

Refined Enumeration of Corners in Tree-like Tableaux

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Tree-like tableaux are certain fillings of Ferrers diagrams originally introduced by Aval et al., which are in simple bijections with permutation tableaux coming from Postnikov's study of totally nonnegative Grassmanian and alternative tableaux introduced by Viennot. In this paper, we confirm two conjectures of Gao et al. on the refined enumeration of non-occupied corners in tree-like tableaux and symmetric tree-like tableaux via intermediate structures of alternative tableaux, linked partitions, type B alternative tableaux and type B linked partitions.

Keywords: tree-like tableau, alternative tableau, linked partition, corner

1 Introduction

This paper is concerned with two conjectures of Gao et al. (2016) on the refined enumeration of tree-like tableaux and symmetric tree-like tableaux with respect to number of non-occupied corners. Tree-like tableaux are certain fillings of Ferrers diagrams originally introduced by Aval et al. (2013), which are in simple bijections with permutation tableaux coming from Postnikov's study of totally nonnegative Grassmanian in Postnikov (2006) and alternative tableaux introduced by Viennot (2008). These three equivalent combinatorial objects are closely related to a statistical physics model called partially asymmetric exclusion process (PASEP); see Corteel (2007); Corteel and Williams (2007a,b).

A *Ferrers diagram* is the left-justified arrangement of square cells with possibly empty rows and columns. The *size* of a Ferrers diagram is the number of rows plus the number of columns. Given a Ferrers diagram of size n , we label the steps in the south-east border with $1, 2, \dots, n$ from north-east to south-west. A row (resp. column) is labeled with i if the row (resp. column) contains the south (resp. west) step labeled with i . Notice that we may place a row label to the left of the first column and place a column label at the top of the first row; see Figure 1. A row (resp. column) labeled with i is called row (resp. column) i . The steps of the south-east border are called *border edges*. The cell (i, j) is the cell in row i and column j unless otherwise stated.

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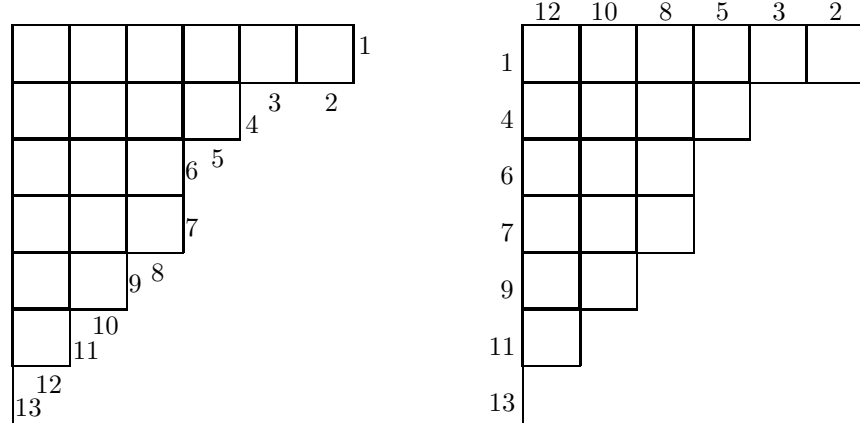


Fig. 1: The labelling of a Ferrers diagram.

A *tree-like* tableau is a filling of a Ferrers diagram without empty rows and columns with points inside some cells, such that the resulting diagram satisfies the following conditions:

- (1) the top left cell contains a point, called the root point;
- (2) for every non-root pointed cell c , there exists either a pointed cell above c in the same column, or a pointed cell to its left in the same row, but not both;
- (3) every row and every column possess at least one pointed cell.

The size of a tree-like tableau is defined to be its number of points. The left subfigure of Figure 2 illustrates a tree-like tableau of size 11. It is well known that the size of a tree-like tableau is equal to the size of its underlying diagram minus one. Denote by \mathcal{T}_n the set of tree-like tableaux of size n . A *symmetric tree-like tableau* is a tree-like tableau unchanged by the reflection with respect to its main diagonal line; see the right subfigure of Figure 2. Since the size of a symmetric tree-like tableau is necessarily odd, we denote by \mathcal{T}_{2n+1}^{sym} the set of symmetric tree-like tableaux of size $2n + 1$.

In a Ferrers diagram, a *corner* is a cell such that its bottom and right edges are border edges. If the corner c is not pointed in a tree-like tableau T , then we say that c is a *non-occupied* corner of T . Recently, Laborde-Zubieta (2015) noticed that the corners in tree-like tableaux could be interpreted in the PASEP as the locations where a jump of particle is possible. In Laborde-Zubieta (2015), he also obtained the enumeration of occupied corners in tree-like tableaux and symmetric tree-like tableaux and posed two conjectures concerning the total number of corners in tree-like tableaux and symmetric tree like tableaux. These two conjectures were independently confirmed by Gao et al. (2016), and Hitczenko and Lohss (2016).

Given a tree-like tableau T , the *weight* of T is defined to be $a^{top(T)}b^{left(T)}$, where $top(T)$ is the number of non-root pointed cells in the topmost row, and $left(T)$ is the number of non-root pointed cells in the

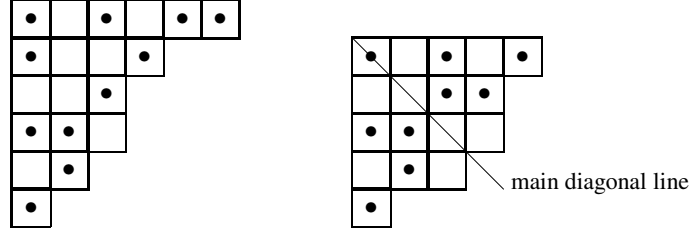


Fig. 2: A tree-like tableau (left) and a symmetric tree-like tableau (right).

leftmost column. The polynomial analogues of the number of tree-like tableaux of size n and symmetric tree-like tableaux of size $2n + 1$ are defined as follows:

$$T_n(a, b) = \sum_{T \in \mathcal{T}_n} a^{\text{top}(T)} b^{\text{left}(T)},$$

$$T_{2n+1}^{\text{sym}}(a, b) = \sum_{T \in \mathcal{T}_{2n+1}^{\text{sym}}} x^{\text{left}(T)-1}.$$

By introducing an elementary insertion procedure, Aval et al. (2013) showed that

$$T_n(a, b) = \sum_{T \in \mathcal{T}_n} a^{\text{top}(T)} b^{\text{left}(T)} = (a + b)_{n-1}, \quad (1.1)$$

and

$$T_{2n+1}^{\text{sym}}(x) = \sum_{T \in \mathcal{T}_{2n+1}^{\text{sym}}} x^{\text{left}(T)-1} = 2^n (x + 1)_{n-1}, \quad (1.2)$$

where $(x)_n$ denotes the rising factorial, that is, $(x)_0 = 1$ and $(x)_n = x(x + 1) \dots (x + n - 1)$ for $n \geq 1$. In order to refine the enumeration of non-occupied corners in tree-like tableaux and symmetric tree like tableaux, Gao et al. (2016) posed the following two conjectures.

Conjecture 1.1 (See Gao et al. (2016), Conjecture 4.10) *For $n \geq 3$, the (a, b) -analogue of the number of non-occupied corners in tree-like tableaux of size n is given by*

$$\text{noc}_n(a, b) = \sum_{T \in \mathcal{T}_n} \text{noc}(T) a^{\text{top}(T)} b^{\text{left}(T)} = ((n - 2)ab + \binom{n - 2}{2}(a + b) + \binom{n - 2}{3}) \cdot T_{n-2}(a, b)$$

where $\text{noc}(T)$ is the number of non-occupied corners of T .

Conjecture 1.2 (See Gao et al. (2016), Conjecture 4.14) *For $n \geq 3$, the x -analogue of the number of non-occupied corners in symmetric tree-like tableaux of size $2n + 1$ is given by*

$$\sum_{T \in \mathcal{T}_{2n+1}^{\text{sym}}} \text{noc}(T) x^{\text{left}(T)-1} = [2nx^2 + 2(2n^2 - 4n + 1)x + \frac{(n - 2)(n - 1)(4n - 3)}{3}] \cdot T_{2n-3}^{\text{sym}}(x).$$

To confirm Conjecture 1.1, we establish a bijection between tree-like tableaux of size n and alternative tableaux of size n in which each column contains an up arrow, and a bijection between alternative tableaux of size n in which each column contains an up arrow and linked partitions of $[n]$. The notion of linked partitions was introduced by Dykema (2007) in the study of the unsymmetrized T-transform in free probability theory. The combination of these two bijections enables us to compute the left-hand side of Conjecture 1.1 in terms of linked partitions.

In order to verify Conjecture 1.2, we introduce the notions of type B alternative tableaux and type B linked partitions. We establish a bijection between symmetric alternative tableaux of size $2n$ and type B alternative tableaux of size n , and a bijection between type B alternative tableaux of size n and type B linked partitions of $[n]$. By combining these two bijections and the bijection between symmetric tree-like tableaux of size $2n + 1$ and symmetric alternative tableaux of size $2n$ given by Aval et al. (2013), we can compute the left-hand side of Conjecture 1.2 in terms of type B linked partitions.

