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On the $k^{th}$ Eigenvalues of Trees with Perfect Matchings

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Let $T_{2p}^+$ be the set of all trees on $2p$ ($p \geq 1$) vertices with perfect matchings. In this paper, we prove that for any tree $T$ in $T_{2p}^+$, the $k$th largest eigenvalue $\lambda_k(T)$ satisfies $\lambda_k(T) \leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{t}{k} \right\rfloor - 1 + \sqrt{\left\lceil \frac{t}{k} \right\rceil + 3}} \right)$ ($k = 1, 2, \ldots, p$). This upper bound is known to be best possible when $k = 1$. The set of trees obtained from a tree on $p$ vertices by joining a pendant vertex to each vertex of the tree is denoted by $T_{2p}^+$. We also prove that for any tree $T$ in $T_{2p}^+$, its $k$th largest eigenvalue $\lambda_k(T)$ satisfies $\lambda_k(T) \leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{t}{k} \right\rfloor - 1 + \sqrt{\left\lceil \frac{t}{k} \right\rceil + 3}} \right)$ ($k = 1, 2, \ldots, p$) and show that this upper bound is best possible when $k = 1$ or $p \not\equiv 0$ (mod $k$). We further give the following inequality

$$\lambda^*_k(2p) > \frac{1}{2} \left( \sqrt{t-1} - \sqrt{\frac{k-1}{t-k}} + \sqrt{t+3 - \sqrt{\frac{k-1}{t-k}}} \right)$$

where $\lambda^*_k(2p)$ is the maximum value of the $k$th largest eigenvalue of the trees in $T_{2p}^+$. By this inequality, it is easy to see that the above upper bound on $\lambda_k(T)$ for $T \in T_{2p}^+$ turns out to be asymptotically tight when $p \equiv 0$ (mod $k$).

Keywords: tree, eigenvalue, perfect matching.

1 Introduction

Let $G$ be a simple graph, i.e., a graph without loops or multiple edges. Suppose the vertex set of $G$ is $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of $G$ is an $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $G$ is denoted by $P(G; \lambda)$. Since $A(G)$ is symmetric, its eigenvalues are real; moreover, they are independent of the ordering of the vertices of $G$. As usual, we write them in non-increasing order as $\lambda_1(G) \geq \lambda_2(G) \geq \lambda_3(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of $G$. If $G$ is a bipartite graph, then $\lambda_i(G) = -\lambda_{n-i+1}(G)$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$ (see [6]), where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$, i.e., the floor function of $x$ when $x$ is a real number. Similarly, $\lceil x \rceil$ denotes the least integer greater than or equal to $x$, i.e., the ceiling function of $x$.
Two distinct edges in a graph $G$ incident with the same vertex will be called \emph{adjacent edges}. A \emph{matching} of $G$ is a set of edges in $G$ such that no two of them are adjacent. A largest matching is called a \emph{maximum matching}. The cardinality of a maximum matching of $G$ is commonly known as its \emph{matching number}, denoted by $\mu(G)$. Let $M$ be a matching of $G$. $M$ is called an $s$-\emph{matching} of $G$ if $M$ contains exactly $s$ edges of $G$. A vertex $v \in V(G)$ is said to be $M$-\emph{saturated} if it is incident with an edge of $M$, otherwise $v$ is called an $M$-\emph{unsaturated vertex}. The matching $M$ of $G$ is called a \emph{perfect matching} if all vertices of $G$ are $M$-saturated. Trees are connected acyclic graphs, and it is obvious that they are also bipartite graphs. So we only need to investigate those eigenvalues $\lambda_k(T)$ of a tree $T$ with $n$ vertices for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$.

Throughout this paper, we denote by $T_n$ and $T_{2p}^*$ the set of trees on $n$ vertices and the set of trees on $2p$ vertices with perfect matchings. For simplicity, a tree with $n$ vertices is often called a tree of order $n$. For symbols and concepts not defined in this paper we refer to the book [2].

