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*The 26 Wilf-equivalence classes of length five quasi-consecutive patterns**

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We present two families of Wilf-equivalences for consecutive and quasi-consecutive vincular patterns. These give new proofs of the classification of consecutive patterns of length 4 and 5. We then prove additional equivalences to explicitly classify all quasi-consecutive patterns of length 5 into 26 Wilf-equivalence classes.

Keywords: permutation pattern, vincular pattern, Wilf-equivalence

1 Introduction

In the theory of pattern avoidance, vincular patterns (generalizing classical patterns) were introduced by Babson and Steingrímsson in [2]. The aim of this paper is to prove several new Wilf-equivalences for certain such patterns of length five.

First, we recall the definition of a classical pattern:

Definition 1.1. We say two sequences $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ of positive integers are *order-isomorphic* if for any $1 \le i, j \le k$, we have $a_i < a_j$ if and only if $b_i < b_j$. In this case, we write $a_1 \cdots a_k \sim b_1 \cdots b_k$.

Definition 1.2. Let $\sigma \in S_k$ (here S_k is the set of permutations of $\{1, \dots, k\}$). A permutation $\pi \in S_n$ $(n \ge k)$ *contains* σ as a *classical pattern* if there exists $1 \le i_1 < \cdots < i_k \le n$ such that

$$\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k} \sim \sigma_1\sigma_2\cdots\sigma_k.$$

Alternatively, if a permutation does not contain σ, it is said to *avoid* σ.

Example 1.3. The permutation 146235 avoids the classical pattern 321.

We now define a so-called vincular pattern, which is a classical pattern with the additional requirement that certain pairs of indices must be consecutive.

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Definition 1.4. A *vincular pattern* is a combination of a permutation and a set of adjacencies, which is formally a pair (σ, T) where $\sigma \in S_k$ and $T \subset \{1, 2, \dots, k-1\}$ (where $i \in T$ signifies that i and $i+1$ should be adjacent).

We generally denote this by a dashed pattern, in which π_i and π_{i+1} are joined by a dash if $i \notin T$. For example, $(2413, \{1, 3\})$ is usually written 24-13.

Definition 1.5. We say a permutation $\pi \in S_n$ *contains* the vincular pattern $\sigma = (\sigma_1 \cdots \sigma_k, T)$ if there exists $1 \le i_1 < \cdots < i_k \le n$ such that

$$\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k} \sim \sigma_1\sigma_2\cdots\sigma_k \qquad \text{and} \quad i_{j+1} = i_j + 1 \text{ for all } j \in T$$

and otherwise π *avoids* σ.

We then define the following special types of vincular patterns:

- A *classical* pattern (as defined above) is then a pattern of the form σ_1-σ_2-$\cdots$-σ_n (all possible dashes). Thus vincular patterns subsume classical patterns.
- We say a pattern of the form $\sigma_1\sigma_2\cdots\sigma_n$ (with no dashes) is a *consecutive* pattern.
- We say a pattern of the form $\sigma_1\sigma_2\cdots\sigma_{n-1}$-$\sigma_n$ (with exactly one dash at the end) is a *quasi-consecutive* pattern.

Example 1.6. The permutation $\pi = 3275164$ contains 24-13 (via $\pi_2\pi_3\pi_5\pi_6$).

The following is the key definition in the study of pattern avoidance.

Definition 1.7. For each n and a vincular pattern σ, we define $S_n(\sigma)$ as the set of permutations in S_n that avoid σ. We say σ and τ are *Wilf-equivalent* (written as $\sigma \equiv \tau$) if

$$|S_n(\sigma)| = |S_n(\tau)|$$

for every positive integer n.

Example 1.8. Trivial examples of Wilf-equivalences include the *reverse* and *complement* of vincular patterns.

For example, if σ = 12-4-3, then the reverse σ^r = 3-4-21 is Wilf-equivalent to σ because there is an obvious bijection $S_n(\sigma) \to S_n(\sigma^r)$ by reversing the letters. Similarly, the complement σ^c = 43-1-2 is also Wilf-equivalent to σ since if $\pi = \pi_1\pi_2\dots\pi_n$ avoids σ, then its complement $\pi^c = (n+1-\pi_1)\cdots(n+1-\pi_n)$ avoids σ^c, and vice versa. Combining both gives us the equivalences 12-4-3 $\equiv$ 43-1-2 $\equiv$ 2-1-34 $\equiv$ 3-4-21.

Wilf-equivalence is clearly an equivalence relation, and thus groups vincular patterns of a fixed length into equivalence classes. The core question in the theory of Wilf equivalences is to classify patterns completely into such equivalence classes.

We comment briefly on what is already known; more details are provided in Section 2. The Wilf-equivalence classification for vincular patterns of length 3 was completed by Claesson [5], and while the classification for vincular patterns of length 4 has not yet been completed, results by Baxter and Shattuck [4] and earlier results from [3, 7, 8, 10, 11] reduced the problem to two remaining specific conjectures. However, results from [8, 13, 9] have fully completed the consecutive case for $k \in \{4, 5, 6\}$. Wilf-equivalences of vincular patterns have also been studied in multisets instead of in permutations, such as in Mansour and Shattuck [12]. Wilf-equivalence classification for classical patterns up to $k = 7$ has been completed, with both the $k = 6$ and $k = 7$ case completed by Stankova and West [14]. Wilf-equivalence classification for involutions avoiding classical patterns up to $k = 7$ has also been completed, where two patterns are considered equivalent if for all n the number of involutions in S_n avoiding the patterns are equal. The cases of $k = 5, 6, 7$ were completed by Dukes et al. [6].

In this paper, we focus on *quasi-consecutive* vincular patterns, which are patterns of the form $\sigma_1\sigma_2\cdots\sigma_k$-$\sigma_{k+1}$. The structure of this paper is as follows. In Section 2 we list the table of equivalence classes and describe exactly which equivalences remain to be shown. In Section 3.3, we provide a bijective result (Theorem 3.5), which gives proofs for Wilf-equivalences for certain consecutive patterns and quasi-consecutive vincular patterns. This leads to a new proof for all Wilf-equivalences for consecutive patterns of lengths 4 and 5. In Section 4, we give a second family of equivalences (Theorem 4.7). Finally, in Section 5 we prove one last equivalence 2153-4 $\equiv$ 3154-2. Taking all this together we have the following theorem.

Theorem 1.9. *There are exactly 26 equivalence classes of quasi-consecutive patterns of length* 5*, as listed in Table 1.*

2 The 26 equivalence classes of quasi-consecutive patterns

2.1 Table of equivalence classes

The claimed 26 equivalence classes are presented in Table 1.

A few remarks are in order. For a pattern and its complement, we only record the lexicographically earliest one (thus halving the number of entries). The naming of the classes above is in ascending order by the number of avoiding permutations when $n = 8$. Distinct equivalence classes which are tied are then sorted according to the number of permutations when $n = 9$, then $n = 10$. (See Appendix A for these totals.) Within each row except L, the permutations are sorted in lexicographically ascending order.

2.2 Equivalences already known

The following theorem was established both by Elizalde [7] and Kitaev [11].

Theorem 2.1 ([7, 11])**.** *Suppose that* $\sigma_1\sigma_2...\sigma_k \equiv \tau_1\tau_2...\tau_k$. *Then:*

$$\sigma_1\cdots\sigma_k\text{-}(k+1) \equiv \tau_1\tau_2\cdots\tau_k\text{-}(k+1).$$

This theorem leads to several corollaries for quasi-consective patterns of length 5.

The following result produces equivalence class Y in Table 1.

