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Complexity of locally-injective homomorphisms to tournaments

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For oriented graphs G and H, a homomorphism $f : G \to H$ is locally-injective if, for every $v \in V(G)$, it is injective when restricted to some combination of the in-neighbourhood and out-neighbourhood of v. Two of the possible definitions of local-injectivity are examined. In each case it is shown that the associated homomorphism problem is NP-complete when H is a reflexive tournament on three or more vertices with a loop at every vertex, and solvable in polynomial time when H is a reflexive tournament on two or fewer vertices.

Keywords: Complexity, Graph homomorphism, Oriented graph, Locally-injective homomorphism

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1 Introduction

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a *homomorphism* from G to H is a function $f : V_G \to V_H$ such that for every $uv \in E_G$, $f(u)f(v) \in E_H$. A homomorphism from G to H is referred to as an H*-colouring* of G and the vertices of H are regarded as *colours*. The graph H is called the *target* of the homomorphism. These definitions extend to directed graphs by requiring that the mapping must preserve the existence as well as the direction of each arc.

A *locally-injective homomorphism* f from G to H is a homomorphism from G to H such that for every $v \in V$ the restriction of f to $N(v)$ (or possibly $N[v] = N(v) \cup \{v\}$) is injective. The complexity of locally-injective homomorphisms for undirected graphs has been examined by a variety of authors and in a variety of contexts Chen et al. (2012); Doyon et al. (2010); Fiala and Kratochvíl (2001, 2002, 2006); Fiala et al. (2008); Hahn et al. (2002); Rzazewski (2014). Locally-injective homomorphisms of graphs find application in a range of areas including bio-informatics Brevier et al. (2007); Fagnot et al. (2008); Fertin et al. (2005) and coding theory Hahn et al. (2002).

Here we consider locally-injective homomorphisms of *oriented graphs*, that is, directed graphs in which any two vertices are joined by at most one arc. Given a vertex v, an arc from v to v is called a *loop*. A

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directed graph with a loop at every vertex is called *reflexive*; a directed graph with no loops is called *irreflexive*. For a vertex v of a directed graph G, let $N^-(v)$, respectively $N^+(v)$, denote the set of vertices u such that uv, respectively vu, is an arc of G. Note that if there is a loop at v, then $v \in N^-(v)$ and $v \in N^+(v)$.

To define locally-injective homomorphisms of oriented graphs, one must choose the neighbourhood(s) on which the homomorphism must be injective. Up to symmetry, there are four natural choices:

1. $N^-(v)$.
2. $N^+(v)$ and also $N^-(v)$.
3. $N^+(v) \cup N^-(v)$.
4. $N^+[v] \cup N^-[v] = N^+(v) \cup N^-(v) \cup \{v\}$.

For irreflexive targets, (2), (3) and (4) are equivalent. Under (4), adjacent vertices must always be assigned different colours, and hence whether or not the target contains loops is irrelevant. Therefore, we may assume that targets are irreflexive when considering (4). Then, a locally-injective homomorphism to an irreflexive target satisfying (4) is equivalent to a locally-injective homomorphism to the same irreflexive target under either (2) or (3). As such, we need not consider (4) and are left with three distinct cases.

Taking (1) as our injectivity requirement defines *in-injective homomorphism*; taking (2) defines *ios-injective homomorphism*; and taking (3) defines *iot-injective homomorphism*. Here "ios" and "iot" stand for "in and out separately" and "in and out together" respectively.

The problem of in-injective homomorphism is examined by MacGillivray, Raspaud, and Swarts in MacGillivray et al. (2014); MacGillivray and Swarts (2010). They give a dichotomy theorem for the problem of in-injective homomorphism to reflexive oriented graphs, and one for the problem of in-injective homomorphism to irreflexive tournaments. The problem of in-injective homomorphism to irreflexive oriented graphs H is shown to be NP-complete when the maximum in-degree of H, $\Delta^-(H)$, is at least 3, and solvable in polynomial time when $\Delta^-(H) = 1$. For the case $\Delta^-(H) = 2$ they show that an instance of directed graph homomorphism polynomially transforms to an instance of in-injective homomorphism to a target with maximum in-degree 2. As such the restriction of in-injective homomorphism to targets H so that $\Delta^-(H) = 2$ constitutes a rich class of problems.

The remaining problems, ios-injective homomorphism and iot-injective homomorphism, are considered by Campbell, Clarke and MacGillivray Campbell (2009); Campbell et al. (2016a,b). In this paper we extend the results of Campbell, Clarke and MacGillivray to provide dichotomy theorems for the restriction of the problems of iot-injective homomorphism and ios-injective homomorphism to reflexive tournaments.

Preliminary results are surveyed in Section 2. In Section 3, we show that ios-injective homomorphism is NP-complete for reflexive tournaments on 4 or more vertices. In Section 4, we show that iot-injective homomorphism is also NP-complete for reflexive tournaments on 4 or more vertices. We close with a brief discussion of injective homomorphisms to irreflexive tournaments.

2 Known Results

For a fixed undirected graph H, the problem of determining whether an undirected graph G admits a homomorphism to H (i.e., *the H-colouring problem*) admits a well-known dichotomy theorem.

Theorem 2.1 (Hell and Neŝetr̂il (1990)) *Let H be an undirected graph. If H is irreflexive and non-bipartite, then H-colouring is NP-complete. If H has a loop, or is bipartite, then H-colouring is solvable in polynomial time.*

A dichotomy theorem for the complexity of H-colouring of directed graphs is given by Bulatov Bulatov (2017) and Zhuk Zhuk (2017).

For fixed small reflexive tournaments T, Campbell, Clarke and MacGillivray give the following result for the complexity of ios-injective T-colouring and iot-injective T-colouring, where ios-injective T-colouring and iot-injective T-colouring are defined analogously to H-colouring.

Theorem 2.2 (Campbell (2009); Campbell et al. (2016a,b)) *If T is a reflexive tournament on 2 or fewer vertices, then ios-injective T-colouring and iot-injective T-colouring are solvable in polynomial time. If T is a reflexive tournament on 3 vertices, then ios-injective T-colouring and iot-injective T-colouring are NP-complete.*

Fig. 1: The two reflexive tournaments on three vertices: the reflexive three-cycle C_3 and the reflexive transitive tournament on three vertices TT_3.

3 Ios-injective homomorphisms

In this section we prove a dichotomy theorem for ios-injective T-colouring, where T is a reflexive tournament. We show that both ios-injective T_4-colouring and ios-injective T_5-colouring are NP-complete (see Figures 2 and 3). We then show that any instance of ios-injective T-colouring, where T is a reflexive tournament on at least 4 vertices, polynomially transforms to an instance of ios-injective T'-colourings, where T' is T_4, T_5, C_3 or TT_3 (see Figures 1, 2 and 3). The dichotomy theorem follows from combining these results with the result in Theorem 2.2.

We begin with a study of ios-injective T_4-colouring. To show ios-injective T_4-colouring is NP-complete we provide a transformation from 3-edge-colouring subcubic graphs. We construct an oriented graph H from a graph G so that G has a 3-edge-colouring if and only if H admits an ios-injective homomorphism to T_4. The key ingredients in this construction are a pair of oriented graphs, H_x and H_e, given in Figures 4 and 5, respectively. Figures 6 and 7 give ios-injective T_4-colourings of H_x and H_e, respectively.

Lemma 3.1 *In any ios-injective T_4-colouring of H_x*

1. *the vertices $3, 13$ and 23 are coloured a; and*

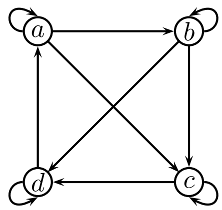

Fig. 2: T_4 – the only strongly connected reflexive tournament on four vertices.

