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Splittability and 1-Amalgamability of Permutation Classes

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A permutation class C is splittable if it is contained in a merge of two of its proper subclasses, and it is 1-amalgamable if given two permutations $\sigma, \tau \in C$, each with a marked element, we can find a permutation $\pi \in C$ containing both σ and τ such that the two marked elements coincide. It was previously shown that unsplittability implies 1-amalgamability. We prove that unsplittability and 1-amalgamability are not equivalent properties of permutation classes by showing that the class Av$(1423, 1342)$ is both splittable and 1-amalgamable. Our construction is based on the concept of LR-inflations, which we introduce here and which may be of independent interest.

Keywords: permutation classes, splittability, amalgamability, 1-amalgamability

1 Introduction

In the study of permutation classes, a notable interest has recently been directed towards the operation of merging. We say that a permutation π is a *merge* of σ and τ if the elements of π can be colored red and blue so that the red elements form a copy of σ and the blue elements form a copy of τ. For instance, Claesson, Jelínek and Steingrímsson [3] showed that every 1324-avoiding permutation can be merged from a 132-avoiding permutation and a 213-avoiding permutation, and used this fact to prove that there are at most 16^n 1324-avoiding permutations of length n.

A general problem that follows naturally is how to identify when a permutation class C has proper subclasses A and B, such that every element of C can be obtained as a merge of an element of A and an element of B. We say that such a permutation class C is *splittable*. Jelínek and Valtr [4] showed that every inflation-closed class is unsplittable and the class of σ-avoiding permutations, where σ is a direct sum of two nonempty permutations and has length at least four, is splittable. Furthermore, they mentioned the connection of splittability to more general structural properties of classes of relational structures studied in the area of Ramsey theory, most notably the notion of 1-amalgamability. We say that a permutation class C is *1-amalgamable* if given two permutations $\sigma, \tau \in C$, each with a marked element, we can find a permutation $\pi \in C$ containing both σ and τ such that the two marked elements coincide.

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Not much is known about 1-amalgamability of permutation classes. Jelínek and Valtr [4, Lemma 1.5], using a more general result from Ramsey theory, showed that unsplittability implies 1-amalgamability, and they raised the question whether there is a permutation class that is both splittable and 1-amalgamable. In this paper, we answer this question by showing that the class Av$(1423, 1342)$ has both properties.

For this task, we will introduce a slightly weaker property than being inflation-closed, that is being closed under inflating just the elements that are left-to-right minima. We say that an element of permutation π is a *left-to-right minimum*, or just LR-minimum, if it is smaller than all the elements preceding it. In Section 4 we shall prove that certain properties of a permutation class C imply that its closure under inflating LR-minima is splittable and 1-amalgamable. And finally in Section 5 we show that the class Av$(1423, 1342)$ is actually equal to the class Av(123) closed under inflating left-to-right minima and that Av(123) has the desired properties.

2 Basics

A *permutation* π of length $n \geq 1$ is a sequence of all the n distinct numbers from the set $[n] = \{1, 2, \ldots, n\}$. We denote the i-th element of π as π_i. Note that we omit all punctuation when writing out short permutations, e.g., we write 123 instead of 1, 2, 3. The set of all permutations of length n is denoted S_n.

We say that two sequences of distinct numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are *order-isomorphic* if for every two indices $i < j$ we have $a_i < a_j$ if and only if $b_i < b_j$. Given two permutations $\pi \in S_n$ and $\sigma \in S_k$, we say that π *contains* σ if there is a k-tuple $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the sequence $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ is order-isomorphic to σ and we say that such a sequence is an *occurrence* of σ in π. Furthermore, we say that the corresponding function $f : [k] \to [n]$ defined as $f(j) = i_j$ is an *embedding* of σ into π. In the context of permutation containment, we often refer to the permutation σ as a *pattern*.

A permutation that does not contain σ is σ-*avoiding* and we let Av(σ) denote the set of all σ-avoiding permutations. Similarly, for a set of permutations F , we let Av(F) denote the set of permutations that avoid all elements of F. Note that for small sets F we omit the curly braces, e.g., we simply write Av(σ, ρ) instead of Av$(\{\sigma, \rho\})$.

We say that a set of permutations C is a *permutation class* if for every $\pi \in C$ and σ contained in π, σ belongs to C as well. Observe that a set of permutations C is a permutation class if and only if there is a set F such that $C =$ Av(F). Moreover, for every permutation class C, there is a unique inclusionwise minimal set F such that $C =$ Av(F); this set F is known as the *basis* of C. A class is said to be *principal* if its basis has a single element, i.e., if the class has the form Av(σ) for a permutation σ.

