

A sufficient condition for a balanced bipartite digraph to be hamiltonian

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We describe a new type of sufficient condition for a balanced bipartite digraph to be hamiltonian. Let D be a balanced bipartite digraph and x, y be distinct vertices in D . $\{x, y\}$ dominates a vertex z if $x \rightarrow z$ and $y \rightarrow z$; in this case, we call the pair $\{x, y\}$ dominating. In this paper, we prove that a strong balanced bipartite digraph D on $2a$ vertices contains a hamiltonian cycle if, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a - 1$ and $d(y) \geq a + 1$ or $d(x) \geq a + 1$ and $d(y) \geq 2a - 1$. The lower bound in the result is sharp.

Keywords: balanced bipartite digraph, degree condition, hamiltonian cycle, dominating pair of vertices

1 Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. For the convenience of the reader, we provide all necessary terminology and notation in one section, Section 2. The cycle problems for digraphs are of the central problems in graph theory and its applications⁵. A digraph D is called hamiltonian if it contains a hamiltonian cycle, i.e., a cycle that includes every vertex of D . There are many degree or degree sum conditions for hamiltonicity^{1,2,3,4,6,9,10}. The following result of Meyniel for existence of hamiltonian cycles in digraphs is basic and famous.

Theorem 1.1.¹⁰ *Let D be a strong digraph on n vertices where $n \geq 3$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices x, y in D , then D is hamiltonian.*

In⁴, Bang-Jensen, Gutin and Li described a type of sufficient condition for a digraph to be hamiltonian. Conditions of this type combine local structure of the digraph with conditions on the degrees of non-adjacent vertices. Let x, y be distinct vertices in D . If there is an arc from x to y then we say that x dominates y and write $x \rightarrow y$. $\{x, y\}$ is dominated by a vertex z if $z \rightarrow x$ and $z \rightarrow y$. Likewise, $\{x, y\}$ dominates a vertex z if $x \rightarrow z$ and $y \rightarrow z$; in this case, we call the pair $\{x, y\}$ dominating.

Theorem 1.2.⁴ *Let D be a strong digraph. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$ and $d(y) \geq n - 1$ or $d(x) \geq n - 1$ and $d(y) \geq n$. Then D is hamiltonian.*

In⁴, Bang-Jensen, Gutin and Li raised the following conjecture.

Conjecture 1.3.⁴ *Let D be a strong digraph. Suppose that $d(x) + d(y) \geq 2n - 1$ for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices $\{x, y\}$. Then D is hamiltonian.*

Bang-Jensen, Guo and Yeo³ proved that, if we replaced the degree condition $d(x) + d(y) \geq 2n - 1$ with $d(x) + d(y) \geq \frac{5}{2}n - 4$ in Conjecture 1.3, then D is hamiltonian. They also proved additional support for Conjecture 1.3 by showing that every digraph satisfying the condition of Conjecture 1.3 has a cycle factor. From now on, Conjecture 1.3 is still open and seems quite difficult.

In², Adamus, Adamus and Yeo gave a Meyniel-type sufficient condition for hamiltonicity of a balanced bipartite digraph.

Theorem 1.4.² *Let D be a strong balanced bipartite digraph on $2a$ vertices. If $d(u) + d(v) \geq 3a$ for every pair of non-adjacent vertices u, v in D , then D is hamiltonian.*

The main purpose of this note is to give a sharp sufficient condition for hamiltonicity of balanced bipartite digraphs similar to Theorem 1.2.

Definition 1.5. Consider a balanced bipartite digraph D on $2a$ vertices with $a \geq 2$. For $k \geq 0$, we will say that D satisfies condition B_k when

$$d(x) \geq 2a - k, d(y) \geq a + k \text{ or } d(y) \geq 2a - k, d(x) \geq a + k,$$

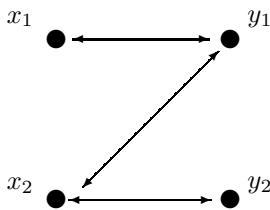
for any dominating pair of vertices $\{x, y\}$ in D .

Theorem 1.6. *Let D be a strong balanced bipartite digraph on $2a$ vertices where $a \geq 2$. If D satisfies condition B_1 , then D is hamiltonian.*

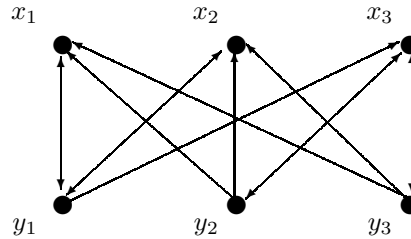
In Section 3, we shall prove Theorem 1.6 as a corollary of Theorem 1.7.

Theorem 1.7. *Let D be a strong balanced bipartite digraph on $2a$ vertices where $a \geq 2$. Suppose that, for every dominating pair of vertices $\{x, y\}$ either $d(x) \geq 2a - 2, d(y) \geq a + 1$ or $d(y) \geq 2a - 2, d(x) \geq a + 1$. Then D is either hamiltonian or isomorphic to one of the digraphs H_1, H_2 and H_3 (see Figure 1).*

It is not difficult to see that none of the digraphs H_1, H_2 and H_3 contains a hamiltonian cycle. Here, in the digraph H_2 , x_3 may dominate y_1 or not.



The digraph H_1



The digraph H_2 .

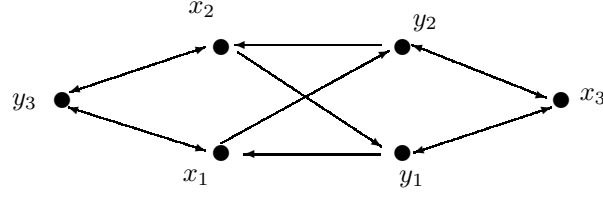


Figure 1. The digraph H_3 .

It seems quite natural to ask whether a similar result to Conjecture 1.3 is true or not in balanced bipartite digraphs. In Section 4, we show that if a strong balanced bipartite digraph on $2a$ vertices such that $d(x) + d(y) \geq 3a$ for every pair of dominating and every pair of dominated vertices $\{x, y\}$, then D must contain a cycle factor but may contain no hamiltonian cycle.

2 Terminology and notation

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to⁵ for terminology not defined here. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For disjoint subsets X and Y of $V(D)$, $X \rightarrow Y$ means that every vertex of X dominates every vertex of Y , $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \rightarrow Y$ and $X \Rightarrow Y$ hold.