2 The bijection between tree-like tableaux and alternative tableaux

In this section, we give an overview of the bijection α between tree-like tableaux of size n and alternative tableaux of size $n - 1$ which was established by Aval et al. (2013). Analogous to the bijection α , we establish a bijection between tree-like tableaux of size n and alternative tableaux of size n in which each column contains an up arrow. To prove Conjecture 1.2, we introduce the notion of type B alternative tableaux and construct a bijection between type B alternative tableaux of size n and symmetric alternative tableaux of size $2n$.

An *alternative* tableau is a Ferrers diagram with a partial filling of the cells with left arrows \leftarrow and up arrows \uparrow , such that all the cells to the left of a left arrow \leftarrow , or above an up arrow \uparrow are empty. In other words, all the cells pointed by an arrow must be empty. Alternative tableaux were first introduced by Viennot (2008) and systematically studied by Nadeau (2011). The size of an alternative tableau is defined to be the size of its underlying diagram. We denote by \mathcal{AT}_n the set of alternative tableaux of size n . In an alternative tableau T , a row is said to be *unrestricted* if it has no $\leftarrow' s$. A corner c is said to be *non-occupied* if the cell c is empty. Denote by $urr(T)$ and $noc(T)$ the number of unrestricted rows and the number of non-occupied corners of T , respectively.

A *symmetric* alternative tableau is an alternative tableau unchanged by the reflection with respect to its main diagonal line. It is obvious that the size of a symmetric alternative tableau is even. Denote by \mathcal{AT}_{2n}^{sym} the set of symmetric alternative tableaux of size $2n$.

Recall that \mathcal{T}_n denotes the set of tree-like tableaux of size n . In Aval et al. (2013), they established a bijection α between the set \mathcal{T}_n and the set \mathcal{AT}_{n-1} . In the following, we give a description of the bijection α and its inverse α^{-1} without proof, see Aval et al. (2013) for more details.

Given a tree-like tableau $T \in \mathcal{T}_n$ with underlying diagram F , we first replace each non-root point p with an \uparrow if there is no point above p in the same column and with a \leftarrow if there is no point to the left of p in the same row. Then, we remove the topmost row and the leftmost column from F . Let $\alpha(T)$ be the resulting filling. For example, Figure 6 illustrates a tree-like tableau T and its corresponding alternative tableau $\alpha(T)$.

The inverse map α^{-1} is defined as follows. Let T' be an alternative tableau of size $n - 1$ with underlying diagram F' . Suppose that F' has k rows and $n - 1 - k$ columns. We first construct the underlying diagram F of $\alpha^{-1}(T')$ from F' by adding a column of k cells to the left of the leftmost column and adding a row of $n - 1 - k$ cells above its topmost row and a cell at its top left corner. It is easily seen that F is of size

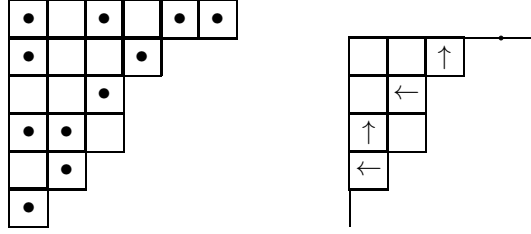


Fig. 3: A tree-like tableau T (left) and its corresponding alternative tableaux tableau $\alpha(T)$ (right).

$n + 1$. Next, we fill the leftmost cell in row i of F' with a point if and only if row $i - 1$ of F' has no \leftarrow' s . Analogously, we fill the topmost cell in column j of F' with a point if and only if column $j - 1$ of F' has no \uparrow' s . Finally, we fill each cell $(i + 1, j + 1)$ with a point if and only if the cell (i, j) is filled with an arrow in F' and fill a point in the top left cell of F . Let $\alpha^{-1}(T')$ be the resulting tableau.

From the definition of the bijection α , it is not difficult to see that α maps a non-root point in the first column of T to an unrestricted row of T' . Moreover, the bijection α preserves the number of non-occupied corners since every row and every column of T possess at least one pointed cell. Hence, we have the following properties of the bijection α .

Theorem 2.1 *Let $n \geq 1$. For any $T \in \mathcal{T}_n$ and $T' \in \mathcal{AT}_{n-1}$ with $\alpha(T) = T'$, we have $left(T) = urr(T')$ and $noc(T) = noc(T')$.*

Theorem 2.2 *Let $n \geq 1$. The bijection α restricted to the set of symmetric tree-like tableaux of size $2n + 1$ induces a bijection between the set \mathcal{T}_{2n+1}^{sym} and the set \mathcal{AT}_{2n}^{sym} . Furthermore, for any $T \in \mathcal{T}_{2n+1}^{sym}$ and $T' \in \mathcal{AT}_{2n}^{sym}$ with $\alpha(T) = T'$, we have $left(T) = urr(T')$ and $noc(T) = noc(T')$.*

In order to prove Conjecture 1.1, we need to consider a subset of $T \in \mathcal{AT}_n$, denoted by \mathcal{AT}_n^* , in which each column of T contains an \uparrow . Analogous to the bijection α , we establish a bijection β between the set \mathcal{T}_n to the set \mathcal{AT}_n^* without proof.

Given a tree-like tableau T of size n with underlying diagram F , we construct an alternative tableau $T' = \beta(T)$ as follows. First replace each non-root point p with an \uparrow if there is no point above p in the same column and with a \leftarrow if there is no point to the left of p . Then, T' is the resulting tableau by removing the leftmost column from F . For example, Figure 4 illustrates a tree-like tableau T and its corresponding alternative tableau $\beta(T)$.

The inverse map β^{-1} is defined as follows. Let T' be an alternative tableau of size n with underlying diagram F' . Suppose that F' has k rows and $n - k$ columns. We first construct the underlying diagram F of $\beta^{-1}(T')$ from F' by adding a column of k cells to the left of the leftmost column. It is easily seen that F is of size $n + 1$. Next, we fill the leftmost cell of row i of F with a point if and only if row i of F' contains no \leftarrow' s . Finally, fill the cell (i, j) of F with a point if and only if the cell (i, j) of F' has an arrow. Let $\beta^{-1}(T')$ be the resulting tableau.

By similar arguments as for the bijection α , we can get the following properties of β analogous to α .

Theorem 2.3 *Let $n \geq 1$. For any $T \in \mathcal{T}_n$ and $T' \in \mathcal{AT}_n^*$ with $\beta(T) = T'$, we have the following relations.*

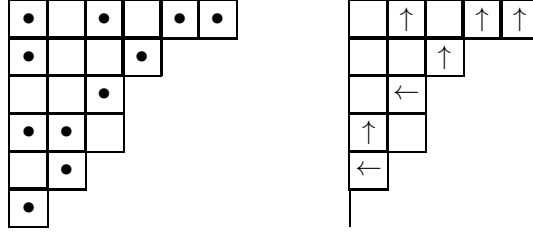


Fig. 4: A tree-like tableau T (left) and its corresponding alternative tableau $\beta(T)$ (right).

- (1) $left(T) = urr(T') - 1$;
- (2) $noc(T) = noc(T')$;
- (3) $top(T) = top(T')$, where $top(T')$ is the number of arrows in the topmost row of T' .

From Theorem 2.3, we deduce that, for $n \geq 1$,

$$\sum_{T \in \mathcal{T}_n} a^{top(T)} b^{left(T)} = \sum_{T \in \mathcal{AT}_n^*} a^{top(T)} b^{urr(T)-1} \quad (2.1)$$

and

$$\sum_{T \in \mathcal{T}_n} noc(T) a^{top(T)} b^{left(T)} = \sum_{T \in \mathcal{AT}_n^*} noc(T) a^{top(T)} b^{urr(T)-1}. \quad (2.2)$$

Let n be a positive integer. Combining (1.1) and (2.1), we have

$$\sum_{T \in \mathcal{AT}_n^*} a^{top(T)} b^{urr(T)-1} = (a + b)_{n-1}. \quad (2.3)$$

Notice that (2.3) was first proved by Corteel and Nadeau (2009) using recurrence relations. Later, Corteel and Kim (2009) provided two bijective proofs of (2.3).

In order to prove Conjecture 1.2, we need to introduce the notion of *type B alternative tableaux*, which are defined based on shifted Ferrers diagrams. For a Ferrers diagram F with k columns, the *shifted* Ferrers diagram of F , denoted by \bar{F} , is the diagram obtained from F by adding k rows of size $1, 2, \dots, k$ above it in increasing order. The size of \bar{F} is defined to be the size of F . In this context, the Ferrers diagram F is called the subdiagram of \bar{F} . The *diagonal* of \bar{F} is the set of rightmost cells in the added rows. A *diagonal* cell is a cell in the diagonal. We label the added rows as follows. If the diagonal cell of an added row is in column i , then the row is labeled with $-i$. The labels of the other rows and columns remain the same with F . Figure 5 illustrates a Ferrers diagram F and its corresponding shifted Ferrers diagram \bar{F} , where the diagonal cells are marked with stars.