Suppose that $\pi \in S_n$ is a permutation, let $\sigma_1, \ldots, \sigma_n$ be an n-tuple of non-empty permutations, and let m_i be the length of σ_i for $i \in [n]$. The *inflation* of π by the sequence $\sigma_1, \ldots, \sigma_n$, denoted by $\pi[\sigma_1, \ldots, \sigma_n]$, is the permutation of length $m_1 + \cdots + m_n$ obtained by concatenating n sequences $\overline{\sigma}_1\overline{\sigma}_2 \cdots \overline{\sigma}_n$ with these properties:

- for each $i \in [n]$, $\overline{\sigma}_i$ is order-isomorphic to σ_i, and
- for each $i, j \in [n]$, if $\pi_i < \pi_j$, then all the elements of $\overline{\sigma}_i$ are smaller than all the elements of $\overline{\sigma}_j$.

For two sets of permutations A and B, we let $A[B]$ denote the set of all the permutations that can be obtained as an inflation of a permutation from A by a sequence of permutations from B. We say that a set of permutations A is $\cdot[B]$-*closed* if $A[B] \subseteq A$, and similarly a set of permutations B is $A[\cdot]$-*closed* if $A[B] \subseteq B$. Finally, we say that a set of permutations C is *inflation-closed* if $C[C] \subseteq C$.

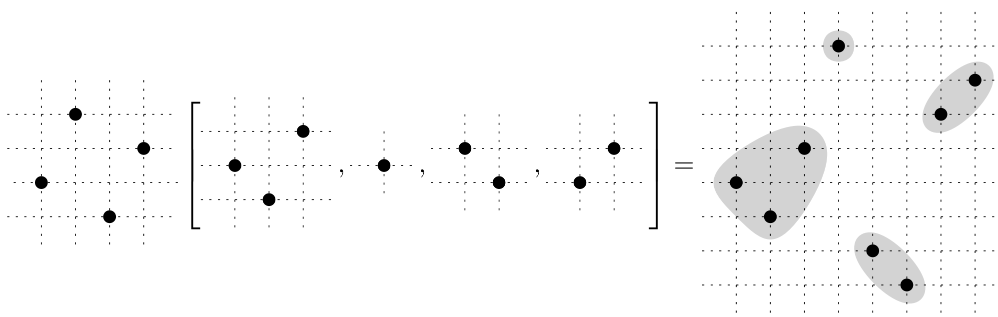

Figure 1: An example of inflation: $2413[213, 1, 21, 12] = 43582167$.

There is a nice way to characterize an inflation-closed class through its basis. We say that a permutation π is *simple* if it cannot be obtained by inflation from smaller permutations, except for the trivial inflations $\pi[1, \ldots, 1]$ and $1[\pi]$. Inflation-closed permutation classes are precisely the classes whose basis only contains simple permutations [1, Proposition 1].

3 Splittability and 1-amalgamability

We now focus on the properties of splittability and 1-amalgamability of permutation classes. Mostly, we state or rephrase results that were already known. For more detailed overview, especially regarding splittability, see Jelínek and Valtr [4].

3.1 Splittability

We say that a permutation π is a *merge* of permutations τ and σ, if it can be partitioned into two disjoint subsequences, one of which is an occurrence of σ and the other is an occurrence of τ. For two permutation classes A and B, we write $A \odot B$ for the class of all merges of a (possibly empty) permutation from A with a (possibly empty) permutation from B. Trivially, $A \odot B$ is again a permutation class.

Conversely, we say that a multiset of permutation classes $\{P_1, \ldots, P_m\}$ forms a *splitting* of a permutation class C if $C \subseteq P_1 \odot \cdots \odot P_m$. We call P_i the *parts* of the splitting. The splitting is *nontrivial* if none of its parts is a superset of C, and the splitting is *irredundant* if no proper submultiset of $\{P_1, \ldots, P_m\}$ forms a splitting of C. A permutation class C is then *splittable* if C admits a nontrivial splitting.

The following simple lemma is due to Jelínek and Valtr [4, Lemma 1.3].

Lemma 3.1. *For a class C of permutations, the following properties are equivalent:*

(a) C is splittable.

(b) C has a nontrivial splitting into two parts.

(c) C has a splitting into two parts, in which each part is a proper subclass of C.

(d) C has a nontrivial splitting into two parts, in which each part is a principal class.