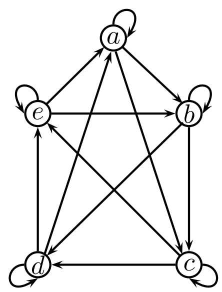

Fig. 3: T_5 – the only reflexive tournament on five vertices where all the vertices have in-degree and out-degree three.

2. *vertex* 31 *is coloured* d*.*

Proof: *(1)* By symmetry, it suffices to show the claim for vertex 3. Let us first note that the vertices 3 and 7 have out-degree 3 and can therefore only be coloured a or b, as these are the only vertices of out-degree 3 in T_4. If vertex 7 is coloured a, then its two in-neighbours, vertices 5 and 6, are coloured d and a. However, this is impossible as no vertex of out-degree three in T_4 has both d and a as out-neighbours. Hence, vertex 7 is coloured b. If vertex 3 is coloured b, then vertices 5 and 6 would be both in- and out-neighbours of vertices coloured b. Thus, each of vertices 5 and 6 are coloured b. This is a violation of the injectivity requirement. Therefore vertex 3 (and by symmetry, the vertices 13 and 23) must be coloured a.

(2) Notice that the square vertices in the H_x (vertices 1, 11 and 21) cannot be coloured a; they each have an in-neighbour that already has an out-neighbour coloured a. These square vertices have a common out-neighbour and so must receive distinct colours by the injectivity requirement. As none is coloured a, these three vertices are coloured b, c and d, in some order. The only vertex that is an out-neighbour of b, c and d in T_4 is d. And so, the common out-neighbour of vertices 1, 11 and 21 (i.e., vertex 31) has colour d. □

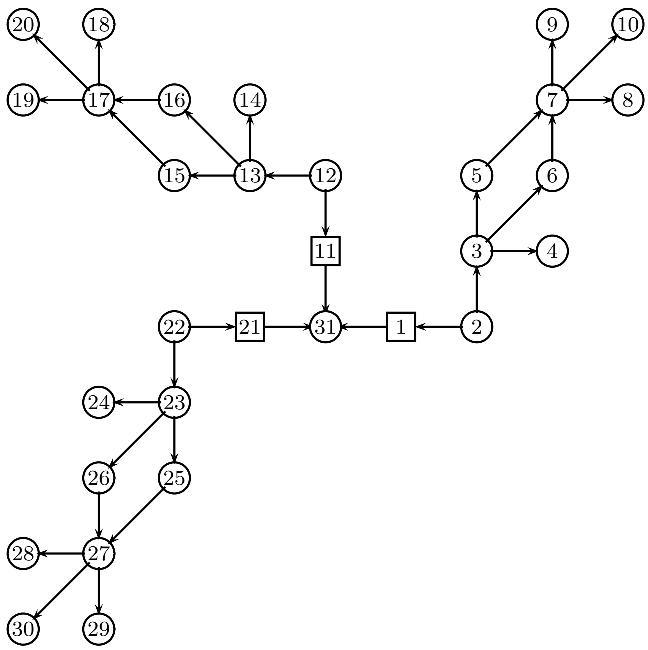

Fig. 4: H_x.

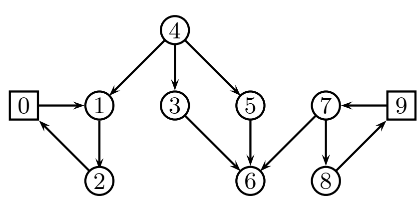

Fig. 5: H_e.

Lemma 3.2 *Let H_e' be an oriented graph formed from a copy of H_e and two copies of H_x by identifying vertex 0 in H_e with any square vertex in one copy of H_x and identifying vertex 9 in H_e with any square vertex in the second copy of H_x. In any ios-injective T_4-colouring of H_e', the vertices 0 and 9 in the subgraph induced by H_e have the same colour.*

Proof: Let H_e' be constructed as described. Consider an ios-injective T_4-colouring of H_e'. We examine the colours of the vertices in the subgraph induced by the copy of H_e. By Lemma 3.1 and the construction of H_e', vertices 0 and 9 each have an in-neighbour that has an out-neightbour coloured a. By the injectivity requirement neither vertex 0 or 9 is coloured a. We proceed in cases to show that vertices 0 and 9 receive the same colour.

Case I: Vertex 0 is coloured b. It cannot be that vertex 1 is coloured d, as vertex 0 has an out-neighbour coloured d – vertex 31 in a copy of H_x. It cannot be that vertex 1 is coloured c as no 3-cycle of T_4 contains both a vertex coloured b and a vertex coloured c. Thus, vertex 1 must be coloured b. The vertex 2 is both an in-neighbour and an out-neighbour of vertices coloured b and is therefore coloured b. The vertex 4 is an in-neighbour of vertex 1, and so cannot be coloured b as vertex 1 already has an in-neighbour coloured b. The vertex 4 must thus be coloured a. By injectivity, the out-neighbours of vertex 4 must receive distinct colours that are out-neighbours of a in T_4. Therefore vertices 3 and 5 are coloured a and c in some order, as vertex 1 is coloured b. The only common out-neighbour of a and c in T_4 is c. As such, vertex 6 must be coloured c. By injectivity, each of the in-neighbours of vertex 6 must receive distinct colours that are in-neighbours of c in T_4. And so vertex 7 must be coloured b. As vertex 9 cannot be coloured a and it has an out-neighbour coloured b, namely vertex 7, we have that vertex 9 must be coloured b. Thus vertices 0 and 9 have the same colour.

Case II: Vertex 0 is coloured c. The out-neighbours of c in T_4 are c and d. It cannot be that vertex 1 is coloured d, as vertex 0 has an out-neighbour coloured d – vertex 31 in a copy of H_x. And so vertex 1 is coloured c.

The vertex 4 has an out-neighbour coloured c, and so must be coloured a or b or c. Since vertex 0 is coloured c, vertex 4 cannot be coloured c without violating injectivity. We claim vertex 4 is coloured b.

If vertex 4 is coloured a, then by injectivity, vertices 3 and 5 are coloured a and b, in some order. The only out-neighbour of a and b in T_4 that has in-degree 3 is c. As such vertex 6 is coloured c. The vertex c in T_4 has three in-neighbours – a, b, and c. As vertex 6 has in-neighbours coloured a and b (namely, vertices 3 and 5), then by injectivity the third in-neighbour of vertex 6 (namely, vertex 7) is coloured c. In T_4, c has two out-neighbours: c and d. Since vertex 7 is coloured c and already has an out-neighbour coloured c, vertex 8 must be coloured d. The vertex 9 has an in-neighbour coloured d. Only vertices a and d in T_4 have d as an in-neighbour. Therefore vertex 9 is coloured with a or d. However, we have shown previously that vertex 9 cannot be coloured a. This implies, that vertex 9 is coloured d. However, vertex 9 has an out-neighbour coloured c. Since c is not an out-neighbour of d in T_4, we arrive at a contradiction. Thus vertex 4 is not coloured a. Therefore vertex 4 is coloured b.

Since vertex 4 is coloured b, vertices 3 and 5 are coloured b and d, in some order. The only common out-neighbour of b and d in T_4 is d. Therefore vertex 6 is coloured d. Hence, by injectivity, vertex 7 is coloured c. Since vertex 7 has an out-neighbour coloured d, vertex 8, another out-neighbour of vertex 7 must be coloured c. Since 9 has both an in-neighbour and an out-neighbour coloured c, vertex 9 must be coloured c. Thus vertices 0 and 9 have the same colour.