A *type B alternative* tableau is a shifted Ferrers diagram with a partial filling of the cells with left arrows \leftarrow and up arrows \uparrow satisfying the following conditions:

- (1) all the cells to the left of a \leftarrow in the same row, or above an \uparrow in the same column are empty;

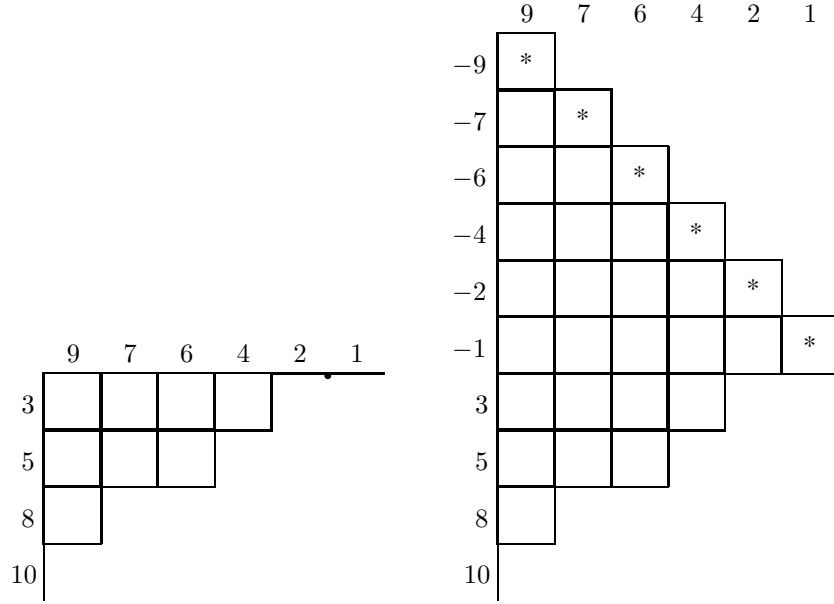


Fig. 5: A Ferrers diagram F (left) and its corresponding shifted Ferrers diagram \bar{F} (right).

- (2) if column i contains an \uparrow , then row $-i$ is empty;
- (3) the diagonal cells are empty.

The size of an alternative tableau of type B is defined to be the size of the corresponding shifted Ferrers diagram. Figure 6 illustrates a type B alternative tableau of size 10. We denote by \mathcal{AT}_n^B the set of alternative tableaux of type B of size n . In an alternative tableau of type B , row i is said to be *unrestricted* if and only if row i has no \leftarrow and column $|i|$, if it exists, has no \uparrow . Denote by $urr(T)$ the number of unrestricted rows of a type B alternative tableau T . Analogous to the ordinary Ferrers diagram, a cell is called a *corner* of a shifted Ferrers diagram if both its bottom and right edges are border edges. If a corner c of a type B alternative tableau is empty, then we say that c is non-occupied. Let $noc(T)$ denote the number of non-occupied corners of a type B alternative tableau T .

In the following, we describe a bijection γ between the set \mathcal{AT}_{2n}^{sym} and the set \mathcal{AT}_n^B . For any symmetric alternative tableau $T \in \mathcal{AT}_{2n}^{sym}$, we can obtain a type B alternative tableau $\gamma(T) \in \mathcal{AT}_n^B$ by removing all the cells strictly to the right of the main diagonal of T . Conversely, given a type B alternative tableau $T' \in \mathcal{AT}_n^B$, we can recover a symmetric alternative tableau $\gamma^{-1}(T') \in \mathcal{AT}_{2n}^{sym}$ by adding the cells obtained by reflecting the non-diagonal cells across its main diagonal. By the definition of the bijection γ , one can easily verify that γ has the following properties.

Theorem 2.4 *Let $n \geq 1$. For any $T \in \mathcal{AT}_{2n}^{sym}$ and $T' \in \mathcal{AT}_n^B$ with $\gamma(T) = T'$, we have the following relations.*

- (1) $urr(T) = urr(T')$;

	9	7	6	4	2	1
-9						
-7	↑					
-6						
-4		←				
-2	←		↑			
-1				←		
3			←			
5		←				
8	←					
10						

Fig. 6: A type B alternative tableau.

(2) $noc(T) = noc'(T') + 2noc''(T')$, where $noc''(T') = noc(T') - noc'(T')$ and $noc'(T')$ is the number of non-occupied diagonal corners of T' .

Combining Theorems 2.2 and 2.4, we have

$$\sum_{T \in \mathcal{T}_{2n+1}^{sym}} x^{left(T)-1} = \sum_{T \in \mathcal{AT}_{2n}^{sym}} x^{urr(T)-1} = \sum_{T \in \mathcal{AT}_n^B} x^{urr(T)-1}, \quad (2.4)$$

and

$$\begin{aligned} \sum_{T \in \mathcal{T}_{2n+1}^{sym}} noc(T) x^{left(T)-1} &= \sum_{T \in \mathcal{AT}_{2n}^{sym}} noc(T) x^{urr(T)-1} \\ &= \sum_{T \in \mathcal{AT}_n^B} (noc'(T) + 2noc''(T)) x^{urr(T)-1}. \end{aligned} \quad (2.5)$$

In view of (1.2) and (2.4), we have

$$AT_n^B(x) = \sum_{T \in \mathcal{AT}_n^B} x^{urr(T)-1} = 2^n (x+1)_{n-1}. \quad (2.6)$$

3 A bijection between \mathcal{AT}_n^* and \mathcal{L}_n

In this section, we will present a bijection between alternative tableaux in which each column contains an \uparrow and linked partitions. We begin with some definitions and notations.

Let $[n] = \{1, 2, \dots, n\}$. A *linked* partition of $[n]$ is a collection of nonempty subsets B_1, B_2, \dots, B_k of $[n]$, called blocks, such that the union of B_1, B_2, \dots, B_k is $[n]$ and any two distinct blocks are nearly

disjoint. Two blocks B_i and B_j are said to be *nearly disjoint* if for any $k \in B_i \cap B_j$ one of the following conditions holds:

- (1) $k = \min(B_i)$, $|B_i| > 1$, and $k \neq \min(B_j)$;
- (2) $k = \min(B_j)$, $|B_j| > 1$, and $k \neq \min(B_i)$.

We denote by \mathcal{L}_n the set of linked partitions of $[n]$. Chen et al. (2008) introduced the linear representation of a linked partition. For a linked partition of $[n]$, we arrange n vertices in a horizontal line and label them with $1, 2, \dots, n$ from left to right. For a block $B = \{b_1, b_2, \dots, b_k\}$ with $k \geq 2$ and $b_1 < b_2 < \dots < b_k$, we draw an arc from b_1 to b_j for $j = 2, \dots, k$. As an example, the linear representation of the linked partition $\{1, 2, 3, 6\}, \{4, 5\}, \{6, 8\}, \{7, 9\}, \{9, 10\}, \{11\}$ is illustrated in Figure 7. For $i < j$, we use a pair (i, j) to denote an arc from i to j , and we call i and j the left-hand endpoint and the right-hand endpoint of arc (i, j) , respectively. A vertex labeled with i is called vertex i . Notice that the linear representation of a linked partition can also be viewed as a simple graph on $[n]$ such that for each vertex i there is at most one vertex j such that $1 \leq j \leq i$ and j is connected to i .

A *type B* linked partition of $[n]$ is a linked partition of $[n]$ in which each integer may be negated. We denote by \mathcal{L}_n^B the set of type B linked partition of $[n]$. The linear representation of a type B linked partitions is the same as that of the ordinary linked partition except that the labels of the vertices maybe negative and the labels are increasing from left to right.

From now on, we will only consider linear representations of the (type B) linked partitions unless otherwise stated. Let τ be a linked partition of $[n]$. A vertex t of τ is called an *origin* if it is only a left-hand endpoint, or a *transient* if it is both a left-hand point and a right-hand endpoint, or a *singleton* if it is an isolated vertex, or a *destination* if it is only a right-hand endpoint. For a destination i_t , a path (i_1, i_2, \dots, i_t) with $i_1 < i_2 < \dots < i_t$ is said to be a *maximal* path with destination i_t if i_1 is an origin. For example, in the linked partition given in Figure 7, vertices 1, 4, 7 are origins, vertices 6, 9 are transients, vertices 2, 3, 5, 8, 10 are destinations, vertex 11 is a singleton and (1,6,8) is a maximal path with destination 8.

Now we define a map $\Phi: \mathcal{AT}_n^* \rightarrow \mathcal{L}_n$. Let T be an alternative tableau of size n without empty columns. We construct a linked partition $\tau = \Psi(T)$ as follows. First, we arrange n vertices in a line and label them by $1, 2, \dots, n$ from left to right. For column j , suppose that the cell (i_1, j) is filled with an \uparrow , and the cells $(i_2, j), (i_3, j), \dots, (i_t, j)$ with $i_2 < i_3 < \dots < i_t$ are cells filled with \leftarrow' s. For $\ell = 1, 2, \dots, t$, let $(i_\ell, i_{\ell+1})$ be an arc in τ , where $i_{\ell+1} = j$. For example, Figure 7 demonstrates an alternative tableau $T \in \mathcal{AT}_{11}^*$ and its corresponding linked partition $\Phi(T) \in \mathcal{L}_{11}$. Notice that there is at most one \leftarrow in each row and exactly one \uparrow in each column. This ensures that for any vertex j in τ , there is at most one arc whose right-hand endpoint is j . Hence, we have $\Phi(T) \in \mathcal{L}_n$.