For a vertex set $S \subset V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ dominated by the vertices of S ; i.e. $N^+(S) = \{u \in V(D) : vu \in A(D) \text{ for some } v \in S\}$. Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ dominating vertices of S ; i.e. $N^-(S) = \{u \in V(D) : uv \in A(D) \text{ for some } v \in S\}$. If $S = \{v\}$ is a single vertex, the cardinality of $N^+(v)$ (resp. $N^-(v)$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the *out-degree* (resp. *in-degree*) of v in D . The degree of v is $d(v) = d^+(v) + d^-(v)$.

Let $P = y_0y_1 \dots y_k$ be a path or a cycle of D . For $i \neq j$, $y_i, y_j \in V(P)$ we denote by $P[y_i, y_j]$ the *subpath* of P from y_i to y_j . If $0 < i \leq k$ then the *predecessor* of x_i on P is the vertex x_{i-1} and is also denoted by x_i^- . If $0 \leq i < k$, then the *successor* of x_i on P is the vertex x_{i+1} and is also denoted by x_i^+ . A digraph D is said to be *strongly connected* or *just strong*, if for every pair x, y of vertices of D , there is an (x, y) -path.

Let C be a cycle in D . An (x, y) -path P is a *C-bypass* if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap V(C) = \{x, y\}$. The length of the path $C[x, y]$ is the *gap* of P with respect to C . A *cycle factor* in D is a collection of vertex-disjoint cycles C_1, C_2, \dots, C_t such that $V(C_1) \cup V(C_2) \cup \dots \cup V(C_t) = V(D)$.

A digraph D is *bipartite* when $V(D)$ is a disjoint union of independent sets V_1 and V_2 . It is called *balanced* if $|V_1| = |V_2|$. A *matching* from V_1 to V_2 is an independent set of arcs with origin in V_1 and terminus in V_2 (u_1v_1 and v_1v_2 are independent arcs when $u_1 \neq v_1$ and $u_2 \neq v_2$). If D is balanced, one says that such a matching is *perfect* if it consists of precisely $|V_1|$ arcs. A digraph D is *semicomplete bipartite*, if the vertices of D can be partitioned into two partite sets such that every partite set is an independent set and for every pair x, y of vertices from distinct partite sets, xy or yx (or both) is in D .

3 The main result

Theorem 3.1.^{7,8} *A semicomplete bipartite digraph D is hamiltonian if and only if it is strong and contains a cycle factor.*

Lemma 3.2. *Let D be a strong balanced bipartite digraph with partite sets V_1 and V_2 of cardinalities a where $a \geq 2$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a - 2$ and $d(y) \geq a + 1$ or $d(x) \geq a + 1$ and $d(y) \geq 2a - 2$ and suppose that D is not isomorphic to the digraph H_2 . Then D contains a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1 . Moreover, D contains a cycle factor.*

Proof: In order to prove that D contains a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1 , by the König-Hall theorem, it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset V_1$ and $|N^+(T)| \geq |T|$ for every $T \subset V_2$.

For a proof by contradiction, suppose that a non-empty set $S \subset V_1$ is such that $|N^+(S)| < |S|$. Then $V_2 \setminus N^+(S) \neq \emptyset$. If $|S| = 1$, write $S = \{x\}$, then $|N^+(S)| < |S|$ implies that $d^+(x) = 0$. It is impossible in a strong digraph. If $|S| = a$, then, for any $w \in V_2 \setminus N^+(S)$, the vertex w is not dominated by any vertex of D contradicting the fact that D is strong. Thus, $2 \leq |S| \leq a - 1$, which implies $a \geq 3$ and $2a - 2 \geq a + 1$ as well. Then $|N^+(S)| < |S|$ implies that there exist $x_1, x_2 \in S$ and $y \in N^+(S)$ such that $\{x_1, x_2\} \rightarrow y$. Thus, $\{x_1, x_2\}$ is a dominating pair of vertices. By the hypothesis of this lemma, we assume, without loss of generality, that $d(x_1) \geq 2a - 2$ and $d(x_2) \geq a + 1$. By

$$|N^+(S)| < |S| \leq a - 1, \quad (3.1)$$

one gets $|N^+(S)| \leq a - 2$. Now we show that $|N^+(S)| = a - 2$ and further $a = 3$. To prove $|N^+(S)| = a - 2$, it suffices to show that $|N^+(S)| \geq a - 2$. Indeed, since there is no arc from S to $V_2 \setminus N^+(S)$, for any $w \in V_2 \setminus N^+(S)$ and $x \in S$, $d(w) \leq 2a - |S|$ and $d(x) \leq 2a - (a - |N^+(S)|)$. Thus,

$$2a - 2 \leq d(x_1) \leq 2a - (a - |N^+(S)|) = a + |N^+(S)|, \quad (3.2)$$

that is, $|N^+(S)| \geq a - 2$ and so $|N^+(S)| = a - 2$, write $V_2 \setminus N^+(S) = \{w_1, w_2\}$. It also follows that there must be equalities in all the estimates that led to (3.2). In other words, $d(x_1) = 2a - 2$ and furthermore $x_1 \rightarrow N^+(S) \rightarrow x_1$ and $\{w_1, w_2\} \mapsto x_1$. This, in turn, implies that $\{w_1, w_2\}$ is a dominating pair of vertices. By the hypothesis of this lemma, we assume, without loss of generality, that

$$d(w_1) \geq 2a - 2 \text{ and } d(w_2) \geq a + 1. \quad (3.3)$$

By (3.1) and $|N^+(S)| = a - 2$, we can obtain that $|S| = a - 1$. So, for any $w \in V_2 \setminus N^+(S)$ and $x \in S$,

$$d(w) \leq a + 1 \text{ and } d(x) \leq 2a - 2. \quad (3.4)$$

According to (3.3) and (3.4), we have that $2a - 2 \leq d(w_1) \leq a + 1$. So $a = 3$. To convenience, write $V_1 \setminus S = \{x_3\}$. By $d(x_2) \geq a + 1 = 4$, (3.3) and (3.4), we get that $d(x_2) = d(w_1) = d(w_2) = 4$. So $y \rightarrow \{x_1, x_2\} \rightarrow y$, $x_3 \rightarrow \{w_1, w_2\} \rightarrow x_3$ and $\{w_1, w_2\} \mapsto \{x_1, x_2\}$. Since D is strong, $y \rightarrow x_3$. Note that D is isomorphic to the digraph H_2 , contrary to our assumption.