Case III: Vertex 0 is coloured d. It cannot be that vertex 1 is coloured d, as vertex 0 has an out-neighbour coloured d – vertex 31 in a copy of H_x. Vertex d has two out-neighbours in T_4: a and d. Therefore vertex

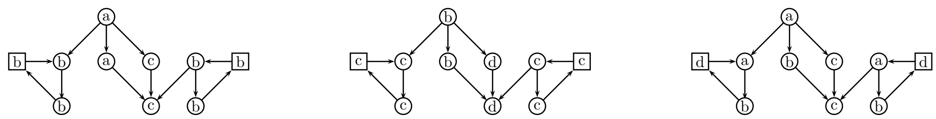

Fig. 6: Colouring the copies of H_e in Theorem 3.3.

1 is coloured a.

The vertex 4 has out-degree 3 and an out-neighbour coloured a. Vertex a is the only vertex in T_4 to have out-degree 3 and have a as an out-neighbour. Therefore vertex 4 is coloured a. By injectivity the vertices 3 and 5, the remaining out-neighbours of vertex 4, are coloured b and c. The vertex 7 cannot be coloured d since 9 already has an out-neighbour coloured d – vertex 37 in a copy of H_x. Moreover, vertex 7 is an in-neighbour of 6, which already has in-neighbours coloured c and b. Hence, vertex 7 must be coloured a. In T_4 the only in-neighbours of a are a and d. Thus vertex 9 is coloured a or d. Since vertex 9 has an out-neighbour coloured d, it cannot be coloured a. Therefore vertex 9 is coloured d, as required. $\square$

Theorem 3.3 *The problem of ios-injective T_4-colouring is NP-complete.*

Proof: The transformation is from 3-edge-colouring subcubic graphs Holyer (1981).

Let G be a graph with maximum degree at most 3 and let $\tilde{G}$ be an arbitrary orientation of G. We create an oriented graph H from $\tilde{G}$ as follows. For every $x \in V(G)$ we add H_x, a copy of the oriented graph given in Figure 4, to H. For every arc $e \in E(\tilde{G})$ we add H_e, a copy of the oriented graph given in Figure 5, to H. To complete the construction of H, for each arc $e = uv \in E(\tilde{G})$ we identify vertex 0 in H_e with one of the three square vertices (i.e., vertices 1, 11, or 21) in H_u and identify vertex 9 in H_e with one of the three square vertices in H_v. We identify these vertices in such a way that each square vertex in a copy of H_x is identified with at most one square vertex from a copy of H_e. We note that this is always possible as vertices in G have degree at most three.

We claim G has a 3-edge-colouring if and only if H has an ios-injective T_4-colouring. Suppose an ios-injective T_4-colouring of H is given. This ios-injective T_4-colouring induces a 3-edge-colouring of G: the colour of an edge in $e \in E(G)$ is given by colour of vertices 0 and 9 in corresponding copy of H_e contained in H. By Lemma 3.2, this colour is well-defined. By Lemma 3.1, vertex 31 in each copy of H_e is coloured d. Therefore, each of the edges incident with any vertex of G receive different colours and no more than 3 colours, namely b, c, and d, are used on the edges of G.

Suppose a 3-edge-colouring of G, $f : E(G) \to \{b, c, d\}$ is given. For each $e \in E(G)$ we colour H_e using one of the ios-injective T_4-colourings given in Figure 6. We choose the colouring of each copy of H_e so that vertices 0 and 9 in that copy are assigned the colour $f(e)$. To complete the proof, we show that such a colouring can be extended to all copies of H_x contained in H.

Recall for each copy of H_x, the vertices 1, 11 and 21 can be respectively identified with either vertex 0 or vertex 9 in some copy of H_e. Since f is a 3-edge-colouring of G, for each $x \in V(G)$, each of the vertices 1, 11 and 21 in H_x are coloured with distinct colours from the set $\{b, c, d\}$ when we colour each copy of H_e using Figure 6.

By symmetry of H_x, we may assume without loss of generality that vertices $1, 11$ and 21 are respectively coloured b, c and d in each copy of H_x. The ios-injective T_4-colouring given in Figure 7 extends a pre-colouring of the vertices 1, 11 and 21 with colours b, c, and d, respectively, to an ios-injective T_4-colouring of H_x. Therefore G has a 3-edge-colouring if and only if H admits an ios-injective T_4-colouring

Since the construction of H can be carried out in polynomial time, ios-injective T_4-colouring is NP-complete □

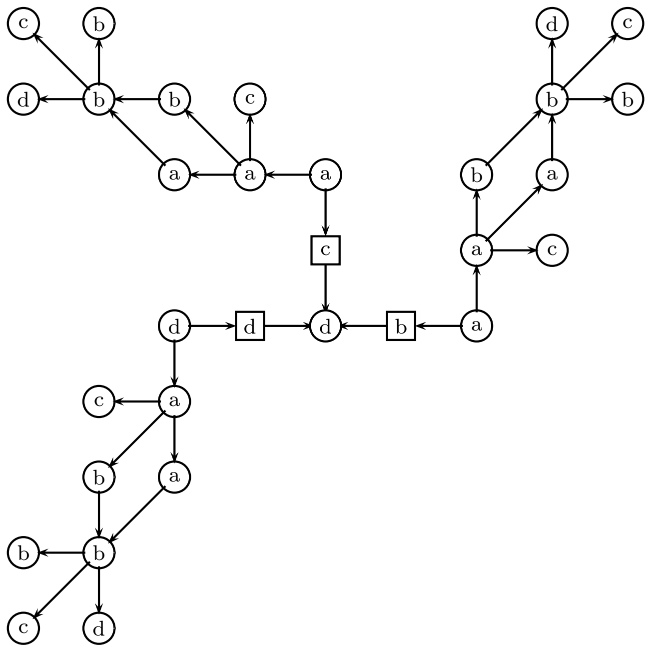

Fig. 7: Colouring the copies of H_x in Theorem 3.3.

We give a similar argument for T_5 (see Figure 3). The transformation is from ios-injective C_3-colouring, which is NP-complete by Theorem 2.2. We construct an oriented graph J from a graph G so that G admits an ios-injective homomorphism to C_3 if and only if J admits an ios-injective homomorphism to T_5. The key ingredient in this construction is the oriented graph J_v, given in Figure 8.

For each $n > 0$ we construct an oriented graph J_n from n copies of J_v, say $J_{v_0}, J_{v_1}, \ldots,$ $J_{v_{n-1}}$, by letting vertices 17, 18 and 19 of J_{v_i} be in-neighbours of vertex 0 in $J_{v_{i+1} \pmod n}$ for all $0 \leq i \leq n-1$.

Lemma 3.4 *For any positive integer n, in an oriented ios-injective T_5-colouring of J_n each of the vertices labelled* 0 *(respectively,* 4*,* 8*,* 12 *and* 16*) receives the same colour.*

Proof: Since T_5 is vertex-transitive, assume without loss of generality that vertex 0 in J_{v_0} receives colour a. If 0 is coloured a, then the vertices 1, 2 and 3 must be coloured a, b and c in some order, these vertices

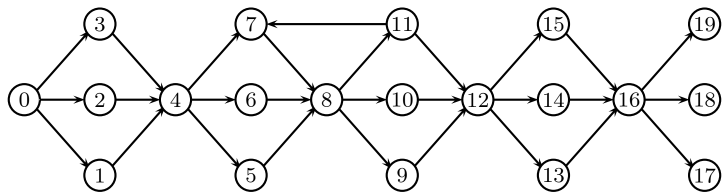

Fig. 8: J_v.

are the only out-neighbours of a in T_5. Since vertex c is the only common out-neighbour of vertices a, b and c in T_5 we have that vertex 4 is coloured c. Since the automorphism of T_5 that maps a to c also maps c to e, we conclude by a similar argument that vertex 8 is coloured e. Similarly we conclude that vertex 12 is coloured b and vertex 16 is coloured d.