The investigation on the eigenvalues of trees in $T_n$ is one of the oldest problems in the spectral theory of graphs and has been intensively studied by many authors (see [1, 6, 11, 12, 13, 15]). A classic result is that for any $T \in T_n$, $\lambda_1(T) \leq \sqrt{n-1}$ and equality holds if and only if $T$ is the star $K_{1,n-1}$. In particular, H. Yuan [12] studied the $k$th eigenvalue of a tree $T \in T_n$ and obtained the following upper bound.

\textbf{Theorem 1.1 ([12])} Let $T$ be a tree in $T_n$. Then

$$\lambda_k(T) \leq \sqrt{\left\lfloor \frac{n-2}{k} \right\rfloor} \quad (2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor)$$

and the upper bound is best possible if $n \equiv 1 \pmod{k}$.

J.Y. Shao [15] improved the above result.

\textbf{Theorem 1.2 ([15])} Let $T$ be a tree in $T_n$. Then

$$\lambda_k(T) \leq \sqrt{\left\lfloor \frac{n}{k} \right\rfloor - 1} \quad (1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor).$$

Moreover, the bound is best possible when $n \equiv 0 \pmod{k}$ and it is an asymptotically tight bound when $n \equiv 0 \pmod{k} \quad (2 \leq k \leq \lfloor n/2 \rfloor)$.

Concerning the trees in $T_{2p}^*$ there are lots of results on the first two largest eigenvalues (see [3, 4, 5, 8, 9, 10, 16, 17, 18]).

Frucht and Harary [7] gave the following construction of graphs. Given two graphs $G$ and $H$, the \emph{corona} of $G$ with $H$, denoted by $G \circ H$, is the graph with

$$V(G \circ H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$

$$E(G \circ H) = E(G) \cup \bigcup_{i \in V(G)} \left( E(H_i) \cup \{iu_i \mid u_i \in V(H_i) \} \right),$$

where $H_i \cong H$ for all $i \in V(G)$.

Let $T_{2p}^* = K_{1,p-1} \circ N_s$ (see Fig. 1.1), where $N_s$ is the null graph (i.e., edgeless graph) of order $s$.

G.H. Xu [17] got the following initial result.
Fig. 1.1: The tree $T_{2p}^1$

Fig. 1.2: Two graphs $T_{2p}^a$ and $T_{2p}^b (= T_{2p}^{2^t})$

Theorem 1.3 ([17]) Let $T$ be a tree in $T_{2p}^+$. Then

$$\lambda_1(T) \leq \frac{1}{2}(\sqrt{p-1} + \sqrt{p+3}) = \lambda_1(T_{2p}^1) \quad p = 1, 2, 3, \ldots.$$

and equality holds if and only if $T \cong T_{2p}^1$.

A. Chang [3] studied bounds for the second largest eigenvalue of trees in $T_{2p}^+$ and proposed the following conjecture:

Let $p$ be a positive integer, and $T$ be a tree in $T_{2p}^+$. Then

$$\lambda_2(T) \leq \begin{cases} r' & \text{if } p = 2t \\ r'' & \text{if } p = 2t + 1 \end{cases} \quad \text{for } t = 1, 2, 3, \ldots,$$

where $r'$ and $r''$ are the maximum positive roots of the equations $x^3 - (t + 1)x + 1 = 0$ and $x^4 - (t + 2)x^2 + x + 1 = 0$, respectively. Equality holds in the first inequality if and only if $T \cong T_{2p}^a$, and equality holds in the second inequality if and only if $T \cong T_{2p}^b$, where $T_{2p}^a$ and $T_{2p}^b$ are the trees shown in Fig. 1.2.