Theorem 3.1 *The map $\Phi: \mathcal{AT}_n^* \rightarrow \mathcal{L}_n$ is a bijection.*

Proof: We will prove the assertion by constructing the inverse map of Φ . Now we define a map $\Psi: \mathcal{L}_n \rightarrow \mathcal{AT}_n^*$. Let τ be a linked partition of $[n]$. Let F be a Ferrers diagram of size n such that for all $j \in [n]$, j is the label of a column if and only if vertex j is a destination.

Suppose that τ has exactly k destinations, which are vertices b_1, b_2, \dots, b_k with $b_1 > b_2 > \dots > b_k$. We set $T^{(0)}$ to be the empty filling of F and $\tau^{(0)} = \tau$. For m from 1 to k , we define $T^{(m)}$ to be the tableau obtained from $T^{(m-1)}$ by filling the cells in column b_m as follows. Suppose that the maximal path in $\tau^{(m-1)}$ with destination b_m is $(a_1, a_2, \dots, a_t, b_m)$. Then we fill the cell (a_1, b_m) with an \uparrow . For

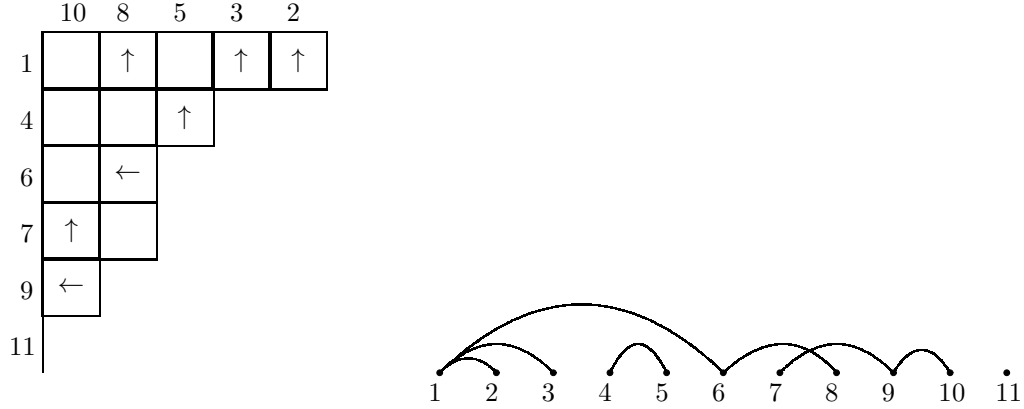


Fig. 7: An alternative tableau T (left) and its corresponding linked partition $\Phi(T)$ (right).

all $2 \leq j \leq t$, fill the cell (a_j, b_m) with a \leftarrow . Let $\tau^{(m)}$ be the linked partition obtained from $\tau^{(m-1)}$ by removing the arcs in the path $(a_1, a_2, \dots, a_t, b_m)$. Set $T = \Psi(\tau) = T^{(k)}$.

It is apparent that each column of F is filled with an \uparrow possibly along with some \leftarrow 's. We claim that the cells pointed by an arrow are empty in $T^{(m)}$ for all $0 \leq m \leq k$. If not, suppose that the cell (a_s, b_p) is non-empty in $T^{(m)}$ for some $2 \leq s \leq t$ and $p < m$. Then the arc (a_{s-1}, a_s) would be in the maximal path with destination b_p . According to the construction of $\tau^{(p)}$, the arc (a_{s-1}, a_s) is not contained in $\tau^{(j)}$ for all $j \geq p$. This contradicts the fact that $\tau^{(m-1)}$ contains the arc (a_{s-1}, a_s) . Hence, the claim is proved. This implies that $\Psi(\tau) \in \mathcal{AT}_n^*$. From the definition of the map Φ , it is clear that $\Phi(T) = \tau$. This implies that the map $\Phi : \mathcal{AT}_n^* \rightarrow \mathcal{L}_n$ is a bijection with inverse $\Phi^{-1} = \Psi$. This completes the proof. \square

From the definition of the map Φ , it is not difficult to see that Φ has the following properties.

Theorem 3.2 For any $T \in \mathcal{AT}_n^*$ and $\tau \in \mathcal{L}_n$ with $\Phi(T) = \tau$, we have the following.

- (1) The labels of the columns of T are exactly the labels of the destinations of τ .
- (2) The labels of the rows of T are exactly the labels of the origins, singletons and transients of τ .
- (3) The labels of the unrestricted rows of T are exactly the labels of the origins and singletons of τ .
- (4) The number of \uparrow 's in the topmost row of T is equal to the number of arcs whose left-hand endpoints are vertex 1 in τ .

To conclude this section, we remark that Chen et al. (2013) established a bijection between permutation tableaux of size n and linked partitions of $[n]$. The combination of their bijection and the bijection between alternative tableau of size n in which each column contains an up arrow and permutation tableaux of size n established by Corteel and Nadeau (2009) is in fact identical with our bijection Φ .

4 Proof of Conjecture 1.1

In this section, we aim to prove Conjecture 1.1. For $2 \leq i \leq n$, denote by $\mathcal{AT}_{n,i}^*$ the set of alternative tableaux $T \in \mathcal{AT}_n^*$ such that the cell $(i-1, i)$ of T is a non-occupied corner. Notice that for any $T \in \mathcal{AT}_n^*$, each column of T contains an up arrow. This implies that the cell $(1, 2)$ cannot be a non-occupied corner of T . Hence we have $\mathcal{AT}_{n,2}^* = \emptyset$.

Recall that for an alternative tableau T , the weight of T is defined to be $a^{\text{top}(T)}b^{\text{urr}(T)-1}$. Analogously, for a linked partition τ , we define the *weight* of τ to be $a^{\text{one}(\tau)}b^{\text{os}(\tau)-1}$, where $\text{os}(T)$ is the total number of origins and singletons of τ and $\text{one}(\tau)$ is the number of arcs whose left-hand endpoints are vertex 1 in τ . Let

$$L_n(a, b) = \sum_{\tau \in \mathcal{L}_n} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1}.$$

By Theorem 3.2 and Formula (2.3), we have

$$L_n(a, b) = \sum_{\tau \in \mathcal{L}_n} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1} = \sum_{T \in \mathcal{AT}_n^*} a^{\text{top}(T)}b^{\text{urr}(T)-1} = (a+b)_{n-1}. \quad (4.1)$$

By Theorems 3.1 and 3.2, the bijection Φ maps an alternative tableau $T \in \mathcal{AT}_{n,i}^*$ to a linked partition $\tau = \Phi(T)$ satisfying that

- (a1) $\text{top}(T) = \text{one}(\tau)$ and $\text{urr}(T) = \text{os}(\tau)$;
- (a2) vertex i is a destination of τ ;
- (a3) vertex $i-1$ is not a destination of τ ;
- (a4) $(i-1, i)$ is not an arc in τ .

For all $2 \leq i \leq n$, denote by $\mathcal{L}_{n,i}$ the set of linked partitions of n satisfying conditions (a2), (a3) and (a4). Thus, the map Φ is a weight preserving bijection between the set $\mathcal{AT}_{n,i}^*$ and the set $\mathcal{L}_{n,i}$. This yields that, for all $3 \leq i \leq n$,

$$\sum_{T \in \mathcal{AT}_{n,i}^*} a^{\text{top}(T)}b^{\text{urr}(T)-1} = \sum_{\tau \in \mathcal{L}_{n,i}} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1}. \quad (4.2)$$

Lemma 4.1 For $n \geq 3$, we have

$$\sum_{i=3}^n \sum_{\tau \in \mathcal{L}_{n,i}} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1} = [(n-2)ab + \binom{n-2}{2}(a+b) + \binom{n-2}{3}] \cdot L_{n-2}(a, b). \quad (4.3)$$

Proof: For $3 \leq i \leq n$, let $\mathcal{M}_{n,i}$ denote the set of linked partitions of $[n]$ satisfying conditions (a2) and (a4). Denote by $\mathcal{N}_{n,i}$ the set of linked partitions of $[n]$ in which both vertex $i-1$ and vertex i are destinations. It is easily seen that

$$\sum_{i=3}^n \sum_{\tau \in \mathcal{L}_{n,i}} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1} = \sum_{i=3}^n \sum_{\tau \in \mathcal{M}_{n,i}} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1} - \sum_{i=3}^n \sum_{\tau \in \mathcal{N}_{n,i}} a^{\text{one}(\tau)}b^{\text{os}(\tau)-1}. \quad (4.4)$$