This completes the proof of existence of a perfect matching from V_1 to V_2 . The proof for a perfect matching in the opposite direction is analogous. Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1 . Hence D contains a cycle factor. \square

Lemma 3.3. *Let D be a strong bipartite digraph with partite sets V_1 and V_2 . Let $C = x_0y_0x_1y_1 \dots x_{m-1}y_{m-1}x_0$ be a longest cycle in D , and let uv be an arc in $D \setminus V(C)$, where u and x_i belong to the same partite set for $i \in \{0, 1, \dots, m-1\}$. Then for every pair of vertices y_i, x_{i+1} from $V(C)$ at most one of the arcs y_iu and vx_{i+1} belongs to $A(D)$. Furthermore, $d_{V(C)}^-(u) + d_{V(C)}^+(v) \leq m$.*

Proof: If there exist $y_i, x_{i+1} \in V(C)$ such that $y_i \rightarrow u$ and $v \rightarrow x_{i+1}$, then C and the arc uv can be merged into a longer cycle than C by deleting the arc y_ix_{i+1} and adding the arcs y_iu and vx_{i+1} . This would contradict the fact that C is a longest cycle in D , so at most one of the arcs y_iu and vx_{i+1} belongs to $A(D)$. There is precisely m of such pairs. By accounting for the arcs y_iu and vx_{i+1} , we get the required estimate $d_{V(C)}^-(u) + d_{V(C)}^+(v) \leq m$. \square

Lemma 3.4. *Let D be a strong balanced bipartite digraph with partite sets V_1 and V_2 of cardinalities a where $a \geq 3$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a - 2$ and $d(y) \geq a + 1$ or $d(x) \geq a + 1$ and $d(y) \geq 2a - 2$. Then D contains a cycle of length at least 4.*

Proof: If D is isomorphic to the digraph H_2 , then obviously D contains a cycle of length 4. If not, then suppose, on the contrary, that D contains no cycle of length more than or equal to 4. So D is non-hamiltonian as $a \geq 3$. By Lemma 3.2, D contains a cycle factor. Let C_1, C_2, \dots, C_t be a minimal cycle factor. Then the length of every C_i is 2 and $t = a$. Write $C_i = x_iy_ix_i$, where $x_i \in V_1$ and $y_i \in V_2$ for $i = 1, 2, \dots, a$. By Lemma 3.3, $d_{V(C_j)}(x_i) + d_{V(C_j)}(y_i) \leq 2$, for every $i \in \{1, 2, \dots, a\}$ and $j \in \{1, 2, \dots, a\} \setminus \{i\}$. Thus $d(x_i) + d(y_i) \leq 2(a-1) + 4 = 2a + 2$.

If there exists a vertex $y_j \in V_2 \setminus \{y_i\}$ such that between x_i and y_j form a 2-cycle. Without loss of generality, assume that $i = 1$ and $j = 2$. Note that $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are both dominating pairs of vertices. Thus, by assumption

$$\begin{aligned} 2(3a-1) &\leq d(x_1) + d(x_2) + d(y_1) + d(y_2) \\ &= d(x_1) + d(y_1) + d(x_2) + d(y_2) \\ &\leq 2(2a+2), \end{aligned}$$

which implies that $a \leq 3$. By the hypothesis of this lemma, $a \geq 3$ and so $a = 3$. Further, there must be equalities in all the estimates, i.e. $d(x_i) = d(y_i) = 4$, for $i = 1, 2$. By Lemma 3.3, x_2 and y_1 are not adjacent, so $y_1 \rightarrow x_3 \rightarrow y_1$ and $y_3 \rightarrow x_2 \rightarrow y_3$. However, $x_1y_1x_3y_3x_2y_2x_1$ is a hamiltonian cycle, a contradiction.

Now assume that there exist no such vertices. Combining this with Lemma 3.3, for any two distinct cycles C_i and C_j , we have that $C_i \Rightarrow C_j$ or $C_j \Rightarrow C_i$. Since D is strong, $t \geq 3$. Without loss of generality, assume that $C_3 \Rightarrow C_1$ and $C_1 \Rightarrow C_2$ and further assume that $x_1 \rightarrow y_2$. Note $\{x_1, x_2\}$ is a dominating pair of vertices. By the hypothesis of this lemma, $d(x_1) \geq 2a - 2$, $d(x_2) \geq a + 1$ or $d(x_2) \geq 2a - 2$, $d(x_1) \geq a + 1$. Observe that for every $i \in \{1, 2, \dots, t\}$, $d(x_i) \leq a + 1$ and $d(y_i) \leq a + 1$. Thus, $2a - 2 \leq a + 1$, i.e. $a \leq 3$ and so $a = 3$. From this, it is not difficult to get that for any dominating pair vertices $\{u, v\}$, $d(u) = d(v) = 4$. So, $d(x_1) = d(x_2) = 4$, which implies that $y_3 \rightarrow x_1$. This, in turn, implies that $\{y_1, y_3\}$ is a dominating pair. Thus, $d(y_1) = d(y_3) = 4$ and further $y_1 \rightarrow x_2$ and $x_3 \rightarrow y_1$. Since D is strong, it must be $x_2 \rightarrow y_3$. However $y_3x_3y_1x_1y_2x_2y_3$ is a hamiltonian cycle. a contradiction. \square

The proof of Theorem 1.7.

Proof: Let V_1 and V_2 denote the partite sets of D . By assumption of this theorem, a dominating pair $\{x, y\}$ means $d(x) \geq 2a - 2$, $d(y) \geq a + 1$, or $d(y) \geq 2a - 2$, $d(x) \geq a + 1$. We will implicitly use this in the remainder of the paper. Suppose $a = 2$. Denote $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2\}$. Since D is strong, the in-degree of every vertex is at least one. If the in-degree of every vertex is one in D , then clearly D contains a hamiltonian cycle. Now assume that there exists $z \in V(D)$ such that $d^-(z) \geq 2$, it is to say that there exist dominating pair of vertices in D . Without loss of generality, assume that $\{x_1, x_2\}$ is a dominating pair and furthermore $d(x_1) \geq 2a - 2 = 2$ and $d(x_2) \geq a + 1 = 3$. This means that x_2 and every vertex of V_2 are adjacent. If x_1 and every vertex of V_2 are adjacent, then D is a semicomplete bipartite digraph. Using Theorem 3.1 and Lemma 3.2, D is hamiltonian. If x_1 and one of y_1 and y_2 , say y_2 , are not adjacent, then by $d(x_1) \geq 2$, we have that $y_1x_1y_1$ is a 2-cycle. Since D is strong, we can deduce that $x_2 \rightarrow \{y_1, y_2\} \rightarrow x_2$. Note that D is isomorphic to the digraph H_1 .