More recently, J-M. Guo and S-W. Tan [9] proved that the second inequality holds but the first one does not hold. A correct version of the first inequality was given by J-M. Guo and S-W. Tan in [10]. Their results can be stated as follows.
Theorem 1.4 ([9][10]) Let \( p \) be a positive integer, and \( T \) be a tree in \( T^+_2 \). Then

\[
\lambda_2(T) \leq \begin{cases} 
  r_1 & \text{if } p = 2t \\
  r_2 & \text{if } p = 2t + 1
\end{cases}
\]

for \( t = 2, 3, \ldots \), where \( r_1 \) and \( r_2 \) are the maximum positive roots of the equations

\[
(x^4 - (t + 1)x^2 + 1)(x^2 + x - 1) + x = 0
\]

and

\[
x^4 - (t + 2)x^2 + x + 1 = 0,
\]

respectively. Equality holds in the first inequality if and only if \( T \cong T^+_2 \), and equality holds in the second inequality if and only if \( T \cong T^+_2 \), where \( T^+_2 \) and \( T^+_2 \) are the trees shown in Fig. 1.3 and 1.2, respectively.

It is natural to consider the problem of determining upper and lower bounds of the \( k \)th eigenvalues of the trees in \( T^+_2 \). This is the purpose of our paper.

2 Main results

We need some groundwork before giving the main result. Before we recall the well-known Cauchy Interlacing Theorem ([6, Theorem 0.10]), we introduce some notation and terminology first. A vertex subset with \( k \) vertices is called a \( k \)-vertex subset. Suppose \( V' \) is a subset of vertices. \( G - V' \) is the subgraph of \( G \) obtained by deleting all vertices in \( V' \) together with their incident edges. Cauchy Interlacing Theorem usually plays an important role in the estimation of the \( k \)th eigenvalue of trees.

Theorem 2.1 (Cauchy Interlacing Theorem) For every graph \( G \) and every \( k \)-vertex subset \( V' \) we have

\[
\lambda_i(G) \geq \lambda_i(G - V') \geq \lambda_{i+k}(G), \quad i = 1, 2, \ldots, n - k.
\]

Lemma 2.2 ([1]) Let \( G \) be a graph and \( H \) a subgraph of \( G \). Then \( \lambda_1(H) \leq \lambda_1(G) \).

Lemma 2.3 ([15]) Let \( T \) be a tree in \( T_n \). Then for any positive integer \( s \), there exists a vertex \( v \in V(T) \) such that the largest component of \( T - v \) has order at most \( \max\{n - 1 - s, s\} \) and all other components of \( T - v \) have orders at most \( s \).

It is worth mentioning that when the tree \( T \) considered in Lemma 2.3 is in \( T^+_2 \), i.e., \( T \) is a tree with a perfect matching, then obviously all components of \( T - v \) but one have perfect matchings. The only component without perfect matching, say \( T_0 \), has matching number \( \mu(T_0) = \frac{1}{2}(V(T_0) - 1) \), and the only unsaturated vertex of \( T_0 \) is the vertex \( w \) which is adjacent with \( v \) in \( T \) and \( vw \) is an edge of the perfect matching of \( T \). This fact leads us to get the following useful lemma.
Lemma 2.4 Let $T \in \mathcal{T}_n^+$, and let $s$ be a positive even integer not greater than $n$. Then there exist a vertex $v \in V(T)$ and a subtree $U$ of $T$ such that

1. $U$ has a perfect matching;
2. either $U$ is a component of $T - v$ when $v \not\in V(U)$, or $U - v$ is a component of $T - v$ when $v \in V(U)$;
3. $|V(U)| \leq \max\{n - s, s\}$;
4. all components of $(T - V(U)) - v$ have order at most $s$, and all but at most one of them have a perfect matching.

Proof: Let $M$ be a perfect matching of $T$. By Lemma 2.3, there exists a vertex $v \in V(T)$ such that one component $T'$ of $T - v$ has order $|V(T')| \leq \max\{n - 1 - s, s\}$, and all other components of $T - v$ have orders not exceeding $s$. We know that only one component, say $T_0$, has no perfect matching and all the others have perfect matchings.