Since vertex 16 is coloured d in J_{v_0}, vertices $17, 18, 19$ are coloured, in some order, a, e, d, as these are the only out-neighbours of d in T_5. The only common out-neighbour of a, e and d in T_5 is a. Therefore vertex 0 in J_{v_1} is coloured a. Repeating this argument, we conclude that vertices 4, 8, 12 and 16 in J_{v_2} receive colours c, e, b and d, respectively. Continuing in this fashion gives that in an oriented ios-injective T_5-colouring of J_n each of the vertices labeled 0 (respectively, 4, 8, 12 and 16) receive the same colour. $\square$

Theorem 3.5 *The problem of ios-injective T_5-colouring is NP-complete.*

Proof: The transformation is from ios-injective C_3-colouring (See Theorem 2.2).

Let G be a graph with vertex set $\{v_0, v_1, \ldots, v_{|V(G)|-1}\}$. Let $\nu_G = |V(G)|$. We construct J from G by first adding a copy of J_{ν_G} to G and then, for each $1 \leq i \leq \nu_G$, adding an arc from vertex 11 in J_{v_i} to v_i.

We show that J has an ios-injective T_5-colouring if and only if G has an ios-injective C_3-colouring.

Consider an ios-injective T_5-colouring of J. Since T_5 is vertex-transitive we can assume without loss of generality that vertex 8 in each copy of J_v is coloured a. Therefore in each J_{v_i}, vertices $9, 10$ and 11 are coloured, in some order, with colours a, b, c, and vertex 12 is coloured c.

We claim that v_i is coloured with b, d or e for all $0 \leq i \leq \nu_G - 1$. If v_i has colour a, then vertex 11 in J_{v_i} has both an in-neighbour and an out-neighbour coloured a and is therefore coloured a. Thus, vertex 7 in J_{v_i} also has both an in-neighbour and an out-neighbour coloured a and must be coloured a. Since vertex 11 already has an out-neighbour coloured a, this contradicts the injectivity requirement. If v_i has colour c, then vertex 11 in J_{v_i} has two out-neighbours coloured c, a violation of the injectivity requirement. Therefore v_i is coloured with one of b, d or e for each $0 \leq i \leq \nu_G - 1$. Since vertices b, d and e of T_5 induce a copy of C_3 in T_5, restricting an ios-injective T_5-colouring of J to the vertices of G yields an ios-injective C_3-colouring of G.

Let β be an ios-injective C_3-colouring of G using colours b, d and e. We extend such a colouring to be an ios-injective T_5-colouring of J by assigning the vertices of each J_{v_i} colours based upon $\beta(v_i)$ as shown in Figure 9. This colouring satisfies the injectivity requirement, as each vertex v_i has only neighbours coloured b, d and e in G, and its additional neighbour in J_{v_i}, vertex 11, has colour a or c.

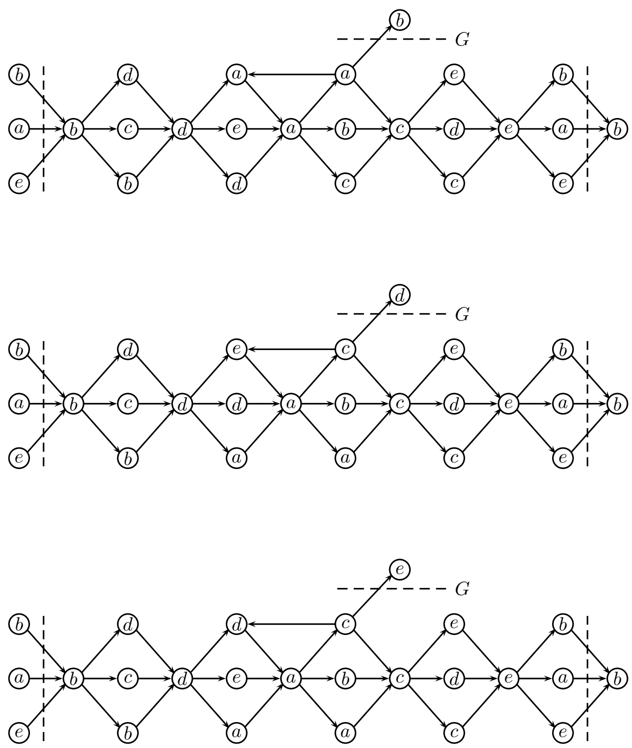

Fig. 9: Colouring the vertices of J_{v_i} using the colour of v_i.

Therefore J has an ios-injective T_5-colouring if and only if G has an ios-injective C_3-colouring. Since J can be constructed in polynomial time, ios-injective T_5-colouring is NP-complete. □

We now present a reduction to instances of ios-injective T-colouring for when T has a vertex v of out-degree at least four. This reduction allows us to polynomially transform an instance of ios-injective T-colouring to an instance of ios-injective T'-colouring, where T' is T_4, T_5, C_3 or TT_3.

Lemma 3.6 *If T is a reflexive tournament on n vertices with a vertex v of out-degree at least four, then ios-injective homomorphism to T' polynomially transforms to ios-injective homomorphism to T, where T' is the tournament induced by the strict out-neighbourhood of v.*

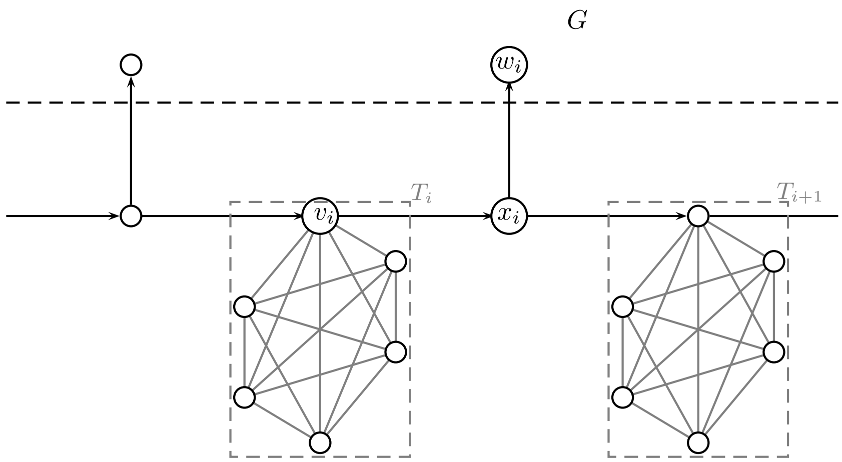

Fig. 10: The construction of H in Lemma 3.6.

Proof: Let T be a reflexive tournament on n vertices with a vertex v of out-degree at least four. Let G be an oriented graph with vertex set $\{w_0, w_1, \ldots, w_{|V(G)|-1}\}$. Let $\nu_G = |V(G)|$. We construct H from G by adding to G

- vertices $x_0, x_1, \ldots, x_{\nu_G - 1}$.
- an arc from x_i to w_i for all $0 \leq i \leq \nu_G - 1$.
- ν_G irreflexive copies of T, labeled T_i, for $0 \leq i \leq \nu_G - 1$.

Let $v_i \in T_i$ be the vertex corresponding to $v \in V(T)$. We complete our construction by adding the arcs $v_i x_i$ and $x_i v_{i+1 \pmod{\nu_G}}$ for all i. See Figure 10.

Claim 1: In an ios-injective T-colouring of H no two vertices of T_i have the same colour. Since T has a vertex of out-degree at least four we observe that T has at least 4 vertices. Let ϕ be an ios-injective T-colouring of H. Suppose there exist $x, y \in T_i$ so that $\phi(x) = \phi(y) = c$, and let z be a third vertex of T_i. By injectivity, x and y cannot be either both in-neighbours or both out-neighbours of z, and therefore z has an in-neighbour and an out-neighbour coloured c. This is only possible if $\phi(z) = c$. Let w be a fourth vertex of T_i. Since x, y and z are all neighbours of w, w has either two in-neighbours or two out-neighbours of the same colour, which is impossible in an ios-injective T-colouring. This proves the claim.