Now suppose that $a \geq 3$. In this case, $2a - 2 \geq a + 1$ and equality holds only if $a = 3$. Suppose that D is non-hamiltonian. Let $C = x_0y_0x_1y_1 \dots x_{m-1}y_{m-1}x_0$ be a longest cycle in D , where $x_i \in V_1$ and $y_i \in V_2$ for $i \in \{0, 1, \dots, m-1\}$. Lemma 3.4 implies that $m \geq 2$. From now on, all subscripts appearing in this proof are taken modulo m .

We first show that D contains a C -bypass. Assume D does not have one. Since D is strong, it must contain a cycle Z such that $|V(Z) \cap V(C)| = 1$. Without loss of generality, assume that $V(Z) \cap V(C) = \{x_0\}$. Let z be the predecessor of x_0 on Z . Since $\{z, y_{m-1}\}$ is a dominating pair of vertices, we have $d(z) \geq 2a - 2$ and $d(y_{m-1}) \geq a + 1$ or $d(z) \geq a + 1$ and $d(y_{m-1}) \geq 2a - 2$. Since D contains no C -bypass, we have $d_{V(C) \setminus \{x_0\}}(z) = 0$ and $d_{V(Z) \setminus \{x_0\}}(y_{m-1}) = 0$. Note that $|(\{z, y_{m-1}\}, x) \cup (x, \{z, y_{m-1}\})| \leq 2$ for every $x \in V_1 \setminus (V(C) \cup V(Z))$. Denote $|V_1 \cap (V(Z) \setminus \{x_0\})| = p$. Hence $d(z) + d(y_{m-1}) \leq 2(a - p - m) + 2p + 2 + 2m = 2a + 2$. This follows $3a - 1 \leq d(y_{m-1}) + d(z) \leq 2a + 2$. Using this inequalities with $a \geq 3$, we obtain that $a = 3$. In addition, the above inequalities become equalities, which implies that $d(z) = 4$. Clearly, $m = 2$, write $V_1 \cap (V(D) \setminus V(C)) = \{x\}$. By $d_{V(C) \setminus \{x_0\}}(z) = 0$ and $d(z) = 4$, we have zxx is a 2-cycle. Since D contains no C -bypass, we have $d_{V(C)}(x) = 0$, that is $d(x) = 2$. However, $\{x, x_0\} \rightarrow z$ implies that $\{x, x_0\}$ is a dominating pair of vertices. By assumption, $d(x) \geq a + 1 = 4$, a contradiction. Therefore, D contains a C -bypass.

Let $P = u_1u_2 \dots u_s$ be a C -bypass ($s \geq 3$). Suppose also that the gap of P is minimum among the gaps of all C -bypass. Denote $C' = C[u_1^+, u_s^-]$, where u_1^+ is the successor of u_1 on C and u_s^- is the predecessor of u_s on C . Since C is a longest cycle of D , we have that $|V(C')| \geq s - 2$. Because D is a bipartite digraph, when s is odd, u_1 and u_s belong to the same partite set; when s is even, u_1 and u_s belong to distinct partite sets. Denote $R = V(D) \setminus (V(C) \cup V(P))$. Noting that $\{u_{s-1}, u_s^-\}$ is a dominating pair of vertices, by assumption,

$$d(u_{s-1}) \geq 2a - 2, d(u_s^-) \geq a + 1 \text{ or } d(u_{s-1}) \geq a + 1, d(u_s^-) \geq 2a - 2. \quad (3.5)$$

The following two claims will be very useful in the remaining proof.

Claim 1. For any $x \in R$, $|(\{u_{s-1}, u_s^-\}, x) \cup (x, \{u_{s-1}, u_s^-\})| \leq 2$.

Proof: Clearly, u_{s-1} and u_s^- belong to the same partite set. It suffices to prove the case when x and u_{s-1} belong to distinct partite sets. If $u_{s-1} \rightarrow x$, then $x \rightarrow u_s^-$, for otherwise $P[u_1, u_{s-1}]xu_s^-$ is also a C -bypass, which gap is strictly less than P , a contradiction. If $x \rightarrow u_{s-1}$, then $u_s^- \rightarrow x$, for otherwise $u_s^-xu_{s-1}C[u_s, u_s^-]$ is a longer cycle than C , a contradiction. Hence the claim holds. \square

Claim 2. Every vertex on C' is not adjacent to any vertex on $P[u_2, u_{s-1}]$.

Proof: Since P has the minimum gap, the claim is obvious. \square

Now we divide the proof into two cases to consider.

Case 1. $|V(C')| \geq 2$.

To complete the proof, we will first give the following useful observation.

Observation 1. For any $x \in V(D) \setminus V(C)$, $d_{V(C)}(x) \leq m$.

Proof: Assume, without loss of generality, that $x \in V_1$. Let $d_{V(C)}^-(x) = t$ and denote $N_{V(C)}^-(x) = \{y_{i_1}, y_{i_2}, \dots, y_{i_t}\}$. According to $|V(C')| \geq 2$ and the fact that P has the minimum gap, $x \rightarrow y_{i_j+1}$ for every $j \in \{1, 2, \dots, t\}$. Hence, $d_{V(C)}(x) = d_{V(C)}^-(x) + d_{V(C)}^+(x) \leq t + (m - t) = m$. \square

We may assume, without loss of generality, that $d(u_s^-) \geq 2a - 2$ and $d(u_{s-1}) \geq a + 1$. In fact, let $x \in V(D) \setminus V(C)$ be any. By Observation 1, $d_{V(C)}(x) \leq m$. So $d(x) \leq m + 2(a - m) = 2a - m$. If $m \geq 3$, then $d(x) \leq 2a - 3$, in particular, $d(u_{s-1}) \leq 2a - 3$. Combining this with (3.5), we have that $d(u_s^-) \geq 2a - 2$ and $d(u_{s-1}) \geq a + 1$. If $m = 2$, then $|V(C')| \geq 2$ implies that $|V(C')| = s - 2 = 2$. By symmetry, we may assume, without loss of generality, that $d(u_s^-) \geq 2a - 2$ and $d(u_{s-1}) \geq a + 1$.

If $s \geq 6$, then, by Claim 2, $d(u_s^-) \leq 2a - 4$, a contradiction to $d(u_s^-) \geq 2a - 2$. So, we assume from now on that $s \leq 5$. If $s \geq 4$, then, by Claim 2, we have that $d(u_s^-) \leq 2a - 2$ and so $d(u_s^-) = 2a - 2$.