Suppose $T' \neq T_0$. Then $M \cap E(T')$ is a perfect matching of $T'$. Since $s$ and $n$ are even, $|V(T')| \leq \max\{n - 2 - s, s\}$. Let $U = T'$. Then $T_0$ is a component of $(T - V(U)) - v$ and its matching number is $\mu(T_0) = \frac{n}{2}(V(T_0) - 1)$. Let $w$ be the only unsaturated vertex of $T_0$ which is adjacent with $v$ in $T$ and $vw$ is an edge of $M$. Now let $T'_0$ be the tree obtained from $T_0$ by joining a pendant vertex $u$ to $w$. Actually, we can view this vertex $u$ as the removed vertex $v$. Obviously, $T'_0$ has a perfect matching $(E(T_0) \cap M) \cup \{vw\}$ and order $|V(T'_0)| = |V(T_0)| + 1 \leq s$, and $T'_0$ is a subtree of $T$.

Suppose $T' = T_0$. Then $T'$ has a maximum matching $M_1 = E(T') \cap M$ and its matching number is $\mu(T') = \frac{n}{2}(|V(T')| - 1)$. Since $s$ is even, $|V(T')| \leq \max\{n - 1 - s, s - 1\}$. Let $w \in V(T')$ be the only $M_1$-unsaturated vertex. Then $vw$ is an edge of $M$. Let $U$ be the tree obtained from $T'$ by joining a pendant vertex $u$ to $w$. Actually, we can view this vertex $u$ as the removed vertex $v$. Then $U$ is a subtree of $T'$ and is of order not greater than $\max\{n - s, s\}$. Clearly, $M_1 \cup \{vw\}$ is a perfect matching of $U$ and $U - v = T'$.

Lemma 2.5 Let $T$ be a tree in $\mathcal{T}_n^+$. Then for any positive integer $k$ with $1 \leq k \leq p$, there exists a $(k - 1)$-vertex subset $V' \subset V(T)$ such that all components of $T - V'$ have the largest eigenvalues not greater than $\lambda_1(T_{2t}^k)$, where $T_{2t}^k$ is the tree shown in Fig. 1.1 and $t = \lceil p/k \rceil$.

Proof: When $k = 1$, the result is actually Theorem 1.3. So we may assume that $k \geq 2$. Let $s = 2t = 2\lceil p/k \rceil$, and $T_0 = T$, $n_0 = 2p$. Since $k \geq 2$, we have $n_0 > s$. We perform the following procedure:

By Lemma 2.4 there are a vertex $v_1 \in V(T)$ and a subtree $T_1$ of order not greater than $\max\{n_0 - s, s\}$ such that $T_1$ has a perfect matching, $T_1 - v_1$ is a component of $T - v_1$ and the other components of $T - v_1$ have orders not greater than $s$. Note that $v_1$ may not be a vertex of $T_1$.

Let $n_1 = |V(T_1)|$. If all components of $T - v_1$ and $T_1$ are of orders not greater than $s$, then we stop the procedure. If not, then $n_1 > s$. By applying Lemma 2.4 to $T_1$ there are a vertex $v_2 \in V(T_1)$ and a subtree $T_2$ of $T_1$ such that the order of $T_2$ is not greater than $\max\{n_1 - s, s\}$, $T_2$ has a perfect matching, $T_2 - v_2$ is a component of $T_1 - v_2$ and the other components of $T_1 - v_2$ have orders not greater than $s$.

Let $n_2 = |V(T_2)|$. If all components of $T - \{v_1, v_2\}$ and $T_2$ are of orders not greater than $s$, then we stop the procedure. If not, we continue to perform the above procedure. Since $n_0$ is finite, there are $h$ subtrees $T_0 \supset T_1 \supset \cdots \supset T_h$ and vertices $v_1, \ldots, v_h$ (not necessary distinct) such that all components of
$T - \{v_1, v_2, \ldots, v_h\}$ are of orders not greater than $s$, $n_i = |V(T_i)| \leq \max\{n_{i-1} - s, s\}$ and $v_i \in V(T_{i-1})$ for $1 \leq i \leq h$. Hence we have $n_i > s$ for $1 \leq i \leq h - 1$. Since $s = 2\lceil p/k \rceil$,

$$ks = 2k\lceil p/k \rceil \geq 2k (p/k) = 2p.$$ 

Since $n_i \leq n_{i-1} - s$, $(i = 1, 2, \ldots, h)$,

$$n_{h-1} - n_0 = \sum_{i=1}^{h-1} (n_i - n_{i-1}) \leq -(h-1)s.$$ 

Hence

$$s \leq n_{h-1} - 2p - (h-1)s \leq ks - hs + s = (k - h + 1)s.$$ 

Thus $h \leq k - 1$.