Claim 2: In an ios-injective T-colouring of H, every vertex in the set

$$\{x_0, x_1, \ldots, x_{\nu_G - 1}\} \cup \{v_0, v_1, \ldots, v_{\nu_G - 1}\}$$

receives the same colour. By the previous claim all the colours of T are used exactly once in each T_i. Therefore the only possible colour for an out-neighbour or an in-neighbour of v_i outside of T_i is $\phi(v_i)$.

Therefore for each i, we have $\phi(v_i) = \phi(x_i) = \phi(x_{i+1 \pmod{\nu_G}})$. Therefore every vertex in the set $\{x_0, x_1, \ldots, x_{\nu_G-1}\} \cup \{v_0, v_1, \ldots, v_{\nu_G-1}\}$ receives the same colour. Since each vertex in T_i maps to a unique vertex in T, if $\phi(v_i) \neq v$ then there is an automorphism of T that maps $\phi(v_i)$ to v. As such, we may assume without loss of generality that $\phi(v_i) = v$ for all $0 \leq i \leq \nu_G - 1$.

Let T' be the reflexive sub-tournament of T induced by the strict out-neighbourhood of v. Note that T' is a reflexive tournament on at least 3 vertices and T' is a proper subgraph of T.

We show that H has an ios-injective T-colouring if and only if G has an ios-injective T'-colouring.

Let ϕ be an ios-injective T-colouring of H. By our previous claim, each x_i has an in-neighbour and an out-neighbour with colour v, namely $v_i \in V(T_i)$ and $v_{i-1} \in V(T_{i-1})$. Therefore $\phi(w_i)$ is an out-neighbour of v in T. That is, $\phi(w_i) \in V(T')$. Therefore the restriction of ϕ to the vertices of G yields an ios-injective T'-colouring of G.

Let β be an ios-injective T'-colouring of G. For all $0 \leq i \leq \nu_G - 1$ and all $u \in V(T)$. Let $u_i \in V(T_i)$ be the vertex corresponding to $u \in V(T)$.

We extend β to be an ios-injective T-colouring of H as follows:

- $\beta(x_i) = \beta(v_i) = v$ for all $0 \leq i \leq \nu_G - 1$; and
- for all $u_i \in T_i$, let $\beta(u_i) = u$.

Hence, ios-injective T'-colouring of G can be polynomially transformed to ios-injective T-colouring of H. $\square$

If T is a reflexive tournament with a vertex of in-degree at least 4, a similar argument holds. We modify the construction by reversing the arc between x_i and w_i in the construction of H.

Lemma 3.7 *If T is a reflexive tournament on n vertices with a vertex v of in-degree at least four, then ios-injective homomorphism to T' polynomially transforms to ios-injective homomorphism to T, where T' is the tournament induced by the strict in-neighbourhood of v.*

Our results compile to give a dichotomy theorem.

Theorem 3.8 *Let T be a reflexive tournament. If T has at least 3 vertices, then the problem of deciding whether a given oriented graph G has an ios-injective homomorphism to T is NP-complete. If T has 1 or 2 vertices, then the problem is solvable in polynomial time.*

Proof: If T is a reflexive tournament on no more than three vertices, the result follows by Theorem 2.2. Suppose then that T has four or more vertices. If $T = T_4$, or if $T = T_5$, then the result follows from Theorem 3.3 or Theorem 3.5. Up to isomorphism, there are 16 distinct reflexive tournaments on 4 or 5 vertices. By inspection, tournaments T_4 and T_5 respectively are the only reflexive tournaments on 4 and 5 vertices respectively with no vertex of out-degree or in-degree four. Since the average out-degree of a reflexive tournament on $n > 5$ vertices is $\frac{n-1}{2} + 1 > 3$, every reflexive tournament on at least six vertices has a vertex with out-degree at least four. Therefore if T has at least four vertices, $T \neq T_4$ and $T \neq T_5$, then T has a vertex with either in-degree or out-degree at least four. By repeated application of Lemma 3.6 and Lemma 3.7 an instance of ios-injective homomorphism to T polynomially transforms to instance of either ios-injective homomorphism to T_4, ios-injective homomorphism to T_5 or ios-injective homomorphism to a target on 3 vertices. $\square$

4 Iot-injective homomorphisms

In this section we prove a dichotomy theorem for iot-injective T-colouring, where T is a reflexive tournament. We employ similar methods as in Section 3. We first show that both iot-injective T_4-colouring and iot-injective T_5-colouring are NP-complete. We then provide a reduction to instances of iot-injective T-colouring to either iot-injective T_4-colouring, iot-injective T_5-colouring, or a case covered by Theorem 2.2. Combining these results yields the desired dichotomy theorem.

We begin with a study of iot-injective T_4-colouring. To show iot-injective T_4-colouring is NP-complete we provide a transformation from 3-edge-colouring. We construct an oriented graph F from a graph G so that G has a 3-edge-colouring if and only if F admits an iot-injective homomorphism to T_4. The key ingredients in this construction are the pair of oriented graphs F_x and F_e, shown in Figure 11.

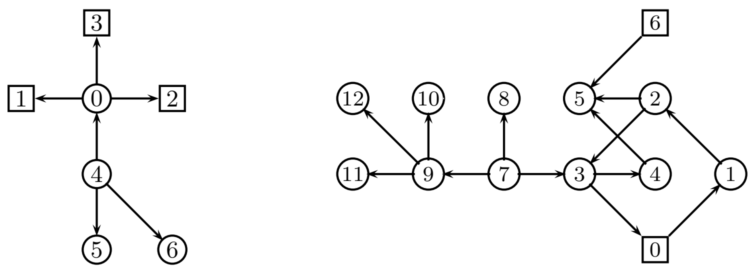

Fig. 11: F_x and F_e, respectively.

Lemma 4.1 *In any iot-injective T_4-colouring of F_x, vertex 0 is coloured b and vertex 4 is coloured a.*

Proof: Consider some iot-injective T_4-colouring of F. Vertex 0 of F_x has out-degree 3. Since each vertex of T_4 has at most three out-neighbours (including itself), vertex 0 must have the same colour as one of its out-neighbours. To satisfy the injectivity constraint, if a colour appears on an out-neighbour of vertex 0, that colour cannot appear on an in-neighbour of vertex 0. Therefore vertex 4 does not have the same colour as vertex 0. Both vertices 4 and 0 have out-degree 3, and there is an arc from 4 to 0. Vertex a in T_4 is the only vertex to have out-degree 3 and have a strict out-neighbour with out-degree 3. Therefore vertex 4 is coloured a and vertex 0 is coloured b. $\square$

Lemma 4.2 *In any iot-injective T_4-colouring of F_e, vertex 7 is coloured a and vertex 9 is coloured b.*

This proof of this lemma follows similarly to the proof of Lemma 4.1. As such, it is omitted.

Lemma 4.3 *Let F_e' be the oriented graph formed from a copy of F_e and two copies of F_x by identifying vertex 0 in the copy of F_e with any square vertex in one copy of F_x and identifying vertex 6 in the copy of F_e with any square vertex in the second copy of F_x. In any iot-injective T_4-colouring of F_e', the vertices 0 and 6 in the subgraph induced by F_e have the same colour, and are coloured with one of b, c or d.*

Proof: Let F_e' be constructed as described. Consider an iot-injective T_4-colouring of F_e'. We examine the colours of the vertices in the subgraph induced by the copy of F_e. By Lemma 4.1 and the construction of F_e', vertices 0 and 6 in F_e have an in-neighbour coloured b – vertex 0 in a copy of F_x. Since b is not an in-neighbour of a in T_4, vertex 0 in the copy of F_e must receive one of the colours b, c, or d. We proceed in cases based on the possible colour of vertex 0 in copy of F_e.