Suppose $s = 5$. In this case, u_1 and u_s belong to the same partite set. Now, without loss of generality, assume that $u_1 = x_0$ and $u_s = x_r$. Clearly, $R \cap V_1 \neq \emptyset$. By $|V(C')| \geq s - 2$, we have that $r \geq 2$. By the above argument, $d(y_{r-1}) = 2a - 2$ and y_{r-1} and every vertex of $V_1 \setminus \{u_3\}$ form a 2-cycle, in particular, for any $x \in R \cap V_1$, x and y_{r-1} form a 2-cycle. By Claim 1, u_4 and x are not adjacent. Combining this with Observation 1, we have $d(x) \leq m + 2(a - m - 1) = 2a - m - 2 \leq 2a - 4$. Note that $\{x_{r-1}, x\} \rightarrow y_{r-1}$, i.e. $\{x, x_{r-1}\}$ is a dominating pair. Using Claim 2, we can obtain that $d(x_{r-1}) \leq 2a - 4$, contradicting the fact that $\{x, x_{r-1}\}$ is a dominating pair.

Suppose $s = 4$. In this case, $|V(C')|$ is even and u_1 and u_s belong to distinct partite sets. Now assume, without loss of generality, that $u_1 = x_0$ and $u_s = y_r$. Obviously, $r \geq 1$. By the above argument, $d(x_r) = 2a - 2$ and x_r and every vertex of $V_2 \setminus \{u_2\}$ form a 2-cycle. By Claim 1, u_3 and every vertex of $R \cap V_2$ are not adjacent. This together with Observation 1 implies that $d(u_3) \leq m + 2$. Combining this with $d(u_3) \geq a + 1$, we have $a = m + 1$. Thus, $R = \emptyset$ and $d_{V(C)}(u_3) = m$.

Assume that $r = 1$. Note that $\{y_0, y_1\}$ is a dominating pair. Since $2a - 2 \geq a + 1$, by assumption, $d(y_0) \geq a + 1$. For any $i \in \{2, \dots, m - 1\}$, if $x_i \rightarrow y_0$, then $x_i y_0 x_1 C[y_{i+1}, x_0] u_2 u_3 C[y_1, x_i]$ is a hamiltonian cycle, a contradiction; if $y_0 \rightarrow x_i$, then $y_{i-1} x_1 y_0 C[x_i, x_0] u_2 u_3 C[y_1, y_{i-1}]$ is a hamiltonian cycle, a contradiction. Thus y_0 and x_i are not adjacent. Furthermore, $y_0 \rightarrow x_0$, for else $u_2 u_3 C[y_1, y_{m-1}] x_1 y_0 x_0 u_2$ is a hamiltonian cycle, a contradiction. From this we have that $d(y_0) \leq 3$. However, $a + 1 \leq d(y_0) \leq 3$ implies that $a \leq 2$, a contradiction.

Assume that $r \geq 2$, i.e. $|V(C')| \geq 4$. Denote $d_{V(C)}^-(u_3) = t$. If $t = 0$, then, by Claim 2, $d_{V(C)}(u_3) \leq m - 2$, a contradiction to $d_{V(C)}(u_3) = m$. Next assume $t \geq 1$. Since P has the minimum gap with respect to C , if $y_i \rightarrow u_3$, then $u_3 \rightarrow y_{i+1}$ and $u_3 \rightarrow y_{i+2}$. Hence $d_{V(C)}(u_3) = d_{V(C)}^-(u_3) + d_{V(C)}^+(u_3) \leq t + (m - 2t) = m - t \leq m - 1$, a contradiction.

Suppose $s = 3$. In this case, u_1 and u_s belong to the same partite set. Now assume that $u_1 = x_0$ and $u_s = x_r$. Clearly, $R \cap V_1 \neq \emptyset$. Since $|V(C')| \geq 2$, we have $r \geq 2$, which means that $m \geq 3$. Let $x \in V(D) \setminus V(C)$ be any. By Observation 1, $d_{V(C)}(x) \leq m$. So $d(x) \leq m + 2(a - m) = 2a - m \leq$

$2a - 3$. This means that any pair of vertices in $V(D) \setminus V(C)$ cannot form a dominating pair. Hence $d_{V(D) \setminus V(C)}^-(x) \leq 1$ and so $d(x) \leq m + a - m + 1 = a + 1$, which, in turn, implies that if x and some vertex of $V(D)$ form a dominating pair, then $d(x) = a + 1$ and further $d_{V(D) \setminus V(C)}^-(x) = 1$ and $d_{V(D) \setminus V(C)}^+(x) = a - m$. So $d(u_2) = a + 1$. Furthermore, $d_R^-(u_2) = 1$ and $d_R^+(u_2) = a - m$. So there exists $x' \in R \cap V_1$ such that u_2 and x' form a 2-cycle. Then $\{x', x_0\} \rightarrow u_2$ implies that $\{x', x_0\}$ is a dominating pair. So $d(x') = a + 1$. Now we show that $d_{V(C)}(u_2) + d_{V(C)}(x') \leq 2(m - 1)$. In fact, since C is a longest cycle and P has the minimum gap with respect to C , we can obtain that, for any $x_i \in N^-(u_2) \cap V(C)$, $x' \rightarrow y_i$ and $x' \rightarrow y_{i+1}$. Denote $d_{V(C)}^-(u_2) = t$. Clearly, $t \geq 1$. Then, $d_{V(C)}^-(u_2) + d_{V(C)}^+(x') \leq t + (m - 2t) = m - t \leq m - 1$. Similarly, for any $x_i \in N^+(u_2) \cap V(C)$, $y_{i-2} \rightarrow x'$ and $y_{i-1} \rightarrow x'$. Hence $d_{V(C)}^+(u_2) + d_{V(C)}^-(x') \leq m - 1$. Add these two inequalities, we obtain $d_{V(C)}(u_2) + d_{V(C)}(x') \leq 2(m - 1)$, which implies the desired inequality. However, $2(a + 1) = d(u_2) + d(x') \leq 2(m - 1) + 2(a - m + 1) = 2a$ is a contradiction.

Case 2. $|V(C')| = 1$.

In this case, $s = 3$ and u_1 and u_s belong to the same partite set. Assume that $u_1 = x_0$ and $u_s = x_1$. By symmetry and (3.5), without loss of generality, assume that $d(u_2) \geq 2a - 2$ and $d(y_0) \geq a + 1$. Clearly, $R \cap V_1 \neq \emptyset$.