Now we may choose a $(k - 1)$-vertex subset $V'$ containing $\{v_1, v_2, \ldots, v_h\}$ such that the components of $T - V'$ are of orders not exceeding $s$. By Lemma 2.2 and Theorem 1.3 all components of $T - V'$ have their largest eigenvalues not great than $\lambda_1(T_{2k})$. The proof is completed. \hfill \Box

Combining Lemma 2.5 with the Cauchy Interlacing Theorem, we obtain the following main result.

**Theorem 2.6** Let $T$ be a tree in $T_{2p}^+$. Then for any positive integer $k$ with $1 \leq k \leq p$, we have

$$\lambda_k(T) \leq \frac{1}{2} \left( \sqrt{\frac{p}{k}} - 1 + \sqrt{\frac{p}{k} + 3} \right) \quad (2.1)$$

and this upper bound is best possible when $k = 1$.

**Proof:** Suppose that $T \in T_{2p}^+$. By Lemma 2.5 we have a $(k - 1)$-vertex subset $V' \subset V(T)$ such that all components, say $T_1, T_2, \ldots, T_q$, of $T - V'$ are trees with the largest eigenvalues not exceeding $\lambda_1(T_{2k})$, $t = \lceil \frac{p}{k} \rceil$. By Theorems 2.1 and 1.3 we obtain

$$\lambda_k(T) \leq \lambda_1(T - V') = \max_{1 \leq i \leq q} \lambda_1(T_i) \leq \max_{1 \leq i \leq s} \frac{1}{2} \left( \sqrt{\frac{|V(T_i)|}{2}} - 1 + \sqrt{\frac{|V(T_i)|}{2} + 3} \right) \leq \frac{1}{2} \left( \sqrt{\frac{p}{k}} - 1 + \sqrt{\frac{p}{k} + 3} \right)$$

This proves the upper bound (2.1). Obviously, for $k = 1$, (2.1) is just the upper bound $\lambda_1(T) \leq \frac{1}{2} (\sqrt{p - 1} + \sqrt{p + 3})$ in Theorem 1.3 and it is best possible upper bound. \hfill \Box

**Example 2.1** For any $T \in T_{10}^+$ from Theorem 2.6 we get that $\lambda_1(T) \leq \frac{1}{2} (\sqrt{4} + \sqrt{8}) \approx 2.414$, $\lambda_2(T) \leq \frac{1}{2} (\sqrt{2} + \sqrt{6}) \approx 1.932$, $\lambda_3(T) \leq \frac{1}{2} (1 + \sqrt{5}) \approx 1.618$, $\lambda_4(T) \leq \frac{1}{2} (1 + \sqrt{3}) \approx 1.618$ and $\lambda_5(T) \leq 1$. We find that $\lambda_1(T)$ and $\lambda_5(T)$ are tight, which can be verified by the table of the spectra of all trees with $n$ vertices ($2 \leq n \leq 10$) in [7]. \hfill \Box
There is a relationship between the characteristic polynomial \( P(G \odot N_s; \lambda) \) of \( G \odot N_s \) and that of \( G \) as follows.

**Lemma 2.7** ([6]) \( P(G \odot N_s; \lambda) = \lambda^p P(G; \lambda - \frac{1}{k}) \).

Let \( T_{2p}^* \) be the set of the coronas of trees of order \( p \) with \( N_1 \), i.e.,

\[
T_{2p}^* = \{ T \odot N_1 \mid T \in T_p \}.
\]

Obviously, any graph in \( T_{2p}^* \) is a tree and has a perfect matching. Thus we have \( T_{2p}^* \subset T_{2p}^* \). Note that for any \( T^* \in T_{2p}^* \), there is a unique tree \( T \) with \( T^* = T \odot N_1 \). The tree \( T \) is called the contracted tree of the tree \( T^* \). Now we prove an upper bound on the \( k \)th eigenvalue of trees in \( T_{2p}^* \).

