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“Trivializing” Generalizations of some Izergin-Korepin-type Determinants

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We generalize (and hence trivialize and routinize) numerous explicit evaluations of determinants and pfaffians due to Kuperberg, as well as a determinant of Tsuchiya. The level of generality of our statements render their proofs easy and routine, by using Dodgson condensation and/or Krattenthaler’s factor exhaustion method.

All our matrices will be assumed to be embedded inside an infinite matrix.

The first theorem adds parameters to the determinant formulas found in Kuperberg [Ku, Theorem 15], as well as older determinants, mentioned there, due to Cauchy, Stembridge, Laksov–Lascoux–Thorup, and Tsuchiya [T]. This way, the formulation is suited to the method of [AZ]. Our proofs are much more succinct and automatable, since their generality enables an easy induction using Dodgson’s rule [D, AZ], or by employing Krattenthaler’s elegant factor exhaustion method [Kr1]. Relevant background for this paper can found in [Ku], and references thereof.

\textbf{Theorem 1}

\[
\det \left( \frac{1}{x_i + y_j + Ax_i y_j} \right)_{i,j}^{1,n} = \prod_{i<j} (x_j - x_i)(y_j - y_i) \prod_{i,j} (x_i + y_j + Ax_i y_j).
\]

\[
\det \left( \frac{1}{x_i + y_j} - \frac{1}{1 + x_i y_j} \right)_{i,j}^{1,n} = \prod_{i<j} (1 - x_i x_j)(1 - y_i y_j)(x_j - x_i)(y_j - y_i) \prod_{i,j} (x_i + y_j)(1 + x_i y_j) \prod_{i=1}^{n} (1 - x_i)(1 - y_i).
\]

\[
\det \left( \frac{A y_j + B x_i}{y_j + x_i} \right)_{i,j}^{1,n} = (A - B)^{n-1} \frac{A \prod_{j} x_j + (-1)^{n-1} B \prod_{i} x_i \prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (x_i + y_j)}.
\]

\[
\det \left( \frac{1 - x_i y_j}{y_j - x_i} \right)_{i,j}^{1,n} = (-1)^{n-1} \left( \prod_{i} (1 + x_j)(1 - y_j) + \prod_{i} (1 - x_i)(1 + y_i) \right) \prod_{i<j} (x_i - x_j)(y_i - y_j) \prod_{i,j} (y_j - x_i) \prod_{i=1}^{n} (1 - x_i)(1 - y_i).
\]

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Corollary 1 (Cauchy, Stembridge, Laksov–Lascoux–Thorup)

Sketch of Proof: An automatic application of Dodgson Condensation \([D][AZ]\).

\[\det\left(\frac{1}{x_i + y_j}\right)_{i,j}^{1,n} = \prod_{i<j} (x_j - x_i)(y_j - y_i) / \prod_{i<j} (x_i + y_j).\]

\[\text{Pf}^2 = \det\left(\frac{x_j - x_i}{x_j + x_i}\right)_{i,j}^{1,2n} = \prod_{i<j \leq 2n} (x_i - x_j)^2 / (x_i + x_j)^2.\]

\[\text{Pf}^2 = \det\left(\frac{x_j - x_i}{1 - x_ix_j}\right)_{i,j}^{1,2n} = \prod_{i<j \leq 2n} (x_i - x_j)^2 / (1 - x_ix_j)^2.\]

Remark 1 Notice that the latter two statements apply only to even-dimensional matrices. An error from \([Ku]\) in the second formula has been corrected here. Pf stands for Pfaffian of a matrix.

The next theorem generalizes, and presents variations of, several of the determinants that appear in Theorems 16 and 17 \([Ku]\) (typos corrected in \([Kr2]\) Theorems 13 and 14). Below, \(Z_1, Z_2, Z_3, Z_4, Z_5\) are defined by (here \(\gamma(a, b) = a - b, \tau(a, b) = a + b\))

\[Z_1(p, q; x, y)_{i,j} = \frac{\gamma(q^{j-i}, x^{j-i})}{\tau(p^{j-i}, y^{j-i})}, \quad Z_2(p, q; x, y)_{i,j} = \frac{\gamma(q^{1+j-i}, x^{1+j-i})}{\tau(p^{1+j-i}, y^{1+j-i})},\]

\[Z_3(p, q; x, y)_{i,j} = \frac{\gamma(q^{-1+j-i}, x^{-1+j-i})}{\tau(p^{-1+j-i}, y^{-1+j-i})}, \quad Z_4(q, q; x, x)_{i,j} = \frac{\gamma(q^{a+j-i}, x^{a+j-i})}{\tau(q^{a+j-i}, y^{a+j-i})},\]

\[Z_5(q, q; x, x)_{i,j} = \frac{\gamma(q^{b+j-i}, x^{b+j-i})}{\gamma(q^{b+j-i}, x^{b+j-i})},\]

for \(a \in \mathbb{Z}, b = \pm n, \pm (n + 1), \ldots\) Let \(\delta_{e,n} = \frac{1+(-1)^n}{2}\) denote Kronecker’s delta function centered at the even integers, and let \(\lambda_{i,j} = 1, \text{ if } i \neq j \text{ and } \lambda_{i,i} = 0.\)

Theorem 2 Write \(\gamma(a, b) = a - b, \tau(a, b) = a + b, \text{ we have the matrix determinants}\)

\[\det\left(\frac{\gamma(q^{j-i}, x^{j-i})}{\gamma(p^{j-i}, y^{j-i})}\right)_{i,j}^{1,n} = (pq)^{\frac{1}{2}}(\gamma(q, x)^n \prod_{j>i} \frac{\gamma(p^{j-i}, x^{j-i})^2 \gamma(q^{j-i}, y^{j-i}) \gamma(x^{j-i}, y^{j-i})}{\prod_{i,j} \gamma(p^{n+j-i}, y^{n+j-i})}).\]

\[\det\left(\frac{\tau(q^{j-i}, x^{j-i})}{\tau(p^{j-i}, y^{j-i})}\right)_{i,j}^{1,n} = \frac{\prod_{2i-j>0} \gamma(p^{j-i}, y^{j-i})^2 \prod_{2j-i>0} \gamma(q^{j-i}, x^{j-i}) \gamma(x^{j-i}, y^{j-i})}{(pq)^{n^2/4} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.\]

\[\det(Z_1)_{i,j}^{1,n} = \delta_{e,n} \frac{\gamma(q, x)^n (pq)^{n/2} \prod_{2i-j>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(q^{j-i}, x^{j-i}) \gamma(x^{j-i}, y^{j-i})}{(pq)^{n^2/4} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.\]
Sketch of Proof: Identities $Z_4$ and $Z_5$ are directly amenable to Dodgson’s Condensation technique [AZ]. For the remaining assertions, use the factor exhaustion method [Kr1] (see also [Ku]): the essential idea is to compare zeros and poles on both sides of the equation at hand. We leave the straightforward details to the reader.

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