”Trivializing” generalizations of some Izergin-Korepin-type determinants
Tewodros Amdeberhan, Doron Zeilberger

To cite this version:

HAL Id: hal-00966504
https://inria.hal.science/hal-00966504
Submitted on 26 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
“Trivializing” Generalizations of some Izergin-Korepin-type Determinants

Tewodros Amdeberhan\textsuperscript{1} and Doron Zeilberger\textsuperscript{2}

\textsuperscript{1} Tulane Univ., Dept of Math., New Orleans, LA 70118, USA
\textsuperscript{2} Rutgers Univ., Dept of Math., Piscataway, NJ 08854, USA


We generalize (and hence trivialize and routinize) numerous explicit evaluations of determinants and pfaffians due to Kuperberg, as well as a determinant of Tsuchiya. The level of generality of our statements render their proofs easy and routine, by using Dodgson condensation and/or Krattenthaler’s factor exhaustion method.

All our matrices will be assumed to be embedded inside an infinite matrix.

The first theorem adds parameters to the determinant formulas found in Kuperberg [Ku, Theorem 15], as well as older determinants, mentioned there, due to Cauchy, Stembridge, Laksov–Lascoux–Thorup, and Tsuchiya [T]. This way, the formulation is suited to the method of [AZ]. Our proofs are much more succinct and automatable, since their generality enables an easy induction using Dodgson’s rule [D, AZ], or by employing Krattenthaler’s elegant factor exhaustion method [Kr1]. Relevant background for this paper can found in [Ku], and references thereof.

**Theorem 1**

\[
\det \left( \frac{1}{x_i + y_j + Ax_i y_j} \right)_{i,j}^{1,n} = \prod_{i<j} (x_j - x_i)(y_j - y_i) \prod_{i,j} (x_i + y_j + Ax_i y_j).
\]

\[
\det \left( \frac{1}{x_i + y_j} - \frac{1}{1 + x_i y_j} \right)_{i,j}^{1,n} = \prod_{i<j} (1 - x_i x_j)(1 - y_i y_j)(x_j - x_i)(y_j - y_i) \prod_{i,j} (x_i + y_j)(1 + x_i y_j).\]

\[
\det \left( \frac{Ay_j + Bx_i}{y_j + x_i} \right)_{i,j}^{1,n} = (A - B)^{n-1} \prod_{i,j} (x_i + y_j + (-1)^{n-1}B \prod_{i,j} x_i) \prod_{i<j} (x_i - x_j)(y_i - y_j) \prod_{i=1}^n (1 - x_i)(1 - y_i).
\]

\[
\det \left( \frac{1 - x_i y_j}{y_j - x_i} \right)_{i,j}^{1,n} = (-1)^n \left( \prod_{i,j} (1 + x_j)(1 - y_j) + \prod_{i,j} (1 - x_i)(1 + y_i) \right) \prod_{i<j} (x_i - x_j)(y_i - y_j) \frac{2}{2 \prod_{i,j} y_j - x_i}.
\]
Sketch of Proof: An automatic application of Dodgson Condensation [D, AZ].

**Corollary 1 (Cauchy, Stembridge, Laksov–Lascoux–Thorup)**

\[
\det \left( \frac{1}{x_i + y_j} \right)_{i,j}^{1,n} = \prod_{i<j}(x_j - x_i)(y_j - y_i) / \prod_{i,j}(x_i + y_j).
\]

\[
\text{Pf}^2 = \det \left( \frac{x_j - x_i}{x_j + x_i} \right)_{i,j}^{1,2n} = \prod_{i<j \leq 2n} (x_i - x_j)^2 / (x_i + x_j)^2.
\]

\[
\text{Pf}^2 = \det \left( \frac{x_j - x_i}{1 - x_i x_j} \right)_{i,j}^{1,2n} = \prod_{i<j \leq 2n} (x_j - x_i)^2 / (1 - x_i x_j)^2.
\]

**Remark 1** Notice that the latter two statements apply only to even-dimensional matrices. An error from [Ku] in the second formula has been corrected here. Pf stands for Pfaffian of a matrix.