Theorem 2.2 (Theorem 9 of [4])**.** *For a fixed k, and for* $1 \le i \le n-1$, *define*

$$\sigma_i = 12\cdots i(i+2)\cdots(k+1)\text{-}(i+1).$$

Then, $\sigma_i \equiv \sigma_j$ *for every* $1 \le i < j \le n-1$.

2.3 Equivalence class F

Here we elaborate on equivalence class F. First, by using [7, 11] alongside with the complement operation, we obtain

$$1243\text{-}5 \equiv 2134\text{-}5 \equiv 2354\text{-}1 \equiv 3245\text{-}1.$$

	#	Elements	Proof
A	2	1254-3, 1354-2	Thm. 4.7
B	2	1453-2, 1543-2	Thm. 3.8
C	2	2135-4, 2145-3	Thm. 4.7
D	1	1534-2	Singleton
E	1	2315-4	Singleton
F	6	1243-5, 1253-4, 2134-5, 2354-1, 3145-2, 3245-1	See §2.3
G	2	3125-4, 3215-4	Thm. 3.8
H	1	1523-4	Singleton
I	1	3251-4	Singleton
J	1	2154-3	Singleton
K	1	1435-2	Singleton
L	16	3124-5, 3214-5, 2543-1, 2453-1, 2341-5, 2431-5, 1342-5, 1432-5, 1352-4, 1452-3, 1532-4, 1542-3, 2531-4, 2541-3, 2351-4, 2451-3,	See §2.4
M	2	2153-4, 3154-2	Thm. 5.4
N	1	2513-4	Singleton
O	1	1524-3	Singleton
P	1	2415-3	Singleton
Q	1	1325-4	Singleton
R	4	1423-5, 2314-5, 2534-1, 3241-5	[7, 11]
S	2	2143-5, 3254-1	[7, 11]
T	1	3152-4	Singleton
U	1	1425-3	Singleton
V	2	1324-5, 2435-1	[7, 11]
W	1	2514-3	Singleton
X	2	2413-5, 3142-5	[7, 11]
Y	3	1235-4, 1245-3, 1345-2	[4, Thm. 9]
Z	2	1234-5, 2345-1	[7, 11]

Tab. 1: The 26 $abcd$-e equivalence classes.

Thus, the class F is complete once we prove

$$1243\text{-}5 \equiv 1253\text{-}4 \tag{Fa}$$

$$3245\text{-}1 \equiv 3145\text{-}2 \tag{Fb}$$

which both follow as corollaries of Theorem 4.7.

2.4 Equivalence class L

We now elaborate on equivalence class L. Again, [7, 11] together with complement operation implies that the first eight patterns (those ending in 1 or 5) in L are equivalent to one another, that is:

$$\begin{aligned} 3124\text{-}5 &\equiv 3214\text{-}5 \equiv 2543\text{-}1 \equiv 2453\text{-}1 \\ &\equiv 2341\text{-}5 \equiv 2431\text{-}5 \equiv 1342\text{-}5 \equiv 1432\text{-}5. \end{aligned}$$

(Here we are using the fact that $1342 \equiv 1432$, which is for example [8], or our Corollary 3.6.)

Next, by Theorem 4.7 we obtain that

$$1342\text{-}5 \equiv 1352\text{-}4 \equiv 1452\text{-}3 \quad \text{(La)}$$
$$1432\text{-}5 \equiv 1532\text{-}4 \equiv 1542\text{-}3 \quad \text{(Lb)}$$
$$2431\text{-}5 \equiv 2531\text{-}4 \equiv 2541\text{-}3 \quad \text{(Lc)}$$
$$2341\text{-}5 \equiv 2351\text{-}4 \equiv 2451\text{-}3. \quad \text{(Ld)}$$

These finish the classification of L.

3 A family of bijective filling-shape-Wilf-equivalences

3.1 Shape Wilf-Equivalence

In order to state the results of Section 3.3 in their full generality, we first define the stronger notions of shape-Wilf-equivalence and filling-shape-Wilf-equivalence (introduced in [1] and [3], respectively).

Consider a Young diagram with m rows and n columns where columns are nonincreasing from left to right and rows are nonincreasing from bottom to top. In other words, the Young diagram can be thought of as a subset of the $m \times n$ grid where $(1,1)$ is at the bottom-left corner and if (a,b) is in the diagram, then so is (a',b') for any $a' \leq a$, $b' \leq b$ (this is the “French notation”). We define a filling of a Young Diagram as follows:

Definition 3.1. A *filling* is a collection of *selected elements* of the form (a_1,b_1), …, (a_k,b_k) such that $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i \neq j \leq k$.

Similarly, a *standard filling* is a filling of a Young diagram such that $k = m = n$, i.e. every row and every column has an element.

Examples of nonstandard and standard fillings are presented in Figure 1.

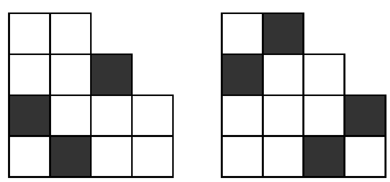

Fig. 1: For the same Young diagram, a nonstandard filling on the left and a standard filling on the right.

We can define containment of a vincular pattern for a filling in the same way as for a permutation.

Definition 3.2. We say that a filling *contains* $\sigma = (\sigma_1 \cdots \sigma_k, T)$ if there exists $i_1, \ldots, i_k$ such that the following conditions hold:

- $(i_1, \pi(i_1)), \ldots, (i_k, \pi(i_k))$ are selected elements
- $\sigma_1 \cdots \sigma_k \sim \pi(i_1) \cdots \pi(i_k)$,
- $(i_j, \pi(i_{j'}))$ is in the Young diagram for any $1 \leq j, j' \leq k$, and
- $i_{j+1} - i_j = 1$ for any $j \in T$.

Else, we say that a filling *avoids* σ.

From here, we can define shape-Wilf-equivalence and filling-shape-Wilf-equivalence.

Definition 3.3. For two vincular patterns σ, τ, we say that σ and τ are shape-Wilf-equivalent if for any Young diagram, the number of standard fillings avoiding σ equals the number of standard fillings avoiding τ. In this case, we write $\sigma \equiv_{\text{s}} \tau$.

Definition 3.4. For two vincular patterns σ, τ, we say that σ and τ are filling-shape-Wilf-equivalent if for any Young diagram and for any fixed set of rows R and fixed set of columns C, the number of fillings having no element in exactly the rows R and columns C and avoiding σ equals the number of such fillings avoiding τ. In this case, we say that $\sigma \equiv_{\text{fs}} \tau$.

It is easy to see that shape-Wilf-equivalence implies Wilf-equivalence by taking the $n \times n$ boards. It is also easy to see that filling-shape-Wilf-equivalence implies shape-Wilf-equivalence since we can take $R = C = \varnothing$.

3.2 Motivating example

To motivate the following results, we briefly outline a proof of the Wilf-equivalence

$$1342 \equiv 1432.$$

As this section is motivational only, it may be skipped without loss of continuity, though we reuse the main idea of bijective swapping expained in this section in the proofs of Theorem 3.5 and Theorem 3.8.

We define a map $\Psi : S_n \to S_n$ as follows. Given a permutation viewed as a standard filling of an $n \times n$ Young tableau, take each instance of 1342 or 1432, and swap the two elements in the middle, for example as in Figure 2.