**Lemma 2.8** Let \( T^* \in T_{2p}^* \) and let \( T \) be the contracted tree of \( T^* \). Then

\[
\lambda_k(T^*) = \frac{1}{2} \left( \sqrt{\lambda_k(T)^2 + 4 + \lambda_k(T)} \right).
\]

**Proof:** By Lemma 2.7 we have \( P(T^*; \lambda) = \lambda^p P(T; \lambda - \frac{1}{k}) \). Since \( \lambda_k(T) \) is the \( k \)th eigenvalue of \( T \) for \( k = 1, 2, \ldots, p \),

\[
P(T^*; \lambda) = \lambda^p \prod_{i=1}^{p} (\lambda - \frac{1}{k} - \lambda_i(T)) = \prod_{i=1}^{p} (\lambda^2 - \lambda_i(T)\lambda - 1).
\]

So the positive eigenvalues of \( T^* \) are \( \lambda^2 \left( \sqrt{\lambda_i(T)^2 + 4 + \lambda_i(T)} \right), \ i = 1, 2, \ldots, p \). Since \( f(x) = \frac{1}{2} \left( x^2 + 4 + x \right) \) is an increasing function of the variable \( x \), the result follows immediately. \( \square \)

**Theorem 2.9** Let \( T^* \) be a tree in \( T_{2p}^* \). Then

\[
\lambda_k(T^*) \leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 + \sqrt{\left\lfloor \frac{p}{k} \right\rfloor + 3} \right), \quad (2.2)
\]

for \( k = 1, 2, \ldots, p \). Moreover, this upper bound is best possible when \( k = 1 \) or \( p \not\equiv 0 \pmod{k} \).

**Proof:** Suppose that \( T \) is the contracted tree of the tree \( T^* \). Then \( T \) is a tree of order \( p \). By Theorem 1.2 Lemma 2.8 and its proof, we have

\[
\lambda_k(T^*) = \frac{1}{2} \left( \sqrt{\lambda_k(T)^2 + 4 + \lambda_k(T)} \right)
\]

\[
\leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 + \sqrt{\left\lfloor \frac{p}{k} \right\rfloor + 3} \right)
\]

for \( k \leq \left\lfloor \frac{p}{2} \right\rfloor \). For \( k > \left\lfloor \frac{p}{2} \right\rfloor \), since \( \lambda_k(T) \leq 0 \), we have \( \lambda_k(T^*) \leq 1 \). The Equation (2.2) holds, since the right hand side of (2.2) is equal to 1.

When \( k = 1 \), it is known that this bound is best possible. To show tightness for \( k \geq 2 \) and \( p \not\equiv 0 \pmod{k} \), we shall construct a corona of a tree with \( N_1 \). First, we write \( p = \left\lfloor \frac{p}{k} \right\rfloor + r \), where \( 1 \leq r \leq k - 1 \). Set \( t = 2 \left\lfloor \frac{p}{k} \right\rfloor \) and thus \( 2p = tk + 2r \). Let \( T \) be the tree obtained by joining edges from the
center \( u \) of a star \( K_{1,r-1} \) to the centers \( v_1, v_2, \ldots, v_k \) of \( k \) disjoint stars \( K_{1,t-1} \) (see Fig. 2.1). Then let \( T^* = T \odot N_1 \in \mathcal{T}_{2p} \). It is easy to see that

\[
\lambda_1(T - u) = \lambda_2(T - u) = \cdots = \lambda_k(T - u) = \lambda_1(K_{1,t-1}) = \sqrt{\frac{t}{2} - 1}.
\]

By Lemma 2.2 and the Cauchy Interlacing Theorem we have

\[
\lambda_k(T - u) \leq \lambda_k(T) \leq \lambda_{k-1}(T - u).
\]