For any $\tau \in \mathcal{M}_{n,i}$, we can obtain a linked partition $\tau' \in \mathcal{L}_{n-1}$ by removing vertex i and the arc whose right-hand endpoint is vertex i , and relabeling the rest vertices by $1, 2, \dots, n-1$. Conversely, given a linked partition $\tau' \in \mathcal{L}_{n-1}$, we can get a linked partition $\tau \in \mathcal{M}_{n,i}$ by inserting a vertex immediately after vertex $i-1$, relabeling the vertices by $1, 2, \dots, n$ from left to right, and adjoining an arc from vertex j to vertex i for some $1 \leq j \leq i-2$. It is apparent that $one(\tau) = one(\tau') + 1$ when $j = 1$ and $one(\tau) = one(\tau')$ otherwise. Moreover, we have $os(\tau) = os(\tau')$. Hence, for $3 \leq i \leq n$, we have

$$\sum_{\tau \in \mathcal{M}_{n,i}} a^{one(\tau)} b^{os(\tau)-1} = \sum_{\tau' \in \mathcal{L}_{n-1}} (i-3+a) \cdot a^{one(\tau')} b^{os(\tau')-1} = (i-3+a) \cdot L_{n-1}(a, b). \quad (4.5)$$

Summing over i from 3 to n , we deduce that

$$\begin{aligned} \sum_{i=3}^n \sum_{\tau \in \mathcal{M}_{n,i}} a^{one(\tau)} b^{os(\tau)-1} &= \sum_{i=3}^n (i-3+a) \cdot L_{n-1}(a, b) \\ &= \binom{n-2}{2} + (n-2)a \cdot L_{n-1}(a, b). \end{aligned} \quad (4.6)$$

From (4.1) we have $L_{n-1}(a, b) = (a+b+n-3)L_{n-2}(a, b)$. Substituting it into the above formula, we get

$$\sum_{i=3}^n \sum_{\tau \in \mathcal{M}_{n,i}} a^{one(\tau)} b^{os(\tau)-1} = \left(\binom{n-2}{2} + (n-2)a \right) \cdot (a+b+n-3) \cdot L_{n-2}(a, b). \quad (4.7)$$

Let $3 \leq i \leq n$. For any $\tau \in \mathcal{N}_{n,i}$, we can obtain a linked partition $\tau' \in \mathcal{L}_{n-2}$ by removing vertices $i-1$ and i and the arcs whose right-hand points are vertices $i-1$ and i , and relabeling the vertices by $1, 2, \dots, n-2$. Conversely, given a linked partition $\tau' \in \mathcal{L}_{n-2}$, we can get a linked partition $\tau \in \mathcal{N}_{n,i}$ by inserting two vertices immediately after vertex $i-2$, relabeling the vertices by $1, 2, \dots, n$, and adjoining an arc from vertex j_1 (resp. j_2) to vertex $i-1$ (resp. vertex i) for some $1 \leq j_1, j_2 \leq i-2$. It is apparent that

$$one(\tau) = \begin{cases} one(\tau') + 2 & \text{if } j_1 = j_2 = 1; \\ one(\tau') + 1 & \text{if either } j_1 = 1 \text{ or } j_2 = 1, \text{ but not both;} \\ one(\tau') & \text{if } j_1, j_2 \neq 1. \end{cases}$$

Moreover, we have $os(\tau) = os(\tau')$. Hence, for $3 \leq i \leq n$, we have

$$\begin{aligned} \sum_{\tau \in \mathcal{N}_{n,i}} a^{one(\tau)} b^{os(\tau)-1} &= \sum_{\tau' \in \mathcal{L}_{n-2}} ((i-3)^2 + 2a(i-3) + a^2) \cdot a^{one(\tau')} b^{os(\tau')-1} \\ &= ((i-3)^2 + 2a(i-3) + a^2) L_{n-2}(a, b). \end{aligned} \quad (4.8)$$

Summing over i from 3 to n , we deduce that

$$\begin{aligned} \sum_{i=3}^n \sum_{\tau \in \mathcal{N}_{n,i}} a^{one(\tau)} b^{os(\tau)-1} &= \sum_{i=3}^n ((i-3)^2 + 2a(i-3) + a^2) \cdot L_{n-2}(a, b) \\ &= \frac{(n-3)(n-2)(2n-5)}{6} + (n-2)a^2 + 2\binom{n-2}{2}a \cdot L_{n-2}(a, b). \end{aligned} \quad (4.9)$$

Combining (4.4), (4.7) and (4.9) and by simple computation, we are led to (4.3). This completes the proof. \square

Now we are in the position to complete the proof of Conjecture 1.1.

Proof of Conjecture 1.1. For $n \geq 3$, we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_n} noc(T) a^{top(T)} b^{left(T)} &= \sum_{T \in \mathcal{AT}_n^*} noc(T) a^{top(T)} b^{urr(T)-1} \quad (\text{by (2.2)}) \\ &= \sum_{i=2}^n \sum_{T \in \mathcal{AT}_{n,i}^*} a^{top(T)} b^{urr(T)-1}. \end{aligned}$$

Recall that $\mathcal{AT}_{n,2}^* = \emptyset$. This yields that

$$\sum_{T \in \mathcal{T}_n} \text{noc}(T) a^{\text{top}(T)} b^{\text{left}(T)} = \sum_{i=3}^n \sum_{\tau \in \mathcal{L}_{n,i}} a^{\text{one}(\tau)} b^{\text{os}(\tau)-1} \quad (\text{by (4.2)}).$$

By Lemma 4.1, we have

$$\sum_{T \in \mathcal{T}_n} \text{noc}(T) a^{\text{top}(T)} b^{\text{left}(T)} = [(n-2)ab + \binom{n-2}{2}(a+b) + \binom{n-2}{3}] \cdot L_{n-2}(a, b).$$

Recall that $L_n(a, b) = (a+b)_{n-1} = T_n(a, b)$. This yields that

$$\sum_{T \in \mathcal{T}_n} \text{noc}(T) a^{\text{top}(T)} b^{\text{left}(T)} = [(n-2)ab + \binom{n-2}{2}(a+b) + \binom{n-2}{3}] \cdot T_{n-2}(a, b)$$

as desired, which completes the proof.

5 A bijection between \mathcal{AT}_n^B and \mathcal{L}_n^B

In this section, we shall establish a bijection between type B alternative tableaux of size n and type B linked partitions of $[n]$. First we describe a map $\Phi_B : \mathcal{AT}_n^B \rightarrow \mathcal{L}_n^B$. Let T be a type B alternative tableau of size n . We construct a type B linked partition $\tau = \Phi_B(T)$ as follows. For all $j \in [n]$, if j is the label of a column, then let j be the label of a vertex of τ if column j contains an \uparrow , and let $-j$ be the label of a vertex of τ , otherwise. If j is the label of a row, then let j be the label of a vertex of τ . For column j with an \uparrow , suppose that the cell (i_1, j) is filled with an \uparrow , and the cells $(i_2, j), (i_3, j), \dots, (i_t, j)$ with $i_2 < i_3 < \dots < i_t$ are cells filled with \leftarrow' s . For $\ell = 1, 2, \dots, t$, let $(i_\ell, i_{\ell+1})$ be an arc in τ , where $i_{t+1} = j$. For non-empty column j without any \uparrow' s , suppose that the cells $(i_1, j), (i_2, j), \dots, (i_t, j)$ with $i_1 < i_2 < \dots < i_t$ are filled with \leftarrow' s . For $\ell = 1, 2, \dots, t$, let $(-j, i_\ell)$ be an arc in τ . As an example, for the type B alternative tableau given in Figure 6, its corresponding type B linked partition is shown in Figure 8.

It is not difficult to check that for all $i \in [n]$, either vertex i or vertex $-i$ is contained in τ , but not both. Notice that there is at most one \leftarrow in each row and at most one \uparrow in each column. Thus, for any vertex j in τ , there is at most one arc whose right-hand endpoint is j . In order to show that the map Φ_B is well defined, that is, $\Phi_B(T) \in \mathcal{L}_n^B$, it remains to show that for $\ell = 1, 2, \dots, t$, vertex i_ℓ is contained in τ . If $i_\ell > 0$, vertex i_ℓ is contained in τ since i_ℓ is the label of a row of T . If $i_\ell < 0$, then it follows from the definition of type B alternative tableaux that column $|i_\ell|$ contains no \uparrow . By the definition of the map Φ_B , vertex i_ℓ is contained in τ . Hence, the map Φ_B is well defined.

The map $\Phi_B : \mathcal{AT}_n^B \rightarrow \mathcal{L}_n^B$ is a generalization of the map $\Phi : \mathcal{AT}_n^* \rightarrow \mathcal{L}_n$ in the sense that for any $T \in \mathcal{AT}_n^B$, if all the cells in a row with a negative label are empty and each column is filled with an \uparrow , we can consider T as an element in \mathcal{AT}_n^* and the map Φ_B on such T is identical with Φ .