Following the previous Lemma 3.1, we can characterize a splittable class C by the splittings of the form $\{\mathrm{Av}(\pi), \mathrm{Av}(\sigma)\}$, where both π and σ are permutations from C. We want to identify permutations inside C that cannot define any such splitting.

Definition 3.2. Let C be a permutation class. We say that a permutation $\pi \in C$ is *unavoidable in* C, if for any permutation $\tau \in C$, there is a permutation $\sigma \in C$ such that any red-blue coloring of σ has a red copy of τ or a blue copy of π. We let U_C denote the set of all unavoidable permutations in C.

It is easy to see that a permutation π is unavoidable in C if and only if C has no nontrivial splitting into two parts with one part being $\mathrm{Av}(\pi)$. A more detailed overview of the properties of unavoidable permutations was provided by Jelínek and Valtr [4, Observation 2.2-3], we will mention only the observations needed for our results.

Note that for a nonempty permutation class C, the set of unavoidable permutations U_C is in fact a nonempty permutation class contained in the class C. We can use the class of unavoidable permutations to characterize the unsplittable permutation classes.

Observation 3.3. *A permutation class C is unsplittable if and only if $U_C = C$.*

Furthermore, we can show that if C is closed under certain inflations then also U_C is closed under the same inflations. Again, the following result is due to Jelínek and Valtr [4, Lemma 2.4].

Lemma 3.4. *Let C be a permutation class. If, for a set of permutations X, the class C is closed under $\cdot[X]$, then U_C is also closed under $\cdot[X]$, and if C is closed under $X[\cdot]$, then so is U_C. Consequently, if C is inflation-closed, then $U_C = C$ and C is unsplittable.*

3.2 Amalgamability

Now let us introduce the concept of amalgamation, which comes from the general study of relational structures.

We say that a permutation class C is π-*amalgamable* if for any two permutations $\tau_1, \tau_2 \in C$ and any two mappings f_1 and f_2, where f_i is an embedding of π into τ_i, there is a permutation $\sigma \in C$ and two mappings g_1 and g_2 such that g_i is an embedding of τ_i into σ and $g_1 \circ f_1 = g_2 \circ f_2$. We also say, for $k \in \mathbb{N}$ that a permutation class C is k-amalgamable if it is π-amalgamable for every π of order at most k. Furthermore, a permutation class C is amalgamable if it is k-amalgamable for every k.

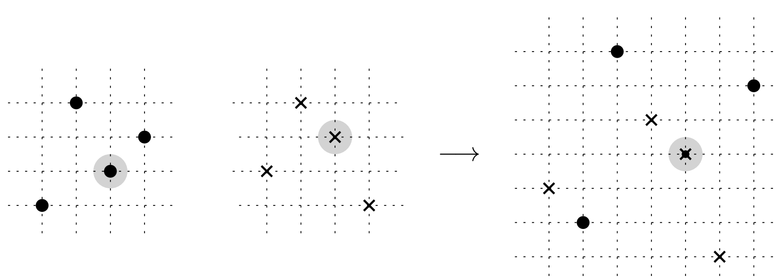

Figure 2: One possible 1-amalgamation of 1423 and 2431 with highlighted embeddings of the singleton permutations is the permutation 3275416.

Note that k-amalgamability implies $(k-1)$-amalgamability, so we have an infinite number of increasingly stronger properties. However, the situation is quite simple in the case of the permutation classes. As shown by Cameron [2], there are only five infinite amalgamable classes, the classes Av(12), Av(21), the class of all layered permutations Av$(231, 312)$, the class of their complements Av$(213, 132)$ and the class of all permutations. These are also the only permutation classes that are 3-amalgamable, implying that for any $k \geq 3$, a permutation class is k-amalgamable if and only if it is amalgamable.

In contrast, very little is known about 1-amalgamable and 2-amalgamable permutation classes. In this paper, we are particularly interested in the 1-amalgamable permutation classes.

Definition 3.5. Let C be a permutation class. We say that a permutation $\pi \in C$ is *1-amalgamable in* C, if for every $\tau \in C$ and every prescribed pair of embeddings f_1 and f_2 of the singleton permutation 1 into π and τ there is a permutation $\sigma \in C$ and embeddings g_1 and g_2 of π and τ into σ such that $g_1 \circ f_1 = g_2 \circ f_2$. We use A_C to denote the set of all 1-amalgamable permutations in C.

Trivially, A_C is a permutation class contained in C. Moreover, the properties of A_C are largely analogous to those of U_C, as shown by the next several results.