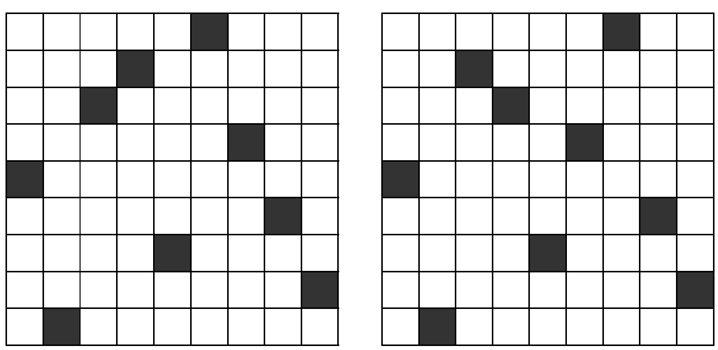

Fig. 2: An example of the bijection Ψ. The consecutive patterns 1342 and 1432 appear starting at columns 2 and 5.

One can check that the pair of consecutive patterns $\{1342, 1432\}$ has the property that any two instances of either in a given permutation overlap in at most one place; from this it follows that Ψ is well-defined, and an involution on S_n. But Ψ also maps 1342-avoiding permutations to 1432-avoiding permutations and vice-versa, which implies the Wilf-equivalence.

3.3 Statement of results

We now prove a family of filling shape-Wilf-equivalences for consecutive and quasi-consecutive patterns, formally stating the ideas in the previous section in their full generality. This will give new proofs of all consecutive Wilf-equivalences of length 4 and 5, answer a filling-shape-Wilf-equivalence conjecture by Baxter, and help complete the Wilf-equivalence classification of quasi-consecutive patterns of length 5.

Theorem 3.5. *Suppose that*

$$\sigma = \sigma_1\sigma_2 \cdots \sigma_i\sigma_{i+1} \cdots \sigma_j\sigma_{j+1} \cdots \sigma_k$$

is a permutation where $1 \leq i < j < k$*. Also suppose that*

$$\tau = \tau_1\tau_2 \cdots \tau_i\tau_{i+1} \cdots \tau_j\tau_{j+1} \cdots \tau_k$$

where $\tau_{i+1}, \ldots, \tau_j$ *is a permutation of* $\sigma_{i+1}, \ldots, \sigma_j$*, and* $\sigma_x = \tau_x$ *for* $x \in [1, i] \cup [j+1, k]$*. Finally, suppose that the nonoverlapping criteria*

$$\begin{aligned} \sigma_1 \cdots \sigma_z &\not\sim \sigma_{k-z+1} \cdots \sigma_k \\ \sigma_1 \cdots \sigma_z &\not\sim \tau_{k-z+1} \cdots \tau_k \\ \tau_1 \cdots \tau_z &\not\sim \sigma_{k-z+1} \cdots \sigma_k \\ \tau_1 \cdots \tau_z &\not\sim \tau_{k-z+1} \cdots \tau_k \end{aligned}$$

all hold for $z > \min(i, k-j)$ *and* $z \neq k$*. Then,*

$$\sigma \equiv_{\text{fs}} \tau.$$

Proof: Fix a Young diagram and a set of rows and columns which are avoided. We present a bijection mapping fillings of containing σ (call the set T) to fillings containing τ (call the set T'.) Note T and T' need not be disjoint.

For each such permutation in S_n, draw a box of width k around every consecutive occurrence of σ or τ. Because of the four equivalences not occurring, no two boxes can intersect in more than $\min(i, k-j)$ places. Also since $i < j$ we cannot have three boxes pairwise intersecting.

Consider the map

$$\Psi : S_n \to S_n$$

defined by permuting every boxed σ permutation to make it a τ permutation and vice versa. Because boxes do not intersect in more than $\min(i, k-j)$ places, the order of permutation does not matter. This also means that Ψ sends T to T' and T' to T, and that Ψ is a well defined map.

Also, our four equivalence restrictions prevent new σ or τ permutations from appearing because a new occurrence of σ or τ would need to have at least one element in a block of k elements changed. Thus, it would need to intersect with a part of a boxed segment in at least one of positions $i+1, \ldots, j$, a clear contradiction. This means for $\pi \in T \cup T'$ the boxes around π will be the same as the boxes around $\Psi(\pi)$. This means Ψ is an involution, as every box will permute from σ to τ to σ again, or vice versa.

Thus, Ψ induces a bijection from T to T', so we are done. □

Corollary 3.6. The following filling-shape-Wilf equivalences hold:

- $1342 \equiv_{\text{fs}} 1432$
- $2341 \equiv_{\text{fs}} 2431$
- $4213 \equiv_{\text{fs}} 4123$
- $3214 \equiv_{\text{fs}} 3124$.

Corollary 3.7. The following filling-shape-Wilf-equivalences hold:

- $13452 \equiv_{\text{fs}} 13542 \equiv_{\text{fs}} 14352 \equiv_{\text{fs}} 14532 \equiv_{\text{fs}} 15342 \equiv_{\text{fs}} 15432$
- $12453 \equiv_{\text{fs}} 12543$
- $24153 \equiv_{\text{fs}} 25143$.

Corollary 3.6 proves all four parts of Conjecture 3 in [3] and represents all nontrivial Wilf-equivalences of consecutive patterns of length 4, first proven in [8]. Corollary 3.7 represents all Wilf-equivalences of consecutive patterns of length 5, first proven in [13]. Thus, our theorem provides a simple bijective proof of the consecutive cases for length 4 and 5 patterns. Our theorem is not sufficient for length 6 patterns, however, which was first proven in [9].

We can extend Theorem 3.5 to permutations of the form $\sigma_1\text{-}\sigma_2\cdots\sigma_{k+1}$.

Theorem 3.8. *Suppose that*

$$\sigma = \sigma_1\text{-}\sigma_2\cdots\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j+1}\sigma_{j+2}\cdots\sigma_{k+1}$$

is a permutation where $1 \le i < j < k$. Also suppose that

$$\tau = \tau_1\text{-}\tau_2\cdots\tau_{i+1}\tau_{i+2}\cdots\tau_{j+1}\tau_{j+2}\cdots\tau_{k+1},$$

where $\tau_{i+2},\ldots,\tau_{j+1}$ is a permutation of $\sigma_{i+2},\ldots,\sigma_{j+1}$, and $\sigma_x = \tau_x$ for $x \in [1, i+1] \cup [j+2, k+1]$. Finally, suppose that the nonoverlapping criteria

$$\begin{aligned}
\sigma_2\cdots\sigma_{z+1} &\not\sim \sigma_{k-z+2}\cdots\sigma_{k+1}\\
\sigma_2\cdots\sigma_{z+1} &\not\sim \tau_{k-z+2}\cdots\tau_{k+1}\\
\tau_2\cdots\tau_{z+1} &\not\sim \sigma_{k-z+2}\cdots\sigma_{k+1}\\
\tau_2\cdots\tau_{z+1} &\not\sim \tau_{k-z+2}\cdots\tau_{k+1}
\end{aligned}$$

all hold for $z > \min(i, k-j)$, $z \neq k$. Then,

$$\sigma \equiv_{\text{fs}} \tau.$$

Proof: The argument is quite similar to that given in Theorem 3.5. Again, let T be the set of $\sigma_1\text{-}\sigma_2\cdots\sigma_{k+1}$-containing permutations and let T' be the set of $\tau_1\text{-}\tau_2\cdots\tau_{k+1}$-containing permutations. Let a map Ψ from S_n to itself in a similar way as in Theorem 3.5, but now we draw boxes of width k around every consecutive occurrence of $\sigma_2\cdots\sigma_{k+1}$ and $\tau_2\cdots\tau_{k+1}$, and then permute every boxed $\sigma_2\cdots\sigma_{k+1}$ to make it a

$\tau_2 \cdots \tau_{k+1}$ permutation and vice versa. For the same reason as in Theorem 3.5, Ψ is an involution. We show that Ψ bijects elements in T containing σ to elements in T' containing τ.