Therefore,

\[
\lambda_k(T) = \sqrt{\frac{t}{2} - 1}.
\]

By Lemma 2.8 we have

\[
\lambda_k(T^*) = \frac{1}{2} \left( \sqrt{\frac{t}{2} - 1} + \sqrt{\frac{t}{2} + 3} \right) = \frac{1}{2} \left( \sqrt{\frac{p}{k}} - 1 + \sqrt{\frac{p}{k} + 3} \right).
\]

This shows that the upper bound \( 2.2 \) is best possible when \( p \not\equiv 0 \pmod{k} \). \( \square \)

\textbf{Example 2.2} For any tree \( T^* \in \mathcal{T}_{10} \), by Theorem 2.6 we have \( \lambda_1(T^*) \leq \frac{1}{2}(\sqrt{4} + \sqrt{8}) \approx 2.414 \), \( \lambda_2(T^*) \leq \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 \), \( \lambda_3(T^*) \leq 1 \), \( \lambda_4(T^*) \leq 1 \) and \( \lambda_5(T^*) \leq 1 \). It can be verified from the table of the spectra of all trees with \( n \) vertices \((2 \leq n \leq 10)\) in [6] that these bounds are tight. \( \square \)

\textbf{Example 2.3} For any tree \( T^* \in \mathcal{T}_{8} \), by Theorem 2.6 we have \( \lambda_1(T^*) \leq \frac{1}{2}(\sqrt{3} + \sqrt{7}) \approx 2.189 \), \( \lambda_2(T^*) \leq \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 \), \( \lambda_3(T^*) \leq 1 \) and \( \lambda_4(T^*) \leq 1 \). We know that the upper bounds on \( \lambda_1 \) and \( \lambda_3 \) are tight but on \( \lambda_2 \) and \( \lambda_4 \) they are not. Actually, the maximum values of \( \lambda_2 \) and \( \lambda_4 \) are approximately 1.356 and 0.477, respectively. \( \square \)
Example 2.3 shows that for those $k$ satisfying $p \equiv 0 \pmod{k}$, we usually only have

$$\lambda_k(T^*) < \frac{1}{2} \left( \sqrt{\left\lceil \frac{p}{k} \right\rceil - 1} + \sqrt{\left\lceil \frac{p}{k} \right\rceil + 3} \right),$$

and especially, the upper bound is not as good as that in Theorem 1.2 when $k = 2$. However, the upper bound in Theorem 2.6 will be shown to be asymptotically tight when $p \equiv 0 \pmod{k}$.

Lemma 2.10 ([14]) Let $v$ be a vertex of $G$, and $C(v)$ be the set of all cycles containing $v$. Then the characteristic polynomial of $G$ satisfies

$$P(G; \lambda) = \lambda P(G - v; \lambda) - \sum_{uv \in E(G)} P(G - v - u; \lambda) - 2 \sum_{Z \in C(v)} P(G - V(Z); \lambda).$$

We first take $k$ copies of the star $K_{1, t-1}$ (say $S_1, S_2, \ldots, S_k$) with centers $u_1, u_2, \ldots, u_k$, respectively, and choose $v_i \in V(S_i) \setminus \{u_i\}$ ($i = 1, 2, \ldots, k$). Then add the edges $v_1v_i$ ($i = 2, 3, \ldots, k$) to obtain tree $T$ with $tk$ vertices as shown in Fig. 2.2.

The next lemma follows by direct calculation from Lemma 2.10 with $v = v_1$, observing that the last term in the lemma becomes zero because there are no cycles containing $v_1$.

Lemma 2.11 ([15]) Denoting $f(x) = x^3 + (t - k)x^2 - 2(k - 1)x - (k - 1)$, the characteristic polynomial of the tree $T$ shown in Fig. 2.2 is

$$P(T; \lambda) = \lambda^{tk-2(k+1)}(\lambda^2 - t + 1)^{k-2}f(\lambda^2 - t + 1)$$

and the $k$th eigenvalue of $T$ satisfies

$$\lambda_k(T) = \sqrt{t - 1 + \lambda_2(f)} > \sqrt{t - 1 - \sqrt{\frac{k - 1}{t - k}}}$$

where $\lambda_2(f)$ is the second largest root of the equation $f(x) = 0$. 