Case I: Vertex 0 *is coloured* b. Since vertex 0 already has a neighbour coloured b (vertex 0 in a copy of F_x), vertex 3, an in-neighbour of vertex 0 in F_e, cannot be coloured b. Since $b \in V(T_4)$ has only a and b as in-neighbours, we have that vertex 3 is coloured a. By Lemma 4.2 vertex 3 has a neighbour coloured a – namely vertex 7. By the injectivity constraint, this colour cannot appear on any other neighbour of vertex 3. As such, vertices 2 and 4 are coloured d and c respectively. The only common out-neighbour of d and c in T_4 is d. Therefore vertex 5 has colour d. In T_4 vertex d has three in-neighbours – b, c and d. Since c and d both appear on an in-neighbour of vertex 5, we have that vertex 6 is coloured with b.

Case II: Vertex 0 *is coloured* c. Vertex c in T_4 has three in-neighbours: a, b and c. Since vertex 0 has an in-neighbour coloured b, namely vertex 0 in a copy of F_x, vertex 3 in F_e must have either colour a or colour c.

Assume vertex 3 is coloured c. In this case, the injectivity constraint implies that vertex 1 is not coloured c. Since c and d are the only out-neighbours of c in T_4, vertex 1 must be coloured d. Vertex 2, an in-neighbour of vertex 3, is coloured with one of a, b or c, the in-neighbours of c in T_4. By Lemma 4.2 vertex 3 has a neighbour coloured a – vertex 7. By assumption vertex 0 has colour c. Therefore by injectivity, vertex 2 has colour b. This is a contradiction, as the arc between vertex 1 and vertex 2 does not have the same direction as the arc between vertex c and vertex b in T_4. Therefore vertex 3 is coloured a.

In T_4, the in-neighbours of a are a and d, and the out-neighbours of a are a, b and c. Since vertex 7 is coloured a, no other neighbour of vertex 3 can be coloured a. Therefore vertex 2, an in-neighbour of vertex 3, must have colour d. Since vertex 0 is coloured c, vertex 4, an out-neighbour of vertex 3 must have colour b. Vertex 5, a common in-neighbour of vertices 2 and 4, must be coloured with a common in-neighbour of d and b in T_4. The only such vertex in T_4 is d. Therefore vertex 5 has colour d.

Vertex d in T_4 has three in-neighbours: b, c and d. Since vertex 2 is coloured d and vertex 4 is coloured b, we have that vertex 6 is coloured c, as required.

Case III: Vertex 0 *is coloured* d. Recall by Lemma 4.2 that vertex 3 has a neighbour coloured a – vertex 7. Since vertex 0 is coloured d, vertex 3 is coloured with a vertex that is an out-neighbour of a and an in-neighbour of d in T_4. The only such vertices are b and c. However, vertex 0 has a neighbour coloured b (vertex 0 in a copy of F_x). Therefore vertex 3 has colour c. Vertex 4 must have a colour that is an out-neighbour of c in T_4. The only such colours are c and d. Since vertex 0, an out-neighbour of vertex 3, is coloured d, we have that vertex 4 has colour c. Vertex 2 must have a colour that is an in-neighbour of c in T_4. The only such colours a, b and c. Vertex 7, an in-neighbour of vertex 3, has colour a. Vertex 4, a neighbour of vertex 3, has colour c. Therefore by injectivity vertex 2 has colour b. Vertex 5 must be coloured with a common out-neighbour of b and c in T_4. The only such colours are c and d. Since vertex 3, an out-neighbour of vertex 2, has colour c, we have by injectivity that vertex 5 has colour d. The in-neighbours of vertex 5 must be coloured with the in-neighbours of d in T_4. Vertex d has three in-neighbours in T_4 – b, c and d. Since vertex 2 has colour b and vertex 4 has colour c, we have by injectivity that vertex 6 has colour d. □

Theorem 4.4 *The problem of iot-injective* T_4*-colouring is NP-complete.*

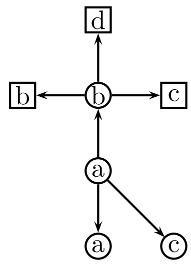

Fig. 12: Colouring F_x.

Proof: The transformation is from 3-edge-colouring subcubic graphs Holyer (1981).

Let G be a graph with maximum degree at most 3 and let $\tilde{G}$ be an arbitrary orientation of G. We create an oriented graph F from $\tilde{G}$ as follows. For every $v \in V(G)$ we add F_v, a copy of the oriented graph F_x given in Figure 11, to F. For every arc $uv \in E(\tilde{G})$ we add F_{uv}, a copy of the oriented graph F_e given in Figure 11, to F. To complete the construction of F, for each arc $uv \in E(\tilde{G})$ we identify vertex 0 in F_{uv} with one of the three square vertices (i.e., vertices 1, 2, or 3) in F_u and identify vertex 6 in F_{uv} with one of the three square vertices in F_v. We identify these vertices in such a way that each square vertex in a copy of F_x is identified with at most one square vertex from a copy of F_e. We note that this is always possible as vertices in G have degree at most three.

We claim that G has a 3-edge-colouring if and only if F has an iot-injective T_4-colouring.

Suppose an iot-injective T_4-colouring of F is given. This iot-injective T_4-colouring induces a 3-edge-colouring of G: the colour of an edge in $uv \in E(G)$ is given by colour of vertices 0 and 6 in corresponding copy of F_{uv} contained in F. By Lemma 4.3 this colour is well-defined, and is one of b, c, or d. Recall for each copy of F_x, the vertices 1,2 and 3 are respectively each identified with either vertex 0 or vertex 6 in some copy of F_e. By Lemma 4.1, vertices 1, 2 and 3 in a copy of F_x cannot be coloured a. By injectivity, vertices 1, 2 and 3 in a copy of F_x all are assigned different colours. Therefore each of the edges incident with any vertex receives different colours and no more than 3 colours, namely b, c, and d, are used on the edges of G. Therefore G has a 3-edge-colouring.

Suppose a 3-edge-colouring of G, $f : E(G) \to \{b, c, d\}$ is given. For each $uv \in E(G)$ we colour F_{uv} using one of the iot-injective T_4-colourings given in Figure 13. We choose the colouring of F_{uv} so that vertices 0 and 6 are assigned the colour $f(uv)$. To complete the proof, we show that such colouring can be extended to all copies of F_x contained in F.

Recall for each copy of F_x, the vertices 1, 2 and 3 are respectively each identified with either vertex 0 or vertex 6 in some copy of F_e. Since f is a 3-edge-colouring of G, for each $x \in V(G)$, each of the vertices 1, 2 and 3 in F_x are coloured with distinct colours from the set $\{b, c, d\}$ when we colour each copy of F_e using Figure 13.

By symmetry of F_x, we may assume without loss of generality that vertices $1, 2$ and 3 are respectively coloured b, c and d in each copy of F_x. The iot-injective T_4-colouring given in Figure 12 extends a pre-colouring of the vertices 1, 2 and 3 with colours b, c, and d, respectively, to an iot-injective T_4-colouring of F_x. Therefore F has an iot-injective T_4-colouring.

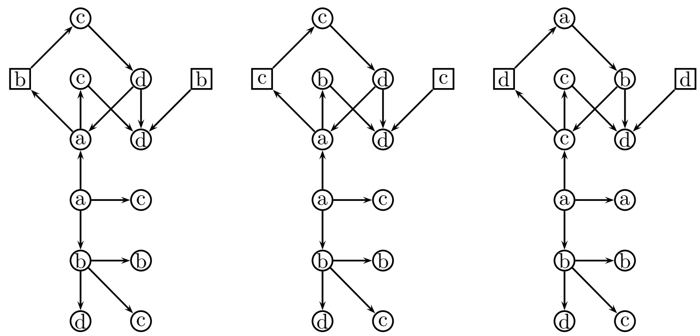

Fig. 13: Colouring the F_e for associated edge colours of b, c, and d.