To see why, suppose that π in T contains σ. Suppose that

$$\pi_s \pi_t \pi_{t+1} \cdots \pi_{t+k-1} \sim \sigma_1 \cdots \sigma_{k+1},$$

where $1 \leq s \leq t \leq n-k+1$ and the sth and $t, (t+1), \ldots, (t+k-1)$th columns all contain the rows corresponding to $\pi_s, \pi_t, \ldots, \pi_{t+k-1}$. Then, Ψ sends $\pi_t \pi_{t+1} \cdots \pi_{t+k-1}$ to a permutation which is order isomorphic to $\tau_1 \cdots \tau_k$, and either π_s maps to itself or the value corresponding to π_s would move to a position still before position t, say column s'. However, since the column now with π_s is before the tth column, and since the tth column contains the rows corresponding to $\pi_s, \pi_t, \ldots, \pi_{t+k-1}$, so will column s'. Therefore, if π contains the vincular pattern σ, then $\Psi(\pi)$ will contain the vincular pattern τ, and for the same reason, if π contains the vincular pattern τ, then $\Psi(\pi)$ will contain the vincular pattern σ. As Ψ is an involution, we are done. □

Corollary 3.9. The following equivalences hold:

(B) 1453-2 $\equiv$ 1543-2

(G) 3125-4 $\equiv$ 3215-4.

These hold due to the complement operation and since filling-shape-Wilf-equivalence implies Wilf-equivalence.

4 A family of inductive equivalences

In this section we prove one main general result for quasi-consecutive patterns by applying an inductive and bijective argument. Baxter and Shattuck [4] prove the following theorem.

Theorem 4.1 (Theorem 6 of [4]). *Consider the quasi-consecutive vincular pattern $\sigma = \sigma_1\sigma_2 \cdots \sigma_k\text{-}\sigma_{k+1}$. Assume*

$$\sigma_1 < \cdots < \sigma_i = k > \sigma_{i+1} > \cdots > \sigma_k \quad \textit{and} \quad \sigma_{k+1} = k+1.$$

Let $\tau = \sigma_1 \cdots \sigma_{i-1}\sigma_{k+1}\sigma_{i+1} \cdots \sigma_k\text{-}\sigma_i$. Then $\sigma \equiv \tau$.

We prove a strong extension of this result by establishing a weaker but still sufficient condition as to when σ_{k+1} and σ_i can be switched, given $\sigma_i + 1 = \sigma_{k+1}$.

Definition 4.2. For a vincular pattern $\sigma = \sigma_1\sigma_2 \cdots \sigma_k\text{-}\sigma_{k+1}$, we define the set $S_n(\sigma)[v]$ where $v \in [n]^r$ for some $0 \leq r \leq n$ as the subset of $S_n(\sigma)$ such that $\pi \in S_n$ is in $S_n(\sigma)[v]$ if and only if

1. π avoids σ, and
2. $\pi_i = v_i$ for all $1 \leq i \leq r$.

We define $T_n(\sigma)[v]$ as the cardinality of $S_n(\sigma)[v]$. Note that if v is the empty string, $S_n(\sigma)[v] = S_n(\sigma)$.

Definition 4.3. For a vincular pattern $\sigma = \sigma_1\sigma_2 \cdots \sigma_k\text{-}\sigma_{k+1}$, we define the set $S_n(\sigma)[x, i, w]$ where $0 \leq x < n-k$, $1 \leq i \leq k$ and $w \in [n]^{x+k-1}$ as the subset of $S_n(\sigma)$ such that $\pi \in S_n$ is in $S_n(\sigma)[x, i, w]$ if and only if

1. π avoids σ, and
2. π first contains the consecutive pattern $\sigma' = \sigma_1\sigma_2\cdots\sigma_k$ at positions $x+1,\ldots,x+k$, and
3. $w = \pi_1\cdots\pi_{x+i-1}\pi_{x+i+1}\cdots\pi_{x+k}$.

We define $T_n(\sigma)[x,i,w]$ as the cardinality of $S_n(\sigma)[x,i,w]$.

Example 4.4. We have $S_5(12\text{-}3)[2,1,524] = \{52143\}$. To spell out the conditions for $\pi \in S_5(12\text{-}3)[2,1,524]$, we need to have $\pi_1\pi_2\pi_4 = 524$, $\pi_3\pi_4$ must be the first occurrence of the consecutive pattern 12, and π must avoid 12-3. The permutation $\pi = 52143$ is the only one.

Lemma 4.5. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k\text{-}\sigma_{k+1}$ and $\tau = \tau_1\tau_2\cdots\tau_k\text{-}\tau_{k+1}$ be vincular patterns. Suppose that some fixed n, i, and for all $0 \le x < n-k$, and $w \in [n]^{x+k-1}$, $T_n(\sigma)[x,i,w] = T_n(\tau)[x,i,w]$. Then, for any $0 \le r < i, v \in [n]^r$, we have that $T_n(\sigma)[v] = T_n(\tau)[v]$.*

Proof: Let $\sigma' = \sigma_1\cdots\sigma_k$ and $\tau' = \tau_1\cdots\tau_k$. We can partition $S_n(\sigma)[v]$ and $S_n(\tau)[v]$ into appropriate sets. Note that any permutation π is in $S_n(\sigma)[v]$ if and only if it is in $S_n(\sigma)[x,i,w]$ for some $x < n-k$ and some $w \in [n]^{x+k-1}$, where $w_1\cdots w_{i-1} = v_1\cdots v_{i-1}$, or $\pi \in S_n(\sigma')$, or π contains σ', but only at positions $n-k+1,...,n$, i.e. at the very end of the permutation. This is clearly a partitioning, since no permutation can be in multiple sets $S_n(\sigma)[x,i,w]$, and if π avoids σ' or only contains σ' at positions $n-k+1,\ldots,n$, it is not in any $S_n(\sigma)[x,i,w]$. We can do the same partitioning for all $\pi \in S_n(\tau)[v]$ also. However, we know that $S_n(\sigma)[x,i,w] = S_n(\tau)[x,i,w]$ for all $0 \le x < n-k, w \in [n]^{x+k-1}$. Also, since $\sigma' \sim \tau'$, the number of permutations avoiding σ' equals the number of permutations avoiding τ' and the number of permutations only containing σ' at positions $n-k+1,\ldots,n$ equals the number of permutations only containing τ' at positions $n-k+1,\ldots,n$. The result follows by summing over the partitions. □

Lemma 4.6. *Let $k \ge 3$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_k\text{-}\sigma_{k+1}$ be a vincular pattern such that $\sigma_{k+1} = \sigma_i + 1$ for some $2 \le i \le k-1$. Suppose that $\tau = \sigma_1\cdots\sigma_{i-1}\sigma_{k+1}\sigma_{i+1}\ldots\sigma_k\text{-}\sigma_i$. Also, suppose that the nonoverlapping criteria*

$$\sigma_1\ldots\sigma_z \not\sim \sigma_{k-z+1}\ldots\sigma_k$$

hold for all $z \ne k, z \ge \min(i, k-i+1)$. Then

$$T_n(\sigma)[x,i,w] = T_n(\tau)[x,i,w].$$

Proof: We start by defining a subset $Y \in [k]$ consisting of the numbers $1 \le Y_1 < \cdots < Y_{\sigma_i - 1} \le k$ such that $\sigma_i > \sigma_{Y_j}$ for any $1 \le j \le \sigma_i - 1$ (note if $\sigma_i = 1$ then $Y = \emptyset$). Define Z similarly as $[k] - Y - \sigma_i$, i.e. the numbers $1 \le Z_1 < \cdots < Z_{k-\sigma_i} \le k$ such that $\sigma_i < \sigma_{Z_j}$ for any $1 \le j \le k - \sigma_i$. Note that $\tau_i > \tau_{Y_j}$ for all $1 \le j \le \sigma_i - 1$ and $\tau_i < \tau_{Z_j}$ for all $1 \le j \le k - \sigma_i$.