Subcase 2.1. There exists $u \in R \cap V_1$ such that u_2 and u are not adjacent.

First $d(u_2) \geq 2a - 2$ and the hypothesis of the subcase imply $d(u_2) = 2a - 2$. Moreover, u_2 and every vertex of $V_1 \setminus \{u\}$ form a 2-cycle, which implies that every pair of vertices of $V_1 \setminus \{u\}$ form a dominating pair. Thus, the degree of every vertex in $V_1 \setminus \{u\}$ is greater or equal to $a + 1$.

If $a = 3$, then $2a - 2 = a + 1 = 4$. Because D is strong, there exists $y_i \in V(C) \cap V_2$ such that $u \rightarrow y_i$. Then $\{x_i, u\} \rightarrow y_i$ means that $\{x_i, u\}$ is a dominating pair and so $d(u) \geq a + 1 = 4$. Since u_2 and u are not adjacent, we obtain that $\{y_0, y_1\} \rightarrow u \rightarrow \{y_0, y_1\}$. If $y_0 \rightarrow x_0$, then $y_0 x_0 u_2 x_1 y_1 u y_0$ is a hamiltonian cycle, a contradiction. Hence $x_0 \mapsto y_0$. Similarly, we can obtain that $y_0 \mapsto x_1$, $x_1 \mapsto y_1$ and $y_1 \mapsto x_0$. Note that D is isomorphic to the digraph H_3 .

Now assume $a \geq 4$. Suppose $|R \cap V_1| = 1$. We first show that $d(u) \leq m + 1$. Indeed, if there exist $y_i, y_{i+1} \in V(C)$ such that $y_i \rightarrow u$ and $u \rightarrow y_{i+1}$, then $d(x_{i+1}) \leq 2a - 3$. For $j \in \{0, 1, \dots, m - 1\} \setminus \{i + 1\}$, $x_{i+1} \rightarrow y_j$ otherwise, $x_{i+1} C[y_j, y_i] u C[y_{i+1}, x_j] u_2 x_{i+1}$ is a hamiltonian cycle, a contradiction. In addition, $y_{i+1} \rightarrow x_{i+1}$, for else $y_{i+1} x_{i+1} u_2 C[x_{i+2}, y_i] u y_{i+1}$ is a hamiltonian cycle, a contradiction. Hence $d(x_{i+1}) \leq 2a - m \leq 2m - 3$. Therefore, there exists at most one pair of $\{y_i, y_{i+1}\}$ such that $y_i \rightarrow u \rightarrow y_{i+1}$, which implies $d_{V(C)}(u) \leq m + 1$. Because D is strong, there exists $y_i \in V(C) \cap V_2$ such that $u \rightarrow y_i$. Then $\{x_i, u\} \rightarrow y_i$ means that $\{x_i, u\}$ is a dominating pair and so $d(u) \geq a + 1$ as $2a - 2 \geq a + 1$, contrary to $d(u) \leq m + 1$.

Suppose $|R \cap V_1| \geq 2$. First we claim that $a = 2m + 1$ and the degree of every vertex of $(R \cap V_1) \setminus \{u\}$ is equal to $a + 1$. In fact, let $x \in (R \cap V_1) \setminus \{u\}$ be any. Similar to Claim 1, we can deduce that y_i and x are not adjacent, for every $y_i \in V(C) \cap V_2$. Recalling that $d(x) \geq a + 1$, we have $a + 1 \leq d(x) \leq 2(a - m)$, i.e. $a \geq 2m + 1$. Recalling that $d(y_0) \geq a + 1$, we have $a + 1 \leq d(y_0) \leq 2m + 2$, i.e. $a \leq 2m + 1$. From these, we get that $a = 2m + 1$ and equality hold everywhere. So $d(x) = a + 1$. Then $a = 2m + 1$ implies that a is odd and further $|R \cap V_1| \geq 3$. This is a contradiction to the fact that every pair of vertices of $(R \cap V_1) \setminus \{u\}$ form a dominating pair and the degree of every vertex in $(R \cap V_1) \setminus \{u\}$ is equal to $a + 1$.

Subcase 2.2. u_2 and every vertex of $R \cap V_1$ are adjacent.

Next we divide the subcase into three subcases.

Subcase 2.2.1. There exists a vertex $u \in R \cap V_1$ such that u_2 and u form a 2-cycle.

Then $\{u, x_0\} \rightarrow u_2$ implies that

$$d(x_0) \geq 2a - 2, d(u) \geq a + 1 \text{ or } d(u) \geq 2a - 2, d(x_0) \geq a + 1. \quad (3.6)$$

Assume $a = 3$. Consider the cycle $u_2x_1y_1x_0u_2$ and note that y_0 and u are not adjacent. Similar to Subcase 2.1, we can obtain that D is isomorphic to the digraph H_3 . Now assume that $a \geq 4$.

If $d(u) \geq 2a - 2$, then, by Lemma 3.3, $4a - 4 \leq d(u_2) + d(u) \leq 2m + 4(a - m) = 4a - 2m \leq 4a - 4$. Hence, equalities hold everywhere, in particular, $m = 2$ and $d_R(u_2) = 2(a - m)$. In other words, u_2 and every vertex of $R \cap V_1$ form a 2-cycle. By Claim 1, y_0 and every vertex of $R \cap V_1$ are not adjacent. Hence $d(y_0) \leq 4$. This together with $d(y_0) \geq a + 1$ implies that $a \leq 3$, a contradiction. Hence, we assume that $d(u) < 2a - 2$. In fact, we may assume that for any $x \in R \cap V_1$, if x and u_2 form a 2-cycle, then $d(x) < 2a - 2$. This also implies that u is the unique vertex in $R \cap V_1$ which forms a 2-cycle with u_2 . Using Lemma 3.3, $3a - 1 \leq d(u_2) + d(u) \leq 2m + 4(a - m) = 4a - 2m$, i.e. $2m \leq a + 1$.

If $2m = a + 1$, then equalities hold everywhere. It follows that $d_R(u_2) = 2(a - m)$. In other words, u_2 and every vertex of $R \cap V_1$ form a 2-cycle. Hence $|R \cap V_1| = 1$. However, $3a - 1 \leq d(u) + d(u_2) \leq 2m + 4$ implies that $a \leq 3$, a contradiction.