Observation 3.6. *A permutation class C is 1-amalgamable if and only if $A_C = C$.*

Similarly to U_C, the set A_C is closed under the same inflations as the original class C.

Lemma 3.7. *Let C be a permutation class. If, for a set of permutations X, the class C is closed under $\cdot[X]$, then A_C is also closed under $\cdot[X]$, and if C is closed under $X[\cdot]$, then so is A_C. Consequently, if C is inflation-closed, then $A_C = C$ and C is 1-amalgamable.*

Proof: Suppose that C is closed under $\cdot[X]$. We can assume that X itself is inflation-closed since if C is closed under $\cdot[X]$, it is also closed under $\cdot[X[X]]$.

Let $\pi \in A_C$ be a 1-amalgamable permutation of order k and let $\rho_1, \dots, \rho_k$ be permutations from X. Our goal is to prove that $\pi[\rho_1, \dots, \rho_k]$ also belongs to A_C. We can assume, without loss of generality, that all ρ_i are actually equal to a single permutation ρ. Otherwise, we could just take $\rho \in X$ that contains every ρ_i (this is possible since X is inflation-closed) and prove the stronger claim that $\pi[\rho, \dots, \rho]$ belongs to A_C. Let us use $\pi[\rho]$ as a shorthand notation for $\pi[\rho, \dots, \rho]$.

It is now sufficient to show that $\pi[\rho]$ belongs to A_C for every $\pi \in A_C$ and $\rho \in X$. Fix a permutation $\tau \in C$ and two embeddings f_1 and f_2 of the singleton permutation into $\pi[\rho]$ and τ. We aim to find a permutation $\sigma \in C$ and two embeddings g_1 and g_2 of $\pi[\rho]$ and τ into σ such that $g_1 \circ f_1 = g_2 \circ f_2$. We can straightforwardly decompose f_1 into an embedding h_1 of the singleton permutation into π, by simply looking to which inflated block order-isomorphic to ρ the image of f_1 belongs, and an embedding h_2 of the singleton permutation into ρ, determined by restricting f_1 only to that copy of ρ. Since π belongs to A_C, there is a permutation σ' with embeddings g'_1 and g'_2 of π and τ such that $g'_1 \circ h_1 = g'_2 \circ f_2$.

Define $\sigma = \sigma'[\rho]$, and view σ as a concatenation of blocks, each a copy of ρ. Let us define mapping g_1 by simply using g'_1 to map blocks of $\pi[\rho]$ to the blocks of σ, each element in $\pi[\rho]$ gets mapped to the same element of the corresponding copy of ρ in σ. Then define mapping g_2 by using g'_2 to map its elements to the blocks of σ and then within the copy of ρ to the single element in the image of h_2. It is easy to see that g_1 and g_2 are in fact embeddings of $\pi[\rho]$ and τ into σ. Also the images of $g_1 \circ f_1$ and $g_2 \circ f_2$ must lie in the same block of σ. And finally these images must be equal since we used h_2 to place the single element from the image of g_2 inside each block of σ.

We now show that if C is closed under $X[\cdot]$ then so is A_C. Fix a permutation $\rho \in X$ of order k, and a k-tuple $\pi_1, \dots, \pi_k$ of permutations from A_C. We will show that $\rho[\pi_1, \dots, \pi_k]$ belongs to A_C.

Fix a permutation $\tau \in C$ and two embeddings f_1 and f_2 of the singleton permutation into $\rho[\pi_1, \ldots, \pi_k]$ and τ. We aim to find a permutation $\sigma \in C$ and two embeddings g_1 and g_2 of $\rho[\pi_1, \ldots, \pi_k]$ and τ into σ such that $g_1 \circ f_1 = g_2 \circ f_2$. We again view $\rho[\pi_1, \ldots, \pi_k]$ as a concatenation of k blocks, the i-th block being order-isomorphic to π_i. Suppose that the image of f_1 is in the j-th block. Let us decompose f_1 into an embedding h_1 of the singleton permutation into ρ whose image is the j-th element of ρ, and an embedding h_2 of the singleton permutation into π_j. Since π_j belongs to A_C, there is a permutation σ' with embeddings g_1' and g_2' of π_j and τ such that $g_1' \circ h_2 = g_2' \circ f_2$.