We prove this by strong induction on the length n of the permutation, starting with base case $1 \le n \le k+1$. If $1 \le n \le k$ then the result is trivial. If $n = k+1$ then it is obvious that $x = 0$ since $0 \le x < n-k = 1$. Also, assume that $w_1\cdots w_{k-1} \sim \sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_k$, or else it is clear that $T_n(\sigma)[0,i,w] = T_n(\tau)[0,i,w] = 0$. Now, any $\pi \in S_n(\sigma)[x,i,w] \cup S_n(\tau)[x,i,w]$ must have $\pi_1,\ldots,\pi_{i-1},\pi_{i+1},\ldots,\pi_k$ fixed due to w, leaving only two remaining elements: $a, b \in [k+1]\backslash\{\pi_1,\ldots,\pi_{i-1},\pi_{i+1},\ldots,\pi_k\}$. Therefore, there are at most two elements in $S_n(\sigma)[x,i,w] \cup S_n(\tau)[x,i,w]$ since we can choose either

$\pi_i = a, \pi_{k+1} = b$ or $\pi_i = b, \pi_{k+1} = a$. If both a and b are larger than all elements of the form π_{Y_j} but smaller than all elements of the form π_{Z_j}, then exactly one assignment of a, b to π_i, π_{k+1} will avoid σ and the other choice will avoid τ, but both will contain the consecutive pattern $\sigma' \sim \tau'$ at positions $1, \dots, k$. In this case, $T_n(\sigma)[x, i, w] = T_n(\tau)[x, i, w] = 1$. If exactly one of a, b (assume WLOG a) is larger than all elements π_{Y_j} but smaller than all elements π_{Z_j} then any element in $S_n(\sigma)[x, i, w]$ or in $S_n(\tau)[x, i, w]$ must have $\pi_i = a$ so that $\pi_1 \cdots \pi_k \sim \sigma_1 \cdots \sigma_k \sim \tau_1 \cdots \tau_k$. But then this permutation will clearly avoid the vincular patterns σ and τ, so again $T_n(\sigma)[x, i, w] = T_n(\tau)[x, i, w] = 1$. Finally, if neither a nor b is larger than all elements π_{Y_j} but smaller than all elements π_{Z_j}, then there is no permutation such that $\pi_1 \cdots \pi_k \sim \sigma_1 \cdots \sigma_k \sim \tau_1 \cdots \tau_k$, so $T_n(\sigma)[x, i, w] = T_n(\tau)[x, i, w] = 0$. This proves the base case.

Now, consider $n > k+1$. Assume that $w_{x+1} \cdots w_{x+i-1} w_{x+i+1} \cdots w_{x+n} \sim \sigma_1 \cdots \sigma_{i-1} \sigma_{i+1} \cdots \sigma_n$, or else $T_n(\sigma)[x, i, w] = T_n(\tau)[x, i, w] = 0$. If $\pi \in S_n(\sigma)[x, i, w] \cup S_n(\tau)[x, i, w]$, then π_{x+i} must be larger than the elements in positions $x + Y_j$ and smaller than the elements in positions $x + Z_j$. Consider the elements among $[n] - \{w_1, \dots, w_{x+k-1}\}$ (recall that $w_1 \cdots w_{x+k-1} = \pi_1 \cdots \pi_{x+i-1} \pi_{x+i+1} \cdots \pi_{x+k}$ for any $\pi \in S_n(\sigma)[x, i, w] \cup S_n(\tau)[x, i, w]$) that are larger than all elements in positions $x + Y_j$ and smaller than all elements in positions $x + Z_j$. Then, a permutation π avoids σ if and only if the largest possible element is chosen (so that a σ permutation doesn't appear where σ' starts at position $x + 1$) and the reduction of $\pi_{x+2} \cdots \pi_n$ avoids σ. However, because $\sigma_1 \cdots \sigma_z \sim \sigma_{k-z+1} \cdots \sigma_k$ only if $i = k$ or if $i < k - z + 1$, (equivalent to $z < k - i + 1$), a later σ' permutation cannot start until at least position $x + i + 1$. Also, because $\sigma_1 \cdots \sigma_z \sim \sigma_{k-z+1} \cdots \sigma_k$ only if $i = k$ or if $z < i$, a later σ' cannot start until at least position $x + k - i + 2$, which means it has at most $i - 1$ elements overlapping with positions 1 to $x + k$. Thus, a permutation π such that $w_1 \cdots w_{x+k-1} = \pi_1 \cdots \pi_{x+i-1} \pi_{x+i+1} \cdots \pi_{x+k}$ avoids σ if and only if the largest possible element less than all π_{x+Z_j} and greater than all π_{x+Y_j} is chosen for π_{x+i} and the reduction of $\pi_{\max(x+k-i+2, x+i+1)}, \dots, \pi_n$ avoids σ. Likewise, a π such that $w_1 \cdots w_{x+k-1} = \pi_1 \cdots \pi_{x+i-1} \pi_{x+i+1} \cdots \pi_{x+k}$ avoids τ if and only if the smallest possible element less than all π_{x+Z_j} and greater than all π_{x+Y_j} is chosen for π_{x+i} and the reduction of $\pi_{\max(x+k-i+2, x+i+1)}, \dots, \pi_n$ avoids τ for the same reason.

Let q_s and q_l be the smallest and largest possible values of π_{x+i}, respectively. Note that $\{\pi_1, \dots, \pi_{x+i-1}, \pi_{x+i+1}, \dots, \pi_{x+k}\}$ have been chosen and that $q_s > S_{x+j}$ if and only if $q_l > S_{x+j}$ for $1 \leq j \leq k, j \neq i$. Thus, if we reduce $\pi_{\max(x+k-i+2, x+i+1)}, \dots, \pi_n$, the resulting reduction of $\pi_{\max(x+k-i+2, x+i+1)}, \dots, \pi_{x+k}$ is always fixed, which is at most $i - 1$ elements. It also does not depend on whether q_s or q_l is chosen. By our induction hypothesis we have that

$$T_{n-\max(x+k-i+2, x+i+1)+1}(\sigma)[x', i, w'] = T_{n-\max(x+k-i+2, x+i+1)+1}(\tau)[x', i, w']$$

for any feasible x', w'. Therefore, by Lemma 4.5, we are done. □

Theorem 4.7. *Retaining the setting of Lemma 4.6, we have $\sigma \equiv \tau$.*

Proof: The theorem follows from Lemmas 4.5 and 4.6 after setting v to be the empty string in Lemma 4.5 . □

We now present the new corollaries that can be directly obtained from applying Theorem 4.7 to the case where $k = 4$.

Corollary 4.8. The following equivalences hold:

(A) 1254-3 $\equiv$ 1354-2

(C) 2135-4 ≡ 2145-3

(Fa) 1243-5 ≡ 1253-4

(Fb) 3245-1 ≡ 3145-2

(La) 1342-5 ≡ 1352-4 ≡ 1452-3

(Lb) 1432-5 ≡ 1532-4 ≡ 1542-3

(Lc) 2431-5 ≡ 2531-4 ≡ 2541-3

(Ld) 2341-5 ≡ 2351-4 ≡ 2451-3.

It should be noted that (Fa) and the first congruence in (La), (Lb), (Lc), and (Ld) are in fact corollaries of Theorem 4.1.

5 Proof of equivalence M, 2153-4 ≡ 3154-2

In the section we prove that 2153-4 ≡ 3154-2.

