92 \pmod{n}, -28e^2 - 52e - 22 \pmod{n}, -28e^2 - 56e - 28 \pmod{n}$ respectively.

(2) When $t = 14e+6$, we have $\gcd(t, 14) = \gcd(14e+6, 14) = \gcd(6, 14) = 2$ and $\gcd(t+1, 14) = \gcd(14e+7, 14) = \gcd(7, 14) = 7$.

When $t = 14e+7$, we have $\gcd(t, 14) = \gcd(14e+7, 14) = 7$ and $\gcd(t+1, 14) = \gcd(14e+8, 14) = \gcd(8, 14) = 2$.

As $2 \times 14 + 1 = 29$ and 29 is a prime number, by Corollary 3.2, we see that when $t = 14e + 6$ or $t = 14e + 7$, $1 \le e \in \mathbb{Z}^+$, $n = 2t^2 + 2t - 14$ can not be optimal.

When $t = 14e + 6$, let $a = 2, b = 2t - 4, u = t + 2$ and $v = t - 3$. As $\frac{4-t}{2} \times b + (t-3) \times v = 1$, let $s = \frac{4-t}{2} \times a - (t-3) \times u \equiv 10 - t^2 \pmod{n} = t^2 + 2t - 4$. By Lemma 2.7, we see that $(t+2, t-3)$ and $(-2, 2t-4)$ are two 0-points. By Lemma 2.5, we have $d(n; 1, s) = t + 1$.

When $t = 14e + 7$, let $a = t - 3, b = t + 2, u = 2t - 4$ and $v = 2$. As $1 \times b - \frac{t+1}{2} \times v = 1$, let $s \equiv 1 \times a + \frac{t+1}{2} \times u \pmod{n} = t^2 - 5$. By Lemma 2.7, we see that $(2t-4, 2)$ and $(-t+3, t+2)$ are two 0-points. Thus by Lemma 2.5 we have $d(n; 1, s) = t + 1$.

From above we see that when $t = 14e + 6$ or $t = 14e + 7$, where $1 \le e \in \mathbb{Z}^+$, $2t^2 + 2t - 14$ is suboptimal. □

Theorem 3.8 *1. Let* $\Theta = \{2t^2 + 2t - 15 \mid t > 15, t \ne 15e + 5$ *and* $t \ne 15e + 9, e \in \mathbb{Z}^+\}$. *Then* Θ *is an optimal infinite family, and when* $t = 15e, 15e + 1, 15e + 2, 15e + 3, 15e + 4, 15e + 6, 15e + 7, 15e + 8, 15e + 10, 15e + 11, 15e + 12, 15e + 13, 15e + 14$ *respectively, the optimal step s is* $30e^2, 240e^2 + 46e - 5, 240e^2 + 78e - 3, 120e^2 + 54e + 3, 120e^2 + 70e + 5, -60e^2 - 54e - 9 \pmod{n}, 60e^2 + 58e + 13, 60e^2 + 66e + 15, 330e^2 + 460e + 150, 330e^2 + 504e + 180, 210e^2 + 348e + 138, -30e^2 - 56e - 24 \pmod{n}, -30e^2 - 60e - 30 \pmod{n}$ *correspondingly.*

2. $\{G(2t^2 + 2t - 15; 1, 2t^2 - 10) \mid t = 15e + 5$ *or* $t = 15e + 9$, *where* $1 \le e \in \mathbb{Z}^+\}$ *is a suboptimal infinite family.*

Proof:

(1) In the following we consider two cases:

(A) $t = 15e$, or $t = 15e + 1$, or $t = 15e + 3$, or $t = 15e + 6$, or $t = 15e + 7$, or $t = 15e + 10$, or $t = 15e + 12$, or $t = 15e + 13$;

(B) $t = 15e + 2$, or $t = 15e + 4$, or $t = 15e + 8$, or $t = 15e + 11$, or $t = 15e + 14$.

Case A When $t = 15e$, or $t = 15e+1$, or $t = 15e+3$, or $t = 15e+6$, or $t = 15e+7$, or $t = 15e+10$, or $t = 15e + 12$, or $t = 15e + 13$, we have $\gcd(t + 1, 15) = 1$.

Case B When $t = 15e + 2$, or $t = 15e + 4$, or $t = 15e + 8$, or $t = 15e + 11$, or $t = 15e + 14$, we have $\gcd(t, 15) = 1$.

As $2 \times 15 + 1 = 31$ and 31 is a prime number, by Corollary 3.2 , we see that Θ is an optimal infinite family.

When $t = 15e$, let $a = t, b = t+1, u = 2t-15$, and $v = 15$. As $1 \times b - e \times v = 1$, $s = 1 \times a + e \times u \pmod n = 30e^2$.

When $t = 15e + 1$ or $15e + 2$ or $15 + 3$ or $15e + 4$ or $15e + 6$ or $15e + 7$ or $15e + 8$ or $15e + 10$ or $15e + 11$ or $15e + 12$ or $15e + 13$ or $15e + 14$, the corresponding optimal step s can be computed similarly. They are $240e^2 + 46e - 5, 240e^2 + 78e - 3, 120e^2 + 54e + 3, 120e^2 + 70e + 5, -60e^2 - 54e - 9 \pmod n), 60e^2 + 58e + 13, 60e^2 + 66e + 15, 330e^2 + 460e + 150, 330e^2 + 504e + 180, 210e^2 + 348e + 138, -30e^2 - 56e - 24 \pmod n), -30e^2 - 60e - 30 \pmod n)$ respectively.

(2) When $t = 15e + 5$, we have $\gcd(t, 15) = \gcd(15e + 5, 15) = \gcd(5, 15) = 5$ and $\gcd(t + 1, 15) = \gcd(15e + 6, 15) = \gcd(6, 15) = 3$.

When $t = 15e + 9$, we have $\gcd(t, 15) = \gcd(15e + 9, 15) = \gcd(9, 15) = 3$ and $\gcd(t + 1, 15) = \gcd(15e + 10, 15) = \gcd(10, 15) = 5$.

As $2 \times 15 + 1 = 31$ and 31 is a prime number, by Corollary 3.2 , we see that when $t = 15e + 5$ or $t = 15e + 9$, $1 \leq e \in \mathbb{Z}^+$, $n = 2t^2 + 2t - 15$ can not be optimal.

When $t = 15e+5$ or $t = 15e+9$, let $a = t, b = t+3, u = 2t-5$ and $v = 1$. As $0 \times b + 1 \times v = 1$, let $s = 0 \times a - 1 \times u \equiv 5 - 2t \pmod n = 2t^2 - 10$. By Lemma 2.7, we see that $(2t - 5, 1)$ and $(-t, t + 3)$ are two 0-points. By Lemma 2.5, we have $d(n; 1, s) = t + 1$.

From above we see that when $t = 15e+5$ or $t = 15e+9$, where $1 \leq e \in \mathbb{Z}^+$, $G(2t^2 + 2t - 15; 1, 2t^2 - 10)$ is suboptimal. □

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