Fig. 2.2: The tree $T$ in Lemma 2.10
Denote the maximum value of the $k$th largest eigenvalue of the trees in $T_{2p}^*$ by $\lambda_k^*(2p)$. Then Theorem 2.9 tells us that $\lambda_k^*(2p) \leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 + \sqrt{\left\lceil \frac{p}{k} \rceil} + 3 \right)$. We shall give a lower bound for $\lambda_k^*(2p)$, which shows that as $k$ gets large, the upper bound in Theorem 2.6 is asymptotically tight for the value of $\lambda_k^*(2p)$ when $p \equiv 0 \mod{k}$.

**Theorem 2.12** Let $p$ and $k$ be integers with $1 \leq k \leq p$. If $t = \left\lfloor \frac{p}{k} \right\rfloor > k$, then

$$\lambda_k^*(2p) > \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 - \sqrt{\frac{k-1}{t-k}} + \sqrt{\frac{p}{k}} + 3 - \sqrt{\frac{k-1}{t-k}} \right).$$

**Proof:** Let $T^* = T \odot N_1 \in T_{2tk}^*$ by taking the tree $T$ with $tk$ vertices described in Fig. 2.2 From Lemma 2.11 it is easy to see that the second largest root $\lambda_2(f)$ of $f(x) = 0$ is negative. Note that $f(0) = -(k-1) < 0$, $f \left( -\sqrt{\frac{k-1}{t-k}} \right) > 0$ and $\lim_{x \to -\infty} f(x) = -\infty$. So we know that $\lambda_2(f) > -\sqrt{\frac{k-1}{t-k}}$. Moreover, the expression $\lambda = \frac{1}{2} \left( \sqrt{t-1} + \alpha + \sqrt{t+3 + \alpha} \right)$ can be regarded as a strictly increasing function of the variable $\alpha$. Thus, by Lemmas 2.8 and 2.11, we have

$$\lambda_k(T^*) = \frac{1}{2} \left( \sqrt{t-1 + \lambda_2(f)} + \sqrt{t+3 + \lambda_2(f)} \right) > \frac{1}{2} \left( \sqrt{t-1 - \frac{k-1}{t-k}} + \sqrt{t+3 - \frac{k-1}{t-k}} \right).$$

There is a tree $U$ of order $p$ containing $T$ described above. Hence $U^* = U \odot N_1 \in T_{2p}$ and

$$\lambda_k(U^*) \geq \lambda_k(T^*) > \frac{1}{2} \left( \sqrt{t-1 - \frac{k-1}{t-k}} + \sqrt{t+3 - \frac{k-1}{t-k}} \right) = \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 - \sqrt{\frac{k-1}{t-k}} + \sqrt{\left\lceil \frac{p}{k} \right\rceil} + 3 - \sqrt{\frac{k-1}{t-k}} \right).$$

Thus we get the theorem. \qed

**Remark:** If we let $t \to \infty$ (that is, $2p \to \infty$) for a fixed $k$, then $\sqrt{\frac{k-1}{t-k}} \to 0$, i.e.,

$$\lambda_k^*(2p) \to \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 + \sqrt{\left\lceil \frac{p}{k} \rceil} + 3 \right) \text{ as } t \to \infty.$$

So we can say that our upper bound (2.2) is asymptotically tight. Of course, if we denote the maximum value of the $k$th eigenvalues of trees in $T_{2p}^*$ by $\lambda_k^*(2p)$, then by Theorem 2.9 we have $\lambda_k^*(2p) \geq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor} - 1 + \sqrt{\left\lceil \frac{p}{k} \rceil} + 3 \right)$. So the upper bound (2.1) is also asymptotically tight in a sense.
References