If $2m < a + 1$, then $d(y_0) \geq a + 1$ implies that y_0 must be adjacent to some vertex of $R \cap V_1$. Since y_0 and u are not adjacent, we have $|R \cap V_1| \geq 2$. Since $d(u_2) \geq 2a - 2$, we have that $|R \cap V_1| \leq 3$.

Suppose $|R \cap V_1| = 3$, i.e. $a = m + 3 \geq 5$. Write $R \cap V_1 = \{u, w_1, w_2\}$ and $R \cap V_2 = \{u_2, v_1, v_2\}$. By the above argument, we can conclude that u_2 and every vertex of $V(C) \cap V_1$ form a 2-cycle. Similar to Claim 1, we can obtain that u are not adjacent to every $y_i \in V(C)$. Thus, $d(u) \leq 6$. This together with $d(u) \geq a + 1$ implies that $a \leq 5$ and so $a = 5$ and $d(u) = a + 1 = 6$. Hence, u and every vertex of $R \cap V_2$ form a 2-cycle. In addition, according to $d(y_0) \geq a + 1 = 6$ and Claim 1, either $\{u_2, y_0\} \mapsto w_1$ or $w_1 \mapsto \{u_2, y_0\}$ and $y_0 \rightarrow \{x_0, x_1\} \rightarrow y_0$. Without loss of generality, assume that $\{u_2, y_0\} \mapsto w_1$. Since D is strong, there exists a vertex in $V_2 \setminus \{u_2, y_0\}$ dominated by w_1 . If $w_1 \rightarrow y_1$, then $w_1y_1x_0u_2x_1y_0w_1$ is a longer cycle than C , a contradiction. If w_1 dominates one of $\{v_1, v_2\}$, say v_1 , then $w_1v_1uu_2x_1y_0w_1$ is a longer cycle than C , a contradiction.

Suppose $|R \cap V_1| = 2$, i.e. $a = m + 2$. Write $R \cap V_1 = \{u, w_1\}$ and $R \cap V_2 = \{u_2, v_1\}$. Note that $a + 1 \leq d(y_0) \leq 2m + 1$, which together with $2m < a + 1$ implies that $a = 2m$ and $d(y_0) = 2m + 1$. Using this with $a = m + 2$, we have that $m = 2$. By Claim 1, either $\{u_2, y_0\} \mapsto w_1$ or $w_1 \mapsto \{u_2, y_0\}$. Without loss of generality, assume that $\{u_2, y_0\} \mapsto w_1$. In addition, $y_0 \rightarrow \{x_0, x_1\} \rightarrow y_0$. Since D is strong, there exists a vertex in $V_2 \setminus \{y_0, u_2\}$ dominated by w_1 . If $w_1 \rightarrow y_1$, then $w_1y_1x_0u_2x_1y_0w_1$ is a longer cycle than C , a contradiction. If $w_1 \rightarrow v_1$, then since D is strong, there exists a vertex in $\{x_0, x_1, u\}$ dominated by v_1 . It is not difficult to check that D would contain a longer cycle than C , a contradiction.

Subcase 2.2.2. $R \cap V_1 \mapsto u_2$.

By $d(u_2) \geq 2a - 2$, we have $|R \cap V_1| \leq 2$. Suppose $|R \cap V_1| = 2$, say $R \cap V_1 = \{x, w\}$. In this case, $d(u_2) = 2a - 2$ and u_2 and every vertex of $V(C) \cap V_1$ form a 2-cycle and $\{w, x\} \mapsto u_2$, which implies that $\{x, w\}$ is a dominating pair. Since C is a longest cycle, we have $(V(C) \cap V_2, \{x, w\}) = \emptyset$.

Hence $d(w) \leq m + 3 = a - 2 + 3 = a + 1$ and $d(x) \leq a + 1$, a contradiction to the fact that $\{x, w\}$ is a dominating pair.

Suppose $|R \cap V_1| = 1$, say $R \cap V_1 = \{x\}$. Observe that for any $y_i \in V(C) \cap V_2$, if $y_i \rightarrow x$, then $u_2 \rightarrow x_{i+1}$. Clearly, $y_0 \rightarrow x$. Since D is strong, there exists $y_i \in V(C) \cap V_2$ such that $y_i \rightarrow x$. Assume, without loss of generality, that $y_1 \rightarrow x$. The above observation implies $u_2 \rightarrow x_2$. This together with $x \mapsto u_2$ implies that $d(u_2) = 2a - 2$ and u_2 and every vertex of $V_2 \setminus \{x, x_2\}$ form a 2-cycle and $x_2 \mapsto u_2$. Continuing using the observation, we have that $y_i \rightarrow x$, for every $i \neq 1$, that is to say, $d^-(x) = 1$. Thus, $a + 1 \leq d(x) \leq 1 + (m + 1) = a + 1$. Note that, for any $x_i \in V(C) \cap V_1$, $\{x_i, x\}$ is a dominating pair and so $d(x_i) \geq 2a - 2$, in particular, $d(x_2) \geq 2a - 2$.

Suppose $a \geq 4$. Denote $Q = y_1 x u_2 C[x_3, y_1]$. It is clear that Q is a cycle of length $2m$. By $\{y_2, u_2\} \rightarrow x_3$, we have that $\{y_2, u_2\}$ is a dominating pair, which implies that $d(y_2) \geq a + 1$. If $y_2 \rightarrow x_2$, then, by Lemma 3.3, $3a - 1 \leq d(x_2) + d(y_2) = 4 + d_{V(Q)}(x_2) + d_{V(Q)}(y_2) \leq 2m + 4 = 2a + 2$ implies that $a \leq 3$, a contradiction. Hence $y_2 \rightarrow x_2$. This together with $d(x_2) \geq 2a - 2$ and $x_2 \mapsto u_2$ imply that $d(x_2) = 2a - 2$ and x_2 and every vertex of $V_2 \setminus \{u_2, y_2\}$ form a 2-cycle. For any $x_i \in (V_1 \cap V(C)) \setminus \{x_2, x_3\}$, $y_2 \rightarrow x_i$, for else $y_2 C[x_i, y_1] x u_2 C[x_3, y_{i-1}] x_2 y_2$ is a hamiltonian cycle, a contradiction. Thus, $d(y_2) \leq 2 + m = a + 1$. Earlier, we showed that $d(y_2) \geq a + 1$. So $d(y_2) = a + 1$. Moreover $d^+(y_2) = 1$ and $d^-(y_2) = a$. However, $y_1 x u_2 x_1 C[y_2, y_0] x_2 y_1$ is a hamiltonian cycle, a contradiction.