The next theorem generalizes, and presents variations of, several of the determinants that appear in Theorems 16 and 17 [Ku] (typos corrected in [Kr2] Theorems 13 and 14). Below, \( Z_1, Z_2, Z_3, Z_4, Z_5 \) are defined by (here \( \gamma(a, b) = a - b, \tau(a, b) = a + b \))

\[
Z_1(p, q; x, y)_{i,j} = \gamma(q^{j-i}, x^{j-i}) / \tau(p^{j-i}, y^{j-i}),
\]

\[
Z_2(p, q; x, y)_{i,j} = \gamma(q^{1+j-i}, x^{1+j-i}) / \tau(p^{1+j-i}, y^{1+j-i}),
\]

\[
Z_3(p, q; x, y)_{i,j} = \gamma(q^{-1+j-i}, x^{-1+j-i}) / \tau(p^{-1+j-i}, y^{-1+j-i}),
\]

\[
Z_4(p, q; x, x)_{i,j} = \gamma(q^{a+j-i}, x^{a+j-i}) / \tau(q^{a+j-i}, x^{a+j-i}),
\]

\[
Z_5(q; x, x)_{i,j} = \gamma(q^{b+j-i}, x^{b+j-i}) / \gamma(q^{b+j-i}, x^{b+j-i}),
\]

for \( a \in \mathbb{Z}, b = \pm n, \pm (n + 1), \ldots \) Let \( \delta_{c,n} = \frac{1+(-1)^n}{2} \) denote Kronecker’s delta function centered at the even integers, and let \( \lambda_{i,j} = 1, \) if \( i \neq j \) and \( \lambda_{i,i} = 0. \)

**Theorem 2** Write \( \gamma(a, b) = a - b, \tau(a, b) = a + b, \) we have the matrix determinants

\[
\det \left( \gamma(q^{n+j-i}, x^{n+j-i}) / \gamma(p^{n+j-i}, y^{n+j-i}) \right)_{i,j}^{1,n} = (py)(z) \gamma(q, x)^n \prod_{j>i} \gamma(p^{j-i}, y^{j-i})^2 \gamma(q^{j-i}, x^{j-i}) \gamma(x^{j-i}, y^{j-i}) \gamma(x^{j-i}, y^{j-i}) / \prod_{i,j} \gamma(p^{n+j-i}, y^{n+j-i}).
\]

\[
\det \left( \tau(q^{j-i}, x^{j-i}) / \tau(p^{j-i}, y^{j-i}) \right)_{i,j}^{1,n} = \frac{\prod_{2i-j>0} \gamma(p^{j-i}, y^{j-i})^2 \prod_{2j-i>0} \gamma(q^{j-i}, x^{j-i}) \gamma(x^{j-i}, y^{j-i}) \gamma(x^{j-i}, y^{j-i}) \gamma(x^{j-i}, y^{j-i})}{(qx)^{n^2/4} \prod_{j<i} \tau(p^{j-i}, y^{j-i})^2}.
\]

\[
\det(Z_1)^{1,n} = \delta_{c,n} \gamma(q, x)^n (py)^{n/2} / (qx)^{n^2/4} \prod_{2j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(q^{j-i}, x^{j-i}) \gamma(x^{j-i}, y^{j-i}) \gamma(x^{j-i}, y^{j-i}) / \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2.
\]
"Trivializing" Generalizations of some Izergin-Korepin-type Determinants

\[ \text{det}(Z_2)_{i,j}^{1,n} = \frac{\gamma(q,x)^n}{(qx)^{(n-1)/2}} \prod_{j>i>0} \gamma(p^{j-i}, y^{j-i}) \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i}) } \tau(p^n, y^n)^{1-\delta_{e,n}} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2. \]

\[ \text{det}(Z_3)_{i,j}^{1,n} = \frac{(-py)^n \gamma(q,x)^n}{(qx)^{(n+1)/2}} \prod_{j>i>0} \gamma(p^{j-i}, y^{j-i}) \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i}) } \tau(p^n, y^n)^{1-\delta_{e,n}} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2. \]

\[ \text{det}(Z_4)_{i,j}^{1,n} = \frac{2^{n-1} q^{na} + (-1)^n x^{na}}{(qx)^{n(n-1)(n+1-3a)/6}} \prod_{i,j} \gamma(q^{j-i}, x^{j-i}) } \tau(q^{a+j-i}, x^{a+j-i})^6. \]

\[ \text{Pf}^2 = \det \left( \frac{\lambda_{i,j} \gamma(q^{j-i}, x^{j-i}) \gamma(x^{j-i}, y^{j-i})}{\gamma(p^{j-i}, y^{j-i})} \right)_{i,j}^{1,2n} = \frac{(yp)^n}{(qyrt)^{2n}} \prod_{i,j} \gamma(q^{j-i}, x^{j-i})^2 } \prod_{i,j} \gamma(p^{j-i}, y^{j-i})^2. \]

Sketch of Proof: Identities \( Z_4 \) and \( Z_5 \) are directly amenable to Dodgson's Condensation technique [AZ]. For the remaining assertions, use the factor exhaustion method [Kr1] (see also [Ku]): the essential idea is to compare zeros and poles on both sides of the equation at hand. We leave the straightforward details to the reader.

Acknowledgments

The authors wish to thank Christian Krattenthaler for helpful suggestions.
References