5.1 Defining the recursion

Similar to Definition 4.2, for a vincular pattern σ, define $T_n(\sigma)$ to equal $|S_n(\sigma)|$. Define $T_n(\sigma)[k]$ denote the number of permutations $\pi \in S_n$ that avoid σ and such that $\pi_1 = k$, and similarly, define $T_n(\sigma)[k, \ell]$ denote the number of permutations avoiding σ such that $\pi_1 = k, \pi_2 = \ell$.

For the remainder of this section, we will fix

$$\sigma = 2153\text{-}4$$
$$\tau = 3154\text{-}2.$$

For convenience, Appendix B lists several values of $T_n(\sigma)[k, \ell]$ and $T_n(\tau)[k, \ell]$.

We prove the following recursion.

Lemma 5.1. *For any* $1 \le k, \ell \le n$ *we have*

$$T_n(\sigma)[k,\ell] = \begin{cases} T_{n-1}(\sigma)[\ell-1] & k < \ell \\ 0 & k = \ell \\ T_{n-1}(\sigma)[\ell] - \sum\limits_{\substack{i \ge j+2 \\ j \ge k-1}} T_{n-2}(\sigma)[i,j] & k > \ell. \end{cases}$$

Proof: Obviously $T_n(\sigma)[k, \ell] = 0$ if $k = \ell$, since permutations cannot contain repeated elements.

We show that a permutation avoids 2153-4 if and only if both the last $n-1$ elements avoid 2153-4 and, if $\pi_1\pi_2\pi_3\pi_4 \sim 2143$, then π_3 and π_4 are consecutive. To see why, note that avoiding 2153-4 clearly means that the last $n-1$ elements avoid 2153-4. Also, if a permutation π avoids 2153-4 but begins with four elements order isomorphic to 2143 and $\pi_3 > \pi_4 + 1$, then $\pi_4 + 1$ appears later, causing an instance of 2153-4. Also, if a 2153-4 occurs, it either occurs in the last $n-1$ elements or the 2153 part occurs at the beginning, which means that $\pi_1\pi_2\pi_3\pi_4 \sim 2143$. But then $\pi_3 = \pi_4 + 1$ means that 2153-4 cannot occur with the 21534 part occurring at the beginning. This proves our claim in the first sentence.

Now, this means that a permutation that begins with k, ℓ avoids 2153-4 only if the reduction of the last $n-1$ elements avoids 2153-4. Since there is clearly a unique map between reductions of the last $n-1$ elements and the permutations, there are $T_{n-1}(\sigma)[l-1]$ such permutations if $k < \ell$ and $T_{n-1}(\sigma)[\ell]$ such permutations otherwise, as it depends on what ℓ reduces to.

However, we must consider the possibility that the last $n-1$ elements avoid 2153-4 but $\pi_1\pi_2\pi_3\pi_4 \sim 2143$ when π_3 and π_4 are not consecutive. This never happens if $k < \ell$, so we have proven our recursion for this case. Now, assume $k > \ell$. Note that if $\pi_1\pi_2\pi_3\pi_4 \sim 2143$, then $\pi_2 < \pi_3$ implies that the last $n-1$ elements avoiding 2153-4 is equivalent to the last $n-2$ elements avoiding 2153-4. Now, we sum up the ways this can occur given a fixed π_3 and π_4. Since $\pi_3, \pi_4 > k \geq 2$, we can say $\pi_3 = i+2, \pi_4 = j+2$. Then, π_3, π_4 reduce to i, j. We now just need to count the number of times when $i+2 > j+2 > k$. Observe that $i+2 \geq (j+2)+2$ as they cannot be consecutive, and the reduction of the last $n-2$ elements avoids 2153-4. Based on definition, this clearly equals $T_{n-2}(\sigma)[i,j]$. Summing over all i, j, where $i \geq j+2$ and $j+2 > k$, or $j \geq k-1$, we get our desired result and we are done with our recursion. □

We now prove a similar recursion for permutations avoiding 3154-2.

Lemma 5.2. *For* $1 \leq k, \ell \leq n$ *we have*

$$T_n(\tau)[k,\ell] = \begin{cases} T_{n-1}(\tau)[\ell-1] & k < \ell \\ 0 & k = \ell \\ T_{n-1}(\tau)[\ell] & k = \ell+1 \\ T_{n-1}(\tau)[\ell] - \displaystyle\sum_{\substack{i \geq j+1 \\ j \geq k-1}} T_{n-2}(\tau)[i,j] & k > \ell+1. \end{cases}$$

Proof: As before $T_n(\tau)[k,\ell] = 0$ if $k = \ell$, since permutations cannot contain repeated elements.

A permutation avoids 3154-2 if and only if both the reduction of the last $n-1$ elements avoid 3154-2 and, if $\pi_1\pi_2\pi_3\pi_4 \sim 2143$, then $k = \ell + 1$. The proof is analogous to the first paragraph in the proof of Lemma 5.1.

Now, this means that a permutation beginning with k, ℓ avoids 3154-2 only if the reduction of the last $n-1$ elements avoids 3154-2. Since there is a natural bijection between reductions of the last $n-1$ elements and the permutations, there are $T_{n-1}(\tau)[\ell-1]$ such permutations if $k < \ell$ and $T_{n-1}(\tau)[\ell]$ such permutations otherwise, as it depends on what ℓ reduces to.

However, we must consider the possibility that the last $n-1$ elements avoid 3154-2 but $\pi_1\pi_2\pi_3\pi_4 \sim 2143$ when π_1 and π_2 are not consecutive. This never happens if $k < \ell$ or if $k = \ell+1$, so we have proven our recursion for this case. Now, assume $k > \ell+1$. Note that if $\pi_1\pi_2\pi_3\pi_4 \sim 2143$, then $\pi_2 < \pi_3$ implies that the last $n-1$ elements avoiding 3154-2 is equivalent to the last $n-2$ elements avoiding 3154-2. Now, we sum up the ways this can occur given a fixed π_3 and π_4. Since $\pi_3, \pi_4 > k \geq 2$, we again say $\pi_3 = i+2, \pi_4 = j+2$. Then, π_3, π_4 reduce to i, j. We now just need to count the number of times when $i+2 > j+2 > k$, and the reduction of the last $n-2$ elements avoids 3154-2. Based on definition, this clearly equals $T_{n-2}(\sigma)[i,j]$. Summing over all i, j, where $i \geq j+2$ and $j+2 > k$, or $j \geq k-1$, we get our desired result and we are done with our recursion. □

5.2 Proof of equivalence

Now that we have proved both Lemmas 5.1 and 5.2, we also make the following remark.

Proposition 5.3. *For any n, we have*

(a) $T_n(\sigma)[1] = T_n(\sigma)[n-1] = T_n(\sigma)[n] = T_{n-1}(\sigma)$.

(b) $T_n(\tau)[1] = T_n(\tau)[2] = T_n(\tau)[n] = T_{n-1}(\tau)$.

(c) $T_n(\sigma)[n,\ell] = T_{n-1}(\sigma)[\ell]$.

Proof: In all cases, the first letter cannot be the first letter of the pattern in question. □

We are now ready to prove the main result.

Theorem 5.4. *For $n \ge 5$ and $k \neq \ell$ the following statements hold.*

(a) For $k < \ell < n$, we have $T_n(\sigma)[k,\ell] = T_{n-1}(\tau)[\ell]$. For $k < \ell = n$ we have we have $T_n(\sigma)[k,\ell] = T_{n-1}(\tau)[1]$.

(b) For $n = k$ and $\ell \neq n-1$, we have $T_n(\sigma)[n,\ell] = T_{n-1}(\tau)[\ell+1]$. Also, $T_n(\sigma)[n,n-1] = T_{n-1}(\tau)[n-1]$.

(c) For $n > k > \ell+1$, we have $T_n(\sigma)[k,\ell] = T_n(\tau)[k+1,\ell+1]$.