In order to show that the map Φ_B is a bijection, we present a map $\Psi_B : \mathcal{L}_n^B \rightarrow \mathcal{AT}_n^B$. To this end, we need the following definitions. Let τ be a type B linked partition of $[n]$. Suppose that (i, j) is an arc of τ and j is a destination of τ . We say that j is a *legal* destination if $|i| < |j|$. Otherwise, it is said to be *illegal*. As an example, in the linked partition given in Figure 8, vertex 6 is a legal destination, while

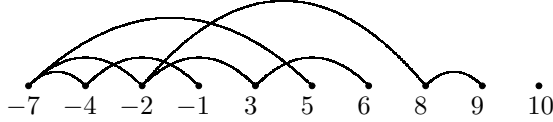


Fig. 8: The type B linked partition corresponding to the type B alternative tableau given in Figure 6.

vertex 5 is an illegal destination. Notice that if vertex j is a legal destination, we must have $j > 0$. A path $(i_1, i_2, \dots, i_t, i)$ with $i_1 < i_2 < \dots < i_t < i$ are said to be a *good path* with destination i if we have $|i_j| < |i|$ for all $1 \leq j \leq t$. When we choose i_1 to be the least such integer, the resulting path is said to be a *maximal good path* with destination i . For example, in the linked partition given in Figure 8, the path $(-2, 3, 6)$ is the maximal good path with destination 6.

Now we are ready to describe the map Ψ_B . Given a type B linked partition τ of $[n]$, let F be a Ferrers diagram of size n such that $|i|$ is the label of a column of F if and only if either $i < 0$ or i is a legal destination. Let \bar{F} be the shifted Ferrers diagram of F . Suppose that τ has exactly k legal destinations, which are vertices b_1, b_2, \dots, b_k with $b_1 > b_2 > \dots > b_k$. We set $T^{(0)}$ to be the empty filling of \bar{F} and $\tau^{(0)} = \tau$. For m from 1 to k , we define $T^{(m)}$ to be a tableau obtained from $T^{(m-1)}$ by filling the cells in column b_m as follows. Suppose that the maximal good path with destination b_m in $\tau^{(m-1)}$ is $(a_1, a_2, \dots, a_t, b_m)$. Then we fill the cell (a_1, b_m) with an \uparrow . For all $2 \leq j \leq t$, fill the cell (a_j, b_m) with a \leftarrow . Let $\tau^{(m)}$ be the type B linked partition obtained from $\tau^{(m-1)}$ by removing the arcs in the path $(a_1, a_2, \dots, a_t, b_m)$. Let $\Psi_B(\tau)$ be the tableau obtained from $T^{(k)}$ by filling the cell $(j, -i)$ of F with a \leftarrow if and only if (i, j) is an arc of $\tau^{(k)}$.

Lemma 5.1 *The map Ψ_B is well defined, that is, we have $\Psi_B(\tau) \in \mathcal{AT}_n^B$ for any $\tau \in \mathcal{L}_n^B$.*

Proof: We proceed the proof by following the notations in the definition of Ψ_B . First we shall show that the map Ψ_B is feasible. To this end, it suffices to show that (1) for all $1 \leq i \leq t$, the cell (a_i, b_m) is contained in \bar{F} and (2) the cell $(j, -i)$ is contained in \bar{F} whenever $\tau^{(k)}$ contains the arc (i, j) . From the construction of \bar{F} , it is not difficult to check that b_m is the label of a column and a_i is the label of a row in \bar{F} . By the definition of the maximal good path with destination b_m , we have $|a_i| < |b_m| = b_m$. Hence \bar{F} contains the cell (a_i, b_m) .

To prove (2), we claim that if (i, j) is an arc in $\tau^{(k)}$, then we have $i < 0$ and $|j| > |i|$. If not, the arc (i, j) will be contained in the maximal good path of a legal destination of τ . This contradicts the definition of Ψ_B . Hence, the claim is proved. This implies that $-i$ is the label of a column in \bar{F} . By the construction of \bar{F} , we have that j is the label of a row. Hence, the cell $(j, -i)$ is contained in F .

Next we aim to show that $\Psi_B(\tau) \in \mathcal{AT}_n^B$. It is apparent that the diagonal cells of F are empty in $\Psi_B(\tau)$. In order to show that $\Psi_B(\tau) \in \mathcal{AT}_n^B$, it remains to show that $\Psi_B(\tau)$ verifies the following properties:

- (i) all the cells pointed by an arrow are empty;

(ii) if column i contains an up arrow \uparrow , then row $-i$ is empty.

We claim that the cells pointed by an arrow are empty in $T^{(m)}$ for all $0 \leq m \leq k$. If not, suppose that the cell (a_s, b_p) is non-empty in $T^{(m)}$ for some $2 \leq s \leq t$ and $p < m$. Then the arc (a_{s-1}, a_s) would be in the maximal good path with destination b_p since $|a_{s-1}| < b_m < b_p$. According to the construction of $\tau^{(p)}$, the arc (a_{s-1}, a_s) is not contained in $\tau^{(j)}$ for all $j \geq p$. This contradicts the fact that $\tau^{(m-1)}$ contains the arc (a_{s-1}, a_s) . Hence, the claim is proved.

In order to prove that $\Psi_B(\tau)$ verifies property (i), it remains to show that if (i, j) is an arc in $\tau^{(k)}$, then there exists no arrow which is pointed by the \leftarrow filled in the cell $(j, -i)$. If not, it follows from the construction of $\Psi_B(\tau)$ that there exists a cell (j, b_ℓ) with $1 \leq \ell \leq k$ which is filled with an arrow. Since $b_\ell > |i|$, the arc (i, j) would be contained in the maximal good path with destination b_ℓ , which is impossible. Thus, we have concluded that $\Psi_B(\tau)$ verifies property (i).

From the construction of $\Psi_B(\tau)$, it is easily seen that if column i is filled with an \uparrow , then vertex i is a legal destination in τ . This implies that $-i$ is not a vertex of τ . Hence, row $-i$ is empty, that is, $\Psi_B(\tau)$ verifies property (ii). This completes the proof. \square

Lemma 5.2 *Following the notations in the definition of Ψ_B , we have $\Phi_B(T) = \tau$, if $\Psi_B(\tau) = T$.*

Proof: From the definition of the map Φ_B , it is easily seen that $\Phi_B(T)$ has the same set of arcs as that of τ . In order to show that $\Phi_B(T) = \tau$, it remains show that $\Phi_B(T)$ has the same set of vertices as that of τ . By the definition of the map Ψ_B , for all $j \in [n]$, vertex j is contained in τ if and only if either j is the label of a column containing an \uparrow or j is the label of a row in T . Moreover, vertex $-j$ is contained in τ if and only if j is the label of a column without any \uparrow in T . Thus, by the definition of the map Φ_B , it follows that $\Phi_B(T)$ has the same set of vertices as that of τ . This completes the proof. \square

By Lemma 5.2, we see that the map Ψ_B is the inverse map of Φ_B . Thus, the map Φ_B is indeed a bijection. From the definition of the map Φ_B , one can easily verify that the bijection Φ_B satisfies the following properties, which are analogous to Φ .

Theorem 5.3 *For any $T \in \mathcal{AT}_n^B$ and $\tau \in \mathcal{L}_n^B$ with $\Phi_B(T) = \tau$, we have the following properties.*

1. *For all $i \in [n]$, i is the label of a column of T if and only if either vertex i is a legal destination of τ , or vertex $-i$ is contained in τ .*
2. *For all $i \in [n]$, i is the label of a row of T if and only if vertex i is contained in τ and is not a legal destination of τ .*
3. *The labels of the unrestricted rows of T are exactly the labels of the origins and singletons of τ .*

For a type B linked partition τ , the *weight* of τ is defined to be $x^{os(\tau)-1}$, where $os(T)$ is the total number of origins and singletons of τ . Define

$$L_n^B(x) = \sum_{\tau \in \mathcal{L}_n^B} x^{os(\tau)-1}.$$

Combining Theorem 5.3 and Formula (2.6), we have

$$L_n^B(x) = \sum_{\tau \in \mathcal{L}_n^B} x^{os(\tau)-1} = \sum_{T \in \mathcal{AT}_n^B} x^{urr(T)-1} = 2^n(x+1)_{n-1}. \quad (5.1)$$

6 Proof of Conjecture 1.2

This section is devoted to the proof of Conjecture 1.2. To this end, we need to consider a subset of type B alternative tableaux of size n . For $2 \leq i \leq n$, denote by $\mathcal{AT}_{n,i}^B$ the set of type B alternative tableaux of size n in which the cell $(i-1, i)$ is a non-occupied corner.

From the construction of the map Φ_B and Lemma 5.3, the map Φ_B maps an type B alternative tableau $T \in \mathcal{AT}_{n,i}^B$ to a type B linked partition $\tau = \Phi_B(T)$ satisfying the following conditions:

- (b1) $urr(T) = os(\tau)$;
- (b2) either vertex i is a legal destination of τ , or vertex $-i$ is contained in τ ;
- (b3) vertex $i-1$ is contained in τ and is not a legal destination;
- (b4) neither $(i-1, i)$ nor $(-i, i-1)$ is an arc in τ .