Since the construction of F can be carried out in polynomial time, iot-injective T_4-colouring is NP-complete. $\square$

We provide a similar argument for iot-injective T_5-colouring. The transformation is from iot-injective C_3-colouring Campbell et al. (2016a). We construct an oriented graph D from a graph G so that G admits an iot-injective homomorphism to C_3 if and only if D admits an iot-injective homomorphism to T_5. The key ingredient in the construction is the oriented graph, D_v, given in Figure 14.

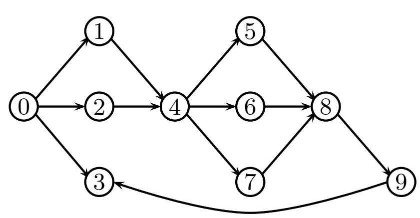

Fig. 14: D_v.

For each $n > 0$ let D_n be the oriented graph constructed from n disjoint copies of D_v, say $D_{v_0}, D_{v_1}, \dots, D_{v_{n-1}}$, by letting vertex 8 of D_{v_i} be an in-neighbour of vertex 0 in $D_{v_{i+1} \pmod n}$ for all $0 \leq i \leq n-1$.

Lemma 4.5 *For any positive integer n, up to automorphism, in an oriented iot-injective T_5-colouring of D_n each of the vertices labeled 0 receive the colour d, each of the vertices labeled 4 receive the colour a, and each of the vertices labeled 8 receive the colour c.*

Proof: Since T_5 is vertex-transitive, assume without loss of generality that vertex 0 in D_{v_0} receives colour d in some iot-injective T_5-colouring of D_n. Observe that vertex 4 has three out-neighbours. Since each vertex of T_5 has at most three out-neighbours (including itself), vertex 4 must have the same colour as one of its out-neighbours. To satisfy the injectivity constraint, no in-neighbour of vertex 4 has the same colour as vertex 4. Further, vertex 4 has two in-neighbours, vertices 1 and 2, that are out-neighbours of a vertex coloured d. Only vertices a and b in T_5 have two in-neighbours that are out-neighbours of d. Therefore vertex 4 has colour a or b.

If vertex 4 has colour b, then vertices 1 and 2 are coloured with the vertices of T_5 that are out-neighbours of d and in-neighbours of b. The only such vertices in T_5 that satisfy these criteria are a and e. Therefore vertices 1 and 2 are coloured with a and e, in some order. Vertex d has three out-neighbours in T_5 – a, d and e. Since vertices 1 and 2 are coloured with a and e, in some order, the third out-neighbour of vertex 0, vertex 3, is coloured with d. Vertex b in T_5 has three out-neighbours – b, c and d. Therefore the out-neighbours of vertex 4, vertices $5, 6$ and 7, are coloured, in some order, with these colours. Vertices b, c and d in T_5 have only d as a common out-neighbour. Therefore the common out-neighbour of vertices $5, 6$ and 7, vertex 8, is coloured d. This is a contradiction, as now vertex 9 has two vertices coloured d in its neighbourhood. Therefore vertex 4 has colour a.

Vertex a in T_5 has three out-neighbours – a, b and c. Thus the out-neighbours of vertex 4 are coloured with a, b and c, in some order. The only common out-neighbour of a, b and c in T_5 is c. Therefore vertex 8 has colour c. This implies that vertex 9 in D_{v_0} and 0 in D_{v_1} have colours from the set $\{c, d, e\}$, the out-neighbours of c in T_5. Since vertex 8 has a neighbour coloured c, neither vertex 0 in D_{v_1} nor 9 (in D_{v_0}) can have this colour. Further, since vertex 3 has a neighbour coloured d, vertex 9 has cannot be coloured d. Thus vertex 9 in D_{v_0} has colour e and vertex 0 in D_{v_1} has colour d.

Repeating this argument implies that every vertex labeled 0 has colour d. □

Theorem 4.6 *The problem of iot-injective T_5-colouring is NP-complete.*

Proof:

The transformation is from iot-injective C_3-colouring (See Theorem 2.2).

Let G be an oriented graph with vertex set $\{v_0, v_1, \dots, v_{|V(G)|-1}\}$. Let $\nu_G = |V(G)|$. We construct D from G by first adding a copy of D_{ν_G} to G and then, for each $1 \leq i \leq \nu_G$, adding an arc from vertex 5 in D_{v_i} to v_i.

We show that D has an iot-injective T_5-colouring if and only if G has an iot-injective C_3-colouring.

Consider ϕ, an iot-injective T_5-colouring of D. Since T_5 is vertex-transitive we may assume that vertex 0 in D_{v_0} has colour d. By Lemma 4.5, for all $0 \leq i \leq \nu_G - 1$, the vertex in D_{v_i} labeled 0 has colour d, the vertex labeled 4 has colour a and the vertex labeled 8 has colour c. By the injectivity requirement, the neighbours of the vertex labeled 5 in each copy of D_v have distinct colours. Since the vertices 4 and 8 have colours a and c, respectively, only colours b, d or e can appear at v_i, for all $0 \leq i \leq \nu_G - 1$. Since

b, d, and e induce a copy of C_3 in T_5, we conclude that the restriction of ϕ to the vertices of G is indeed an iot-injective C_3-colouring.

Let β be an iot-injective C_3-colouring of G using colours b, d and e. We extend such a colouring to be an ios-injective T_5-colouring of D by assigning the vertices of each D_{v_i} colours based upon $\beta(v_i)$ as shown in Figure 15. This colouring satisfies the injectivity requirement, as each vertex v_i has only neighbours coloured b, d and e in G, and its additional neighbour in D_{v_i}, vertex 5, has colour a or c.

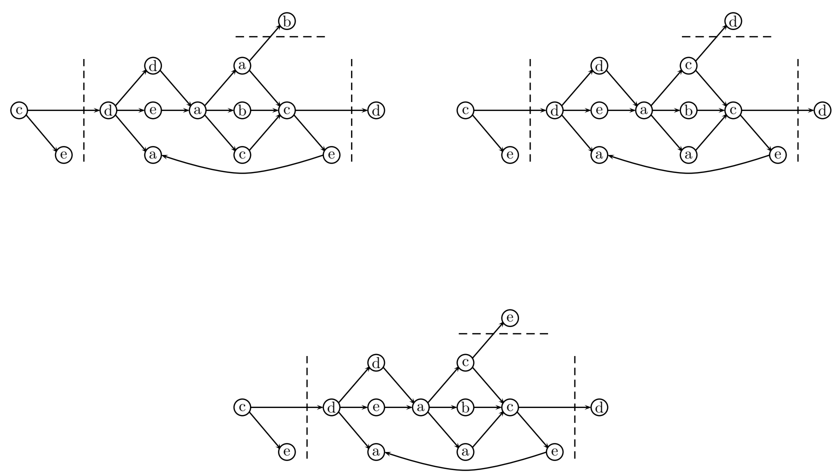

Fig. 15: Colouring the vertices of D_{v_i} using the colour of v_i.

Therefore D has an iot-injective T_5-colouring if and only if G has an iot-injective C_3-colouring. As D can be constructed in polynomial time, iot-injective T_5-colouring is NP-complete. □

We now present a reduction to instances of iot-injective T-colouring for when T has a vertex v of out-degree at least four. This reduction allows us to polynomially transform an instance of iot-injective T-colouring to an instance of iot-injective T'-colouring, where T' is T_4, T_5, C_3 or TT_3.