(d) For any $1 \le k < n$, we have $T_n(\sigma)[k] = T_n(\tau)[k+1]$ and $T_n(\sigma)[n] = T_n(\tau)[1]$.

(e) $T_n(\sigma) = T_n(\tau)$.

Thus 2153-4 $\equiv$ 3154-2.

Proof: We prove all parts simultaneously by induction on n. The base cases $n = 5$ and $n = 6$ can be checked manually; see Appendix B.

For part (a), when $\ell \neq n$ we have $T_n(\sigma)[k,\ell] = T_{n-1}(\sigma)[\ell-1] = T_{n-1}(\tau)[\ell]$, the first equality from the recursion and the second from (d). If $\ell = n$ we instead get $T_{n-1}(\sigma)[n-1] = T_{n-1}(\tau)[1]$ again from (d).

When $k = n$ the sum in the recursion is empty and $T_n(\sigma)[n,\ell] = T_{n-1}(\sigma)[\ell]$, so part (b) follows in the same way as part (a) if $\ell \neq n-1$. As for $k = n, \ell = n-1$, note that both $T_n(\sigma)[n,n-1]$ and $T_{n-1}(\tau)[n-1]$ are equal to $T_{n-2}(\sigma) = T_{n-2}(\tau)$ by Proposition 5.3.

For part (c), note that

$$\begin{aligned} T_n(\sigma)[k,\ell] &= T_{n-1}(\sigma)[\ell] - \sum_{\substack{i \ge j+2 \\ j \ge k-1}} T_{n-2}(\sigma)[i,j] \\ &= T_{n-1}(\sigma)[\ell] - \sum_{\substack{i \ge j+2 \\ i \neq n-2 \\ j \ge k-1}} T_{n-2}(\sigma)[i,j] - \sum_{n-4 \ge j \ge k-1} T_{n-2}(\sigma)[n-2,j]. \end{aligned}$$

By the inductive hypothesis, we now obtain

$$\begin{aligned} T_n(\sigma)[k,\ell] &= T_{n-1}(\tau)[\ell+1] - \sum_{\substack{i\geq j+2\\ i\neq n-2\\ j\geq k-1}} T_{n-2}(\tau)[i+1,j+1] - \sum_{n-4\geq j\geq k-1} T_{n-3}(\sigma)[j] \\ &= T_{n-1}(\tau)[\ell+1] - \sum_{\substack{i\geq j+2\\ i\neq n-2\\ j\geq k-1}} T_{n-2}(\tau)[i+1,j+1] - \sum_{n-4\geq j\geq k-1} T_{n-3}(\tau)[j+1] \\ &= T_{n-1}(\tau)[\ell+1] - \sum_{\substack{i\geq j+2\\ i\neq n-2\\ j\geq k-1}} T_{n-2}(\tau)[i+1,j+1] - \sum_{n-3\geq j\geq k} T_{n-3}(\tau)[j] \\ &= T_{n-1}(\tau)[\ell+1] - \sum_{\substack{i\geq j+2\\ j\geq k}} T_{n-2}(\tau)[i,j] - \sum_{n-3\geq j\geq k} T_{n-2}(\tau)[j+1,j] \end{aligned}$$

Merging the sums,

$$\begin{aligned} T_n(\sigma)[k,\ell] &= T_{n-1}(\tau)[\ell+1] - \sum_{\substack{i\geq j+1\\ j\geq k}} T_{n-2}(\tau)[i,j] \\ &= T_{n-1}(\tau)[\ell+1] - \sum_{i>j\geq(k+1)-1} T_{n-2}(\tau)[i,j] \\ &= T_n(\tau)[k+1,\ell+1]. \end{aligned}$$

Finally we prove part (d). From Proposition 5.3, we have the equalities.

$$T_n(\sigma)[1] = T_n(\tau)[2], \qquad T_n(\sigma)[n-1] = T_n(\tau)[n], \qquad T_n(\sigma)[n] = T_n(\tau)[1].$$

This eliminates the cases $k = 1, n-1, n$.

So now assume $2 \leq k \leq n-2$; we wish to show that $T_n(\sigma)[k] = T_n(\tau)[k+1]$. Notice we have the equalities

$$\begin{aligned} T_n(\sigma)[k,1] &= T_n(\tau)[k+1,2] \\ T_n(\sigma)[k,2] &= T_n(\tau)[k+1,3] \\ &\vdots \\ T_n(\sigma)[k,k-2] &= T_n(\tau)[k+1,k-1] \\ T_n(\sigma)[k,k+1] &= T_{n-1}(\tau)[k+1] = T_n(\tau)[k+1,k+2] \\ T_n(\sigma)[k,k+2] &= T_{n-1}(\tau)[k+2] = T_n(\tau)[k+1,k+3] \\ T_n(\sigma)[k,k+3] &= T_{n-1}(\tau)[k+3] = T_n(\tau)[k+1,k+4] \\ &\vdots \\ T_n(\sigma)[k,n-1] &= T_{n-1}(\tau)[n-1] = T_n(\tau)[k+1,n]. \end{aligned}$$

By matching together all these terms, we see that it only remains to show

$$T_n(\sigma)[k,k-1]+T_n(\sigma)[k,n]=T_n(\tau)[k+1,1]+T_n(\tau)[k+1,k].$$

Notice that by Lemma 5.1 and our induction hypothesis,

$$\begin{aligned}T_n(\sigma)[k,k-1] &= T_{n-1}(\sigma)[k-1]-\sum_{\substack{i\ge j+2\\ j\ge k-1}} T_{n-2}(\sigma)[i,j]\\ &= T_{n-1}(\tau)[k]-\sum_{\substack{n-3\ge i\ge j+2\\ j\ge k-1}} T_{n-2}(\sigma)[i,j]-\sum_{n-4\ge j\ge k-1} T_{n-2}(\sigma)[n-2,j]\\ &= T_{n-1}(\tau)[k]-\sum_{\substack{i\ge j+2\\ j\ge k}} T_{n-2}(\tau)[i,j]-\sum_{n-4\ge j\ge k-1} T_{n-2}(\sigma)[n-2,j].\end{aligned}$$

However, for any $j\le n-4$, $T_{n-2}(\sigma)[n-2,j]=T_{n-3}(\sigma)[j]$ by Proposition 5.3, and by our induction hypothesis, this equals $T_{n-3}(\tau)[j+1]$, which in turn equals $T_{n-2}(\tau)[j+2,j+1]$ by our recursion. Therefore,

$$\begin{aligned}T_n(\sigma)[k,k-1] &= T_{n-1}(\tau)[k]-\sum_{\substack{i\ge j+2\\ j\ge k}} T_{n-2}(\tau)[i,j]-\sum_{n-4\ge j\ge k-1} T_{n-2}(\tau)[j+2,j+1]\\ &= T_{n-1}(\tau)[k]-\sum_{i>j\ge k} T_{n-2}(\tau)[i,j].\end{aligned}$$

In light of $T_{n-1}(\tau)[k+1,k]=T_{n-1}(\tau)[k]$, it suffices then to show that

$$T_n(\sigma)[k,n]=T_n(\tau)[k+1,1]+\sum_{i>j\ge k} T_{n-2}(\tau)[i,j].$$

Finally, we note $T_n(\sigma)[k,n]=T_{n-1}(\tau)[1]$ by (a); substituting this above completes the proof of (d).

Part (e) is immediate by part (d); this completes the induction, and the Wilf-equivalence follows. □

A Enumeration

The following tables list the number of avoiding permutations in each equivalence class for $5 \leq n \leq 11$.