Now assume that $a = 3$. In this case, $2a - 2 = a + 1 = 4$. By the above argument, we have obtained that $\{x_0, x\} \mapsto u_2$, $x \mapsto y_0$, $x \rightarrow y_1 \rightarrow x$, $u_2 \rightarrow x_1 \rightarrow u_2$, $d(y_0) \geq 4$ and $d(x_0) \geq 4$. If $y_0 \rightarrow x_0$, then $y_0 x_0 u_2 x_1 y_1 x y_0$ is a hamiltonian cycle, a contradiction. Thus, $x_0 \mapsto y_0$. This together with $d(x_0) \geq 4$ and $d(y_0) \geq 4$ imply that $x_0 \rightarrow y_1$ and $x_1 \rightarrow y_0$. Note that D is isomorphic to the digraph H_2 .

Subcase 2.2.3. $u_2 \mapsto R \cap V_1$.

Since the proof is similar to Subcase 2.2.2, we omit them. This completes the proof of the theorem. \square

The proof of Theorem 1.6.

Proof: Using Theorem 1.7, it suffices to consider that D is isomorphic to one of the digraphs H_1 , H_2 and H_3 . Now we consider the three cases. Suppose that D is isomorphic to the digraph H_1 . Then $\{x_1, x_2\} \rightarrow y_2$ implies that $\{x_1, x_2\}$ is a dominating pair. Since $a = 2$, we have $2a - 1 = a + 1 = 3$. By assumption, $d(x_1) \geq 3$. So $x_1 \rightarrow y_2$ or $y_2 \rightarrow x_1$, say $x_1 \rightarrow y_2$. Clearly, D is hamiltonian. Suppose that D is isomorphic to the digraph H_2 . Then $\{x_1, x_2\} \rightarrow y_1$ implies that $\{x_1, x_2\}$ is a dominating pair. By assumption, without loss of generality, assume that $d(x_1) \geq 2a - 1 = 5$ and $d(x_2) \geq a + 1 = 4$. So x_1 must dominate one of $\{y_2, y_3\}$, say y_2 . Note that $x_1 y_2 x_2 y_1 x_3 y_3 x_1$ is a hamiltonian cycle. Suppose that D is isomorphic to the digraph H_3 . Then $\{x_1, x_3\} \rightarrow y_2$ implies that $\{x_1, x_3\}$ is a dominating pair. By assumption, without loss of generality, assume that $d(x_1) \geq 2a - 1 = 5$ and $d(x_3) \geq a + 1 = 4$. So $y_2 x_1 \in A(D)$ or $x_1 y_1 \in A(D)$. If $y_2 x_1 \in A(D)$, then $y_2 x_1 y_3 x_2 y_1 x_3 y_2$ is a hamiltonian cycle; if $x_1 y_1 \in A(D)$, then $x_1 y_1 x_3 y_2 x_2 y_3 x_1$ is a hamiltonian cycle. \square

It is natural to propose the following problem.

Problem 3.5. Consider a strong balanced bipartite digraph on $2a$ vertices where $a \geq 4$. Suppose that D satisfies the condition B_k with $2 \leq k \leq a/2$. Is D hamiltonian?

4 Remark

If D is a strong balanced bipartite digraph on $2a$ vertices such that $d(x) + d(y) \geq 3a$ for every pair of dominating and every pair of dominated vertices $\{x, y\}$, then the following theorem shows that D contains a cycle factor. However D may contain no hamiltonian cycle, see the digraph H_1 . Here $a = 2$. Note that $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are both dominating and dominated pairs, where $d(x_1) + d(x_2) = 3a$ and $d(y_1) + d(y_2) = 3a$. Clearly, D has no hamiltonian cycle.

Proposition 4.1. *Let D be a strong balanced bipartite digraph on $2a$ vertices. Suppose that $d(x) + d(y) \geq 3a$ for every pair of dominating and every pair of dominated vertices $\{x, y\}$. Then D contains a cycle factor.*

Proof: Let V_1 and V_2 denote the partite sets of D . Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1 . Therefore, by König-Hall theorem, it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset V_1$ and $|N^+(T)| \geq |T|$ for every $T \subset V_2$.

For a proof by contradiction, suppose that a non-empty set $S \subset V_1$ is such that $|N^+(S)| < |S|$. Then $V_2 \setminus N^+(S) \neq \emptyset$ and for every $y \in V_2 \setminus N^+(S)$, we have $d^-(y) \leq a - |S|$, hence

$$d(y) \leq 2a - |S|. \quad (4.1)$$

If $|S| = 1$, say $S = \{x\}$, then $|N^+(S)| = 0$, which means $d^+(x) = 0$, a contradiction to the fact that D is strong. If $|S| = a$, then $d^-(y) = 0$ for every $y \in V_2 \setminus N^+(S)$, a contradiction to the fact that D is strong. Thus $2 \leq |S| \leq a - 1$. We now consider the following two cases.

Case 1. $\frac{a}{2} < |S| \leq a - 1$.

In this case, $|V_2 \setminus N^+(S)| \geq 2$ and $|V_1 \setminus S| < |V_2 \setminus N^+(S)|$. Since D is strong, for every $y \in V_2 \setminus N^+(S)$, $d^-(y) \geq 1$. This together with $|V_1 \setminus S| < |V_2 \setminus N^+(S)|$ implies that there exist $y_1, y_2 \in V_2 \setminus N^+(S)$ such that $\{y_1, y_2\}$ is a dominated pair. By assumption, $d(y_1) + d(y_2) \geq 3a$. But, by (4.1), $d(y_1) + d(y_2) \leq 2(2a - |S|) = 4a - 2|S| < 3a$, a contradiction.

Case 2. $2 \leq |S| \leq \frac{a}{2}$.

If this is so, then, for every $x \in S$, we have

$$d(x) = d^-(x) + d^+(x) \leq a + (|S| - 1) \leq \frac{3a}{2} - 1. \quad (4.2)$$

Since D is strong, for every $x \in S$, $d^+(x) \geq 1$. This together with $|N^+(S)| < |S|$ implies that there exist $x_1, x_2 \in S$ such that $\{x_1, x_2\}$ is a dominating pair. By assumption, $d(x_1) + d(x_2) \geq 3a$. But, by (4.2), $d(x_1) + d(x_2) \leq 3a - 2$, a contradiction.

This completes the proof of existence of a perfect matching from V_1 to V_2 . The proof for a matching in the opposite direction is analogous. The proof of the theorem is complete. \square

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