Lemma 4.7 *If T is a reflexive tournament on n vertices with a vertex v of out-degree at least four, then iot-injective homomorphism to T' polynomially transforms to iot-injective homomorphism to T, where T' is the tournament induced by the strict out-neighbourhood of v.*

Proof: Let T be a reflexive tournament on n vertices with a fixed vertex v of out-degree four or more. Let $T^\star$ be the graph obtained by removing from T all the arcs with their tail at v.

Let G be an oriented graph with vertex set $\{w_0, w_1, \ldots, w_{|V(G)|-1}\}$. Let $\nu_G = |V(G)|$. We construct C from G by adding to G

- ν_G disjoint irreflexive copies of $T : T_0, T_1, \ldots, T_{\nu_G-1}$;
- ν_G disjoint irreflexive copies of $T^\star : T_0^\star, T_1^\star, \ldots, T_{\nu_G-1}^\star$;
- and for all $u \in V(T)$ where $u \neq v$, an arc from the vertex corresponding to u in $T_{i-1}^\star$ to the vertex corresponding to u in T_i, for all $0 \leq i \leq \nu_G - 1$

Let v_i and $v_i^\star$ be the vertices corresponding to v in T_i and $T_i^\star$, respectively. We complete the construction of C by adding an arc from v_i to $v_i^\star$ for all $0 \leq i \leq \nu_G - 1$. See Figure 16

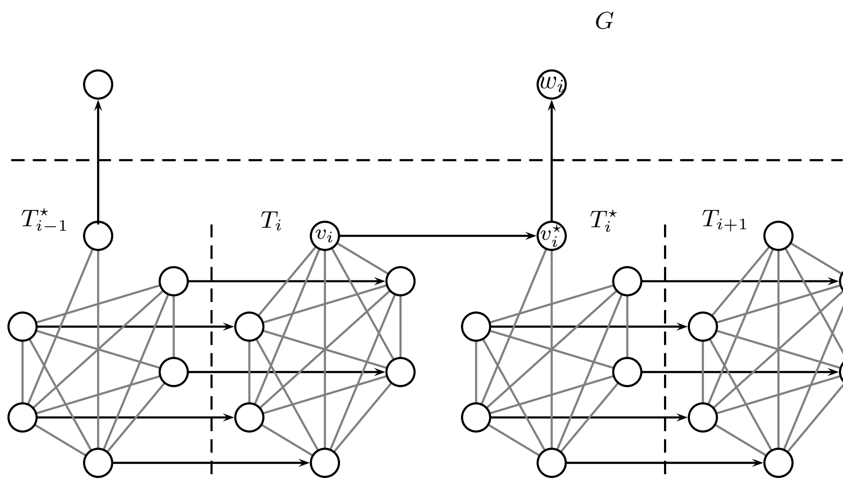

Fig. 16: The construction of C in Lemma 4.7.

Claim 1: In an iot-injective T-colouring of C, no two vertices of T_i have the same colour. If two vertices of T_i are assigned the same colour, then a common neighbour of such vertices in T_i has a pair of neighbours with the same colour. This is a violation of the injectivity requirement. Therefore no two vertices of T_i are assigned the same colour.

Claim 2: In an iot-injective T-colouring of C, v_i and $v_i^\star$ have the same colour. Since v_i has n neighbours, in any iot-injectve T-colouring of C, v_i is assigned the same colour as one of its neighbours. By the previous claim, the neighbour of v_i that has the same colour as v_i must be $v_i^\star$.

Claim 3: In an iot-injective T-colouring of C, v_i and v_{i+1} have the same colour. We show that v_{i+1} has the same colour as $v_i^\star$. If v has out-degree n in T, then by construction v_i and v_{i+1} each have out-degree n in C. Since no two out-neighbours of any vertex can receive the same colour, and since there can be at most one vertex of out-degree n in T, it must be that v_i and v_{i+1} have colour v. Suppose then that v has at least one in-neighbour distinct from itself, say y, in T. Let $y_i^\star$ be the vertex corresponding to y in $T_i^\star$. Let u_{i+1} be a vertex in $T_{i+1} \setminus \{v_{i+1}\}$, and let $u_i^\star$ be the vertex of $T_i^\star$ which has u_{i+1} as an out-neighbour. By the first claim, no two vertices in T_{i+1} share a colour, and since u_{i+1} has $n-1$ neighbours in T_{i+1}, it must be that u_{i+1} and $u_i^\star$ share a colour. This implies that no two vertices of $T_i^\star \setminus \{v_i^\star\}$ have the same colour, and the colours used to colour $T_i^\star \setminus \{v_i^\star\}$ are the same colours as those used to colour $T_{i+1} \setminus \{v_{i+1}\}$. The vertex $y_i^\star$ has $v_i^\star$ as an out-neighbour, and each colour except the colour of v_{i+1} is used to colour a vertex distinct from $v_i^\star$ which is a neighbour of $y_i^\star$. Therefore, $v_i^\star$ must have the same colour as v_{i+1}. The result now follows from the previous claim.

Let T' be the reflexive sub-tournament of T induced by the strict out-neighbourhood of v. We show G has an iot-injective T'-colouring if and only if C has an iot-injective T-colouring.

Consider an iot-injective T-colouring of C, ϕ. By the claims above, $\phi(v_i^\star) = \phi(v_i) = \phi(v_j) = \phi(v_j^\star)$ for all $1 \leq i, j \leq \nu_G - 1$. Since each vertex in T_i is assigned a distinct colour from T and $T \cong T_i$, if $\phi(v_i) \neq v$, then there is an automorphism of T that maps v to $\phi(v_1)$. As such we may assume, without loss of generality that $\phi(v_i) = v$ for all $1 \leq i \leq \nu_G - 1$. Since w_i is an out-neighbour of $v_i^\star$ for each $1 \leq i, \leq \nu_G - 1$ we have that $\phi(w_i)$ is contained in the out-neighbourhood of v for all $1 \leq i \leq \nu_G - 1$. That is, $\phi(w_i) \in V(T')$ for all $1 \leq i \leq \nu_G - 1$. Therefore the restriction of ϕ to G is an iot-injective homomorphism to T'.

Consider now an iot-injective T'-colouring of G, β. We extend β to be an iot-injective T-colouring of C as follows. For each $z \in V(T)$ let z_i and $z_i^\star$ be the corresponding vertices in T_i and $T_i^\star$, respectively. We extend β so that $\beta(z_i) = \beta(z_i^\star) = z$. It is easily verified that β is an iot-injective T-colouring of C. $\square$

The construction of C can be modified to give the corresponding result for reflexive tournaments T with a vertex of in-degree at least four.

Lemma 4.8 *If T is a reflexive tournament on n vertices with a vertex v of in-degree at least four, then iot-injective homomorphism to T' polynomially transforms to iot-injective homomorphism to T, where T' is the tournament induced by the strict in-neighbourhood of v.*

Similar to the case of ios-injective colouring, our results compile to give a dichotomy theorem.

Theorem 4.9 *Let T be a reflexive tournament. If T has at least 3 vertices, then the problem of deciding whether a given oriented graph G has an iot-injective homomorphism to T is NP-complete. If T has 1 or 2 vertices, then the problem is solvable in polynomial time.*

5 A note on irreflexive-injective homomorphisms

No dichotomy theorem has emerged yet for iot-injective homomorphism, and hence ios-injective homomorphism, to irreflexive tournaments. The results of Campbell et al. (2016a); Campbell (2009) tell us that the problem is not only solvable in polynomial time for the irreflexive tournaments on two vertices or less but also for the irreflexive tournaments on three vertices. Preliminary work suggests that the problem is solvable in polynomial time on two of the irreflexive tournaments on four vertices but NP-complete on the remaining two, and on many irreflexive tournaments on more vertices (including at least ten of the twelve irreflexive tournaments on five vertices). The problem has not been proven solvable in polynomial time on any irreflexive tournament on five vertices or more.

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