	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
A	119	704	4838	37864	332456	3236090	34585138
B	119	704	4838	37864	332476	3236740	34599992
C	119	704	4838	37865	332477	3236426	34590177
D	119	704	4838	37866	332516	3237362	34609120
E	119	704	4838	37867	332537	3237698	34614147
F	119	704	4838	37868	332546	3237620	34609332
G	119	704	4838	37868	332558	3238028	34618998
H	119	704	4838	37870	332606	3238886	34633106
I	119	704	4838	37870	332606	3238891	34633233
J	119	704	4838	37870	332620	3239356	34644052
K	119	704	4838	37870	332622	3239412	34645000
L	119	704	4838	37874	332696	3240416	34657116
M	119	704	4838	37875	332731	3241219	34672985
N	119	704	4838	37879	332845	3243505	34713895
O	119	704	4839	37886	332821	3241738	34671733
P	119	704	4839	37887	332842	3242069	34676639
Q	119	704	4839	37888	332873	3242738	34689337
R	119	704	4839	37890	332911	3243252	34695475
S	119	704	4839	37895	333036	3245568	34734865
T	119	704	4839	37897	333096	3246798	34757387
U	119	704	4840	37908	333175	3247036	34750496
V	119	704	4840	37909	333198	3247430	34756773
W	119	704	4840	37912	333287	3249227	34789373
X	119	704	4840	37917	333398	3251054	34817364
Y	119	704	4840	37918	333474	3253240	34865094
Z	119	705	4857	38142	336291	3289057	35337067

B Table of values of $S_n(2153\text{-}4)[k,\ell]$ and $S_n(3154\text{-}2)[k,\ell]$

B.1 Recursion for $S_n(\sigma)[k,\ell]$, where $\sigma = 2153\text{-}4$.

$S_5(\sigma)[k,\ell]$	1	2	3	4	5	Σ
$k=1$	0	6	6	6	6	24
$k=2$	5	0	6	6	6	23
$k=3$	6	6	0	6	6	24
$k=4$	6	6	6	0	6	24
$k=5$	6	6	6	6	0	24
Σ	23	24	24	24	24	**119**

$S_6(\sigma)[k,\ell]$	1	2	3	4	5	6	Σ
$k=1$	0	24	23	24	24	24	119
$k=2$	18	0	23	24	24	24	113
$k=3$	22	21	0	24	24	24	115
$k=4$	24	23	24	0	24	24	119
$k=5$	24	23	24	24	0	24	119
$k=6$	24	23	24	24	24	0	119
Σ	112	114	118	120	120	120	**704**

$S_7(\sigma)[k,\ell]$	1	2	3	4	5	6	7	Σ
$k=1$	0	119	113	115	119	119	119	704
$k=2$	83	0	113	115	119	119	119	668
$k=3$	101	95	0	115	119	119	119	668
$k=4$	113	107	109	0	119	119	119	686
$k=5$	119	113	115	119	0	119	119	704
$k=6$	119	113	115	119	119	0	119	704
$k=7$	119	113	115	119	119	119	0	704
Σ	654	660	680	702	714	714	714	**4838**

$S_8(\sigma)[k,\ell]$	1	2	3	4	5	6	7	8	Σ
$k=1$	0	704	668	668	686	704	704	704	4838
$k=2$	469	0	668	668	686	704	704	704	4603
$k=3$	563	527	0	668	686	704	704	704	4556
$k=4$	632	596	596	0	686	704	704	704	4622
$k=5$	680	644	644	662	0	704	704	704	4742
$k=6$	704	668	668	686	704	0	704	704	4838
$k=7$	704	668	668	686	704	704	0	704	4838
$k=8$	704	668	668	686	704	704	704	0	4838
Σ	4456	4475	4580	4724	4856	4928	4928	4928	**37875**

$S_9(\sigma)[k,\ell]$	1	2	3	4	5	6	7	8	9	Σ
$k=1$	0	4838	4603	4556	4622	4742	4838	4838	4838	37875
$k=2$	3119	0	4603	4556	4622	4742	4838	4838	4838	36156
$k=3$	3690	3455	0	4556	4622	4742	4838	4838	4838	35579
$k=4$	4136	3901	3854	0	4622	4742	4838	4838	4838	35769
$k=5$	4481	4246	4199	4265	0	4742	4838	4838	4838	36447
$k=6$	4719	4484	4437	4503	4623	0	4838	4838	4838	37280
$k=7$	4838	4603	4556	4622	4742	4838	0	4838	4838	37875
$k=8$	4838	4603	4556	4622	4742	4838	4838	0	4838	37875
$k=9$	4838	4603	4556	4622	4742	4838	4838	4838	0	37875
Σ	34659	34733	35364	36302	37337	38224	38704	38704	38704	**332731**

B.2 *Recursion for* $S_n(\tau)[k,\ell]$, *where* $\tau = 3154\text{-}2$.

$S_5(\tau)[k,\ell]$	1	2	3	4	5	Σ
$k=1$	0	6	6	6	6	24
$k=2$	6	0	6	6	6	24
$k=3$	5	6	0	6	6	23
$k=4$	6	6	6	0	6	24
$k=5$	6	6	6	6	0	24
Σ	23	24	24	24	24	**119**

$S_6(\tau)[k,\ell]$	1	2	3	4	5	6	Σ
$k=1$	0	24	24	23	24	24	119
$k=2$	24	0	24	23	24	24	119
$k=3$	18	24	0	23	24	24	113
$k=4$	22	22	23	0	24	24	115
$k=5$	24	24	23	24	0	24	119
$k=6$	24	24	23	24	24	0	119
Σ	112	118	117	117	120	120	**704**

$S_7(\tau)[k,\ell]$	1	2	3	4	5	6	7	Σ
$k=1$	0	119	119	113	115	119	119	704
$k=2$	119	0	119	113	115	119	119	704
$k=3$	83	119	0	113	115	119	119	668
$k=4$	101	101	113	0	115	119	119	668
$k=5$	113	113	107	115	0	119	119	686
$k=6$	119	119	113	115	119	0	119	704
$k=7$	119	119	113	115	119	119	0	704
Σ	654	690	684	684	698	714	714	**4838**

$S_8(\tau)[k,\ell]$	1	2	3	4	5	6	7	8	Σ
$k=1$	0	704	704	668	668	686	704	704	4838
$k=2$	704	0	704	668	668	686	704	704	4838
$k=3$	469	704	0	668	668	686	704	704	4603
$k=4$	563	563	668	0	668	686	704	704	4556
$k=5$	632	632	596	668	0	686	704	704	4622
$k=6$	680	680	644	644	686	0	704	704	4742
$k=7$	704	704	668	668	686	704	0	704	4838
$k=8$	704	704	668	668	686	704	704	0	4838
Σ	4456	4691	4652	4652	4730	4838	4928	4928	**37875**

$S_9(\tau)[k,\ell]$	1	2	3	4	5	6	7	8	9	Σ
$k=1$	0	4838	4838	4603	4556	4622	4742	4838	4838	37875
$k=2$	4838	0	4838	4603	4556	4622	4742	4838	4838	37875
$k=3$	3119	4838	0	4603	4556	4622	4742	4838	4838	36156
$k=4$	3690	3690	4603	0	4556	4622	4742	4838	4838	35579
$k=5$	4136	4136	3901	4556	0	4622	4742	4838	4838	35769
$k=6$	4481	4481	4246	4199	4622	0	4742	4838	4838	36447
$k=7$	4719	4719	4484	4437	4503	4742	0	4838	4838	37280
$k=8$	4838	4838	4603	4556	4622	4742	4838	0	4838	37875
$k=9$	4838	4838	4603	4556	4622	4742	4838	4838	0	37875
Σ	34659	36378	36116	36113	36593	37336	38128	38704	38704	**332731**

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