For all $2 \leq i \leq n$, denote by $\mathcal{L}_{n,i}^B$ the set of all type B linked partitions of $[n]$ satisfying conditions (b2), (b3) and (b4). Thus, the map Φ_B is a weight preserving bijection between the set $\mathcal{AT}_{n,i}^B$ and the set $\mathcal{L}_{n,i}^B$. This yields that for all $2 \leq i \leq n$

$$\sum_{T \in \mathcal{AT}_{n,i}^B} x^{urr(T)-1} = \sum_{\tau \in \mathcal{L}_{n,i}^B} x^{os(\tau)-1}. \quad (6.1)$$

In order to compute the right-hand side of Formula (6.1), we partition the set $\mathcal{L}_{n,i}^B$ into two subsets $\mathcal{X}_{n,i}$ and $\mathcal{Y}_{n,i}$, where $\mathcal{X}_{n,i}$ is the set of type B linked partitions τ satisfying conditions (b3) and (b4) in which vertex $-i$ is contained in τ , and $\mathcal{Y}_{n,i}$ is the set of type B linked partitions τ satisfying conditions (b3) and (b4) in which vertex i is a legal destination of τ . It is apparent that for $2 \leq i \leq n$, we have

$$\sum_{\tau \in \mathcal{L}_{n,i}^B} x^{os(\tau)-1} = \sum_{\tau \in \mathcal{X}_{n,i}} x^{os(\tau)-1} + \sum_{\tau \in \mathcal{Y}_{n,i}} x^{os(\tau)-1}. \quad (6.2)$$

Denote by

$$L_n(x) = L_n(1, x) = \sum_{\tau \in \mathcal{L}_n} x^{os(\tau)-1}.$$

Recall that $L_n(a, b) = \sum_{\tau \in \mathcal{L}_n} a^{one(\tau)} b^{os(\tau)-1} = (a+b)_{n-1}$. Hence, we have

$$L_n(x) = (x+1)_{n-1} \quad (6.3)$$

Lemma 6.1 For $n \geq 2$ and $2 \leq i \leq n$, we have

$$\sum_{\tau \in \mathcal{X}_{n,i}} w(\tau) = 2^{n-2} L_n(x) - (i-1) 2^{n-2} L_{n-1}(x). \quad (6.4)$$

Proof: For $2 \leq i \leq n$, denote by $\mathcal{X}_{n,i}^{(1)}$ the set of type B linked partitions τ of $[n]$ such that vertices $-i$ and $i-1$ are contained in τ . Let $\mathcal{X}_{n,i}^{(2)}$ denote the subset of type B linked partitions $\tau \in \mathcal{X}_{n,i}^{(1)}$ such that

vertex $i - 1$ is a legal destination of τ . Let $\mathcal{X}_{n,i}^{(3)}$ denote the subset of type B linked partitions $\tau \in \mathcal{X}_{n,i}^{(1)}$ such that τ contains an arc $(-i, i - 1)$. Obviously, we have

$$\sum_{\tau \in \mathcal{X}_{n,i}} x^{os(\tau)-1} = \sum_{\tau \in \mathcal{X}_{n,i}^{(1)}} x^{os(\tau)-1} - \sum_{\tau \in \mathcal{X}_{n,i}^{(2)}} x^{os(\tau)-1} - \sum_{\tau \in \mathcal{X}_{n,i}^{(3)}} x^{os(\tau)-1}. \quad (6.5)$$

For any type B linked partition $\tau \in \mathcal{X}_{n,i}^{(1)}$, it can be uniquely represented by an ordered pair (τ', I) , where I is the vertex set of τ and τ' is an ordinary linked partition obtained from τ by relabeling the vertices of τ by $1, 2, \dots, n$ from left to right. Obviously, we have $\{-i, i - 1\} \subset I$. Conversely, given such an ordered pair (τ', I) , we can recover a type B linked partition $\tau \in \mathcal{X}_{n,i}^{(1)}$ by reversing the above procedure. Moreover, the total number of origins and singletons of τ is equal to that of τ' .

Given a linked partition $\tau' \in \mathcal{L}_n$, there are 2^{n-2} different ways to choose a set I such that $\{-i, i - 1\} \subset I$ to construct a type B linked partition $\tau \in \mathcal{X}_{n,i}^{(1)}$. Hence, we have

$$\sum_{\tau \in \mathcal{X}_{n,i}^{(1)}} x^{os(\tau)-1} = 2^{n-2} \sum_{\tau' \in \mathcal{L}_n} x^{os(\tau')-1} = 2^{n-2} L_n(x). \quad (6.6)$$

Let $\mathcal{Z}_{n,i}$ be the set of type B linked partitions τ of $[n]$ such that vertex $-i$ is contained in τ . By similar arguments as above, we have

$$\sum_{\tau \in \mathcal{Z}_{n,i}} x^{os(\tau)-1} = 2^{n-1} \sum_{\tau' \in \mathcal{L}_n} x^{os(\tau')-1} = 2^{n-1} L_n(x). \quad (6.7)$$

For a type B linked partition $\tau \in \mathcal{X}_{n,i}^{(2)}$, we can obtain a linked partition $\tau' \in \mathcal{Z}_{n-1,i-1}$ by removing vertex $i - 1$ from τ and relabeling vertex j by $j - 1$ and vertex $-j$ by $-j + 1$ for all $j \geq i - 1$. Conversely, given a linked partition $\tau' \in \mathcal{Z}_{n-1,i-1}$, we can recover a linked partition $\tau \in \mathcal{X}_{n,i}^{(2)}$ from τ' by the following procedure.

- We first insert a vertex labeled by $i - 1$ immediately after vertex j where j is the maximum integer which is smaller than $i - 1$;
- Then relabel vertex ℓ by $\ell + 1$ and vertex $-\ell$ by $-\ell - 1$ for all $\ell > i - 1$;
- Finally, adjoin an arc from vertex s to vertex $i - 1$ for some $|s| < i - 1$.

It is not difficult to see that the total number of origins and singletons of τ remains the same as that of τ' . Hence, we have

$$\sum_{\tau \in \mathcal{X}_{n,i}^{(2)}} x^{os(\tau)-1} = \sum_{\tau' \in \mathcal{Z}_{n-1,i-1}} (i - 2)x^{os(\tau')-1}. \quad (6.8)$$

From (6.7), it follows that

$$\sum_{\tau \in \mathcal{X}_{n,i}^{(2)}} x^{os(\tau)-1} = (i - 2)2^{n-2} L_{n-1}(x). \quad (6.9)$$

We claim that for any type B linked partition $\tau \in \mathcal{X}_{n,i}^{(3)}$, vertex $i - 1$ is a destination of τ . If not, then according to the definition of Φ_B , the cell $(i - 1, i)$ is filled with a \leftarrow in T with $\Phi_B(T) = \tau$ and there

exists one additional non-empty cell, say c , in row $i - 1$. Recall that the cell $(i - 1, i)$ is a corner of T . This implies that the cell c is pointed by the \leftarrow filled in the cell $(i - 1, i)$, which contradicts the definition of alternative tableaux. Hence, the claim is proved.

By similar arguments in the proof of (6.8), we can deduce that

$$\sum_{\tau \in \mathcal{X}_{n,i}^{(3)}} x^{os(\tau)-1} = \sum_{\tau' \in \mathcal{Z}_{n-1,i-1}} x^{os(\tau')-1}. \quad (6.10)$$

From (6.7), we deduce that

$$\sum_{\tau \in \mathcal{X}_{n,i}^{(3)}} x^{os(\tau)-1} = 2^{n-2} L_{n-1}(x). \quad (6.11)$$

Combining (6.5), (6.6), (6.9) and (6.11), we are led to the desired Formula (6.4), completing the proof. \square

Lemma 6.2 For $n \geq 2$ and $2 \leq i \leq n$, we have

$$\sum_{\tau \in \mathcal{Y}_{n,i}} x^{os(\tau)-1} = (i-2)2^{n-2} L_{n-1}(x) - (i-2)^2 2^{n-2} L_{n-2}(x) \quad (6.12)$$

Proof: For $2 \leq i \leq n$, denote by $\mathcal{Y}_{n,i}^{(1)}$ the set of type B linked partitions τ such that vertex $i - 1$ is contained in τ , vertex i is a legal destination of τ , and $(i - 1, i)$ is not an arc of τ . Let $\mathcal{Y}_{n,i}^{(2)}$ denote the subset of type B linked partitions $\tau \in \mathcal{Y}_{n,i}^{(1)}$ such that vertex $i - 1$ is a legal destination of τ . Obviously, we have

$$\sum_{\tau \in \mathcal{Y}_{n,i}} x^{os(\tau)-1} = \sum_{\tau \in \mathcal{Y}_{n,i}^{(1)}} x^{os(\tau)-1} - \sum_{\tau \in \mathcal{Y}_{n,i}^{(2)}} x^{os(\tau)-1}. \quad (6.13)$$

Analogous to the proof of (6.9), we can deduce that

$$\sum_{\tau \in \mathcal{Y}_{n,i}^{(1)}} x^{os(\tau)-1} = \sum_{\tau' \in \mathcal{Z}_{n-1,i-1}} (i-2)x^{os(\tau')-1} = (i-2)2^{n-2} L_{n-1}(x). \quad (6.14)$$

For a type B linked partition $\tau \in \mathcal{Y}_{n,i}^{(2)}$, we can obtain a type B linked partition $\tau' \in \mathcal{L}_{n-2}^B$ by removing vertices $i - 1$ and i , and relabeling vertex j by $j - 2$ and vertex $-j$ by $-j + 2$ for all $j > i$. Conversely, given a linked partition $\tau' \in \mathcal{L}_{n-2}^B$, we can recover a type B linked partition $\tau \in \mathcal{Y}_{n,i}^{(2)}$ from τ' by the following procedures.

- we first relabel vertex j by $j + 2$ and vertex $-j$ by $-j - 2$ for all $j \geq i - 1$;
- Then we insert a vertex labeled with $i - 1$ and a vertex labeled with i immediately after vertex j where j is the maximum integer which is smaller than $i - 1$;
- Finally, adjoin an arc from vertex ℓ (resp. s) to vertex $i - 1$ (resp. i) for some $|\ell| < i - 1$ (resp. $|s| < i - 1$).

It is not difficult to see that the total number of origins and singletons of τ is the same as that of τ' . Thus, we have

$$\begin{aligned} \sum_{\tau \in \mathcal{Y}_{n,i}^{(2)}} x^{os(\tau)-1} &= \sum_{\tau' \in \mathcal{L}_{n-2}^B} (i-2)^2 x^{os(\tau')-1} \\ &= (i-2)^2 L_{n-2}^B(x) \\ &= (i-2)^2 2^{n-2} L_{n-2}(x). \end{aligned} \quad (6.15)$$

The last equality in the above formula follows from (5.1) and (6.3). Combining (6.13), (6.14) and (6.15), we are led to (6.12). This completes the proof. \square

Denote by $\mathcal{AT}_{n,1}^B$ the set of type B alternative tableaux of size n in which the cell $(-1, 1)$ is a non-occupied corner. It is easy to check that the map Φ_B sends an alternative tableau $T \in \mathcal{AT}_{n,1}^B$ to a type B linked partition $\tau \in \mathcal{L}_n^B$ in which -1 is the label of a vertex. Let $\mathcal{L}_{n,1}^B$ denote the set of such type B linked partitions. Hence, the map Φ_B is a weight preserving bijection between the set $\mathcal{AT}_{n,1}^B$ and the set $\mathcal{L}_{n,1}^B$. This yields that

$$\sum_{T \in \mathcal{AT}_{n,1}^B} x^{urr(T)-1} = \sum_{\tau \in \mathcal{L}_{n,1}^B} x^{os(\tau)-1}. \quad (6.16)$$

By similar arguments as in the proof of Formula (6.6), we can deduce the following result, and the proof is omitted.

Lemma 6.3 For $n \geq 1$, we have

$$\sum_{\tau \in \mathcal{L}_{n,1}^B} w(\tau) = 2^{n-1} L_n(x). \quad (6.17)$$

In the following, we proceed to complete the proof of Conjecture 1.2.

Proof of Conjecture 1.2. In view of (2.5), we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_{2n+1}^{sym}} noc(T) x^{left(T)-1} &= \sum_{T \in \mathcal{AT}_{2n}^{sym}} noc(T) x^{urr(T)-1} \\ &= \sum_{T \in \mathcal{AT}_n^B} (noc'(T) + 2noc''(T)) x^{urr(T)-1} \\ &= \sum_{T \in \mathcal{AT}_{n,1}^B} x^{urr(T)-1} + 2 \sum_{i=2}^n \sum_{T \in \mathcal{AT}_{n,i}^B} x^{urr(T)-1}. \end{aligned}$$

From (6.16) and (6.17), we deduce that

$$\sum_{T \in \mathcal{AT}_{n,1}^B} x^{urr(T)-1} = \sum_{\tau \in \mathcal{L}_{n,1}^B} x^{os(\tau)-1} = 2^{n-1} L_n(x).$$

From (6.1) and (6.2), we deduce that

$$\begin{aligned} \sum_{i=2}^n \sum_{T \in \mathcal{AT}_{n,i}^B} x^{urr(T)-1} &= \sum_{i=2}^n \sum_{\tau \in \mathcal{L}_{n,i}^B} x^{os(\tau)-1} \\ &= \sum_{i=2}^n \sum_{\tau \in \mathcal{X}_{n,i}} x^{os(\tau)-1} + \sum_{i=2}^n \sum_{\tau \in \mathcal{Y}_{n,i}} x^{os(\tau)-1}. \end{aligned}$$

By (6.4) and simple computation, we have

$$\sum_{i=2}^n \sum_{\tau \in \mathcal{X}_{n,i}} w(\tau) = (n-1)2^{n-2} L_n(x) - \binom{n}{2} 2^{n-2} L_{n-1}(x).$$

By (6.12) and simple computation, we have

$$\sum_{i=2}^n \sum_{\tau \in \mathcal{Y}_{n,i}} w(\tau) = \binom{n-1}{2} 2^{n-2} L_{n-1}(x) - \frac{(n-2)(n-1)(2n-3)}{6} 2^{n-2} L_{n-2}(x)$$

Substituting $L_n(x) = (x+1)_{n-1}$ and $T_{2n+1}^{sym}(x) = L_n^B(x) = 2^n L_n(x)$ into the above formulae, we conclude the proof of Conjecture 1.2.

7 Concluding notes

In this paper, we obtain the polynomial analogues of the number of non-occupied corners in tree-like tableaux and symmetric tree-like tableaux, confirming two conjectures posed by Gao et al. (2016).

In Laborde-Zubieta (2015), he obtained the following results on the number of occupied corners in tree-like tableaux and symmetric tree-like tableaux.

Theorem 7.1 (See Laborde-Zubieta (2015), Theorem 3.2) *For $n \geq 1$, the number of occupied corners in tree-like tableaux of size n is given by $n!$.*

Theorem 7.2 (See Laborde-Zubieta (2015), Theorem 3.7) *For $n \geq 1$, the number of occupied corners in symmetric tree-like tableaux of size $2n+1$ is given by $2^n n!$.*

Denote by $oc(T)$ the number of occupied corners in (symmetric) tree-like tableau T . As remarked in Gao et al. (2016), we can adapt Laborde-Zubieta's easy proof of Theorems 7.1 and 7.2 to deduce the following results by keeping track of left and top points.

Theorem 7.3 *For $n \geq 1$, we have*

$$oc_n(a, b) = \sum_{T \in \mathcal{T}_n} oc(T) a^{top(T)} b^{left(T)} = T_n(a, b).$$

Theorem 7.4 *For $n \geq 1$, we have*

$$\sum_{T \in \mathcal{T}_{2n+1}^{sym}} oc(T) x^{left(T)-1} = T_{2n+1}^{sym}(x).$$

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References

- Alexander Postnikov. Total positivity, grassmannians, and networks. *arXiv: math.CO 0609764v1*, 2006.
- Alice L.L. Gao, Emily X.L. Gao, Patxi Laborde-Zubieta, Brian Y. Sun. Enumeration of corners in tree-like tableaux. *Discrete Math. Theor. Comput. Sci.*, 18:3:17, 2016.
- Kenneth J. Dykema. Multilinear function series and transforms in free probability theory. *Adv. Math.*, 208:351–407, 2007.

- Patxi Laborde-Zubieta. Occupied corners in tree-like tableaux. *Sém. Lothar. Combin.*, 14:74:Art. B74b., 2015.
- Pawel Hitzenko, Amanda Lohss. Corners in tree-like tableaux. *Electronic J. Combin.*, 23(4):P4.26, 2016.
- Philippe Nadeau. The structure of alternative tableaux. *J. Combin. Theory Ser. A*, 118:1638–1660, 2011.
- Xavier Viennot. Alternative tableaux, permutations and partially asymmetric exclusion process. *Slides of a talk at the Isaac Newton Institute in Cambridge*, 2008.
- Jean-Christophe Aval, Adrien Boussicault, Philippe Nadeau. Tree-like tableaux. *Electronic J. Combin.*, 20(4):Paper 34, 2013.
- Sylvie Corteel. Crossings and alignments of permutations. *Adv. in Appl. Math.*, 38:149–163, 2007.
- Sylvie Corteel, Jang Soo Kim. Combinatorics on permutation tableaux of type a and type b. *European J. Combin.*, 32:563–579, 2009.
- Sylvie Corteel, Lauren K. Williams. Tableaux combinatorics for the asymmetric exclusion process. *Adv. in Appl. Math.*, 39:293–310, 2007a.
- Sylvie Corteel, Lauren K. Williams. A markov chain on permutations which projects to the pasep. *Int. Math. Res. Not. IMRN*, 17:Art. ID rnm055, 27, 2007b.
- Sylvie Corteel, Philippe Nadeau. Bijections for permutation tableaux. *European J. Combin.*, 30:295–310, 2009.
- William Y.C. Chen, Lewis H. Liu, Carol J. Wang. Linked partitions and permutation tableaux. *Electronic J. Combin.*, 20(3):P53, 2013.
- William Y.C. Chen, Susan Y.J. Wu, Catherine H. Yan. Linked partitions and linked cycles. *European J. Combin.*, 29:1408–1426, 2008.