Define $\sigma = \rho[\pi_1, \ldots, \pi_{j-1}, \sigma', \pi_{j+1}, \ldots, \pi_k]$ and let us define mapping g_1 in the following way. Every block of $\rho[\pi_1, \ldots, \pi_k]$ except for the j-th one gets mapped to the corresponding block of σ, and the j-th block is mapped using the embedding g_1' to the j-th block of σ. Then define mapping g_2 simply by mapping τ to the j-th block of σ using g_2'. It is easy to see that both g_1 and g_2 are in fact embeddings of $\rho[\pi_1, \ldots, \pi_k]$ and τ into σ. Furthermore, the images of $g_1 \circ f_1$ and $g_2 \circ f_2$ both lie in the j-th block of σ. Their equality then follows from the construction since $g_1' \circ h_2 = g_2' \circ f_2$.

It remains to show that if C is inflation-closed then $A_C = C$. But if C is inflation-closed, then it is closed under $\cdot[C]$, so A_C is also closed under $\cdot[C]$. And since A_C trivially contains the singleton permutation, for every $\pi \in C$ we have that $\pi = 1[\pi]$ also belongs to A_C. □

As noted by Jelínek and Valtr [4, Lemma 1.5], it follows from the results of Nešetřil [5] that if a permutation class C is unsplittable then C is also 1-amalgamable. Using the same argument, we get the following stronger proposition relating the classes U_C and A_C.

Proposition 3.8. *Let C be a permutation class, then $U_C \subseteq A_C$.*

Proof: Let π be an unavoidable permutation in C and let τ be a permutation from C. By the definition of U_C, there is a permutation $\sigma \in C$ such that any red-blue coloring of σ has a red copy of τ or a blue copy of π. We claim that σ contains every 1-amalgamation of π and τ. Suppose for a contradiction that there are two embeddings f_1 and f_2 of the singleton permutation 1 into π and τ such that there are no embeddings g_1 and g_2 of π and τ into σ that would satisfy $g_1 \circ f_1 = g_2 \circ f_2$.

Let $f_1(1) = a$ and $f_2(1) = b$. We aim to color the elements of σ to avoid both a red copy of τ and a blue copy of π. We color an element σ_i red if and only if there is an embedding of π which maps π_a to σ_i. Trivially, we cannot obtain a blue copy of π, since we must have colored the image of π_a red. On the other hand, suppose we obtained a red copy of τ. Then the image of τ_b was painted red which means that there is an embedding of π which maps π_a to the same element. We assumed that such a pair of embeddings does not exist, therefore we defined a coloring of σ that contains neither a red copy of τ nor a blue copy of π. □

4 Left-to-right minima

We say that the element π_i *covers the element* π_j if $i < j$ and simultaneously $\pi_i < \pi_j$. The i-th element of a permutation π is then a *left-to-right minimum*, or shortly LR-minimum, if it is not covered by any other element.

Similarly we could define LR-maxima, RL-minima and RL-maxima. However we can easily translate between right-to-left and left-to-right orientation by looking at the reverses of the permutations, and similarly between maxima and minima by looking at the complements of the permutations. Therefore we restrict ourselves to dealing only with LR-minima from now on.

Definition 4.1. Suppose that $\pi \in S_n$ is a permutation with k LR-minima and let $\sigma_1, \ldots, \sigma_k$ be a k-tuple of non-empty permutations. The ***LR-inflation*** of π by the sequence $\sigma_1, \ldots, \sigma_k$ is the permutation resulting from the inflation of the LR-minima of π by $\sigma_1, \ldots, \sigma_k$. We denote this by $\pi\langle\sigma_1, \ldots, \sigma_k\rangle$.

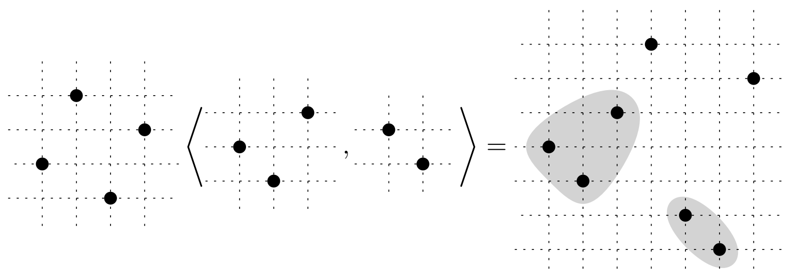

Figure 3: An example of LR-inflation: $2413\langle 213, 21\rangle = 4357216$.

Definition 4.2. We say that a permutation class C is *closed under LR-inflations* if for every $\pi \in C$ with k LR-minima, and for every k-tuple $\sigma_1, \ldots, \sigma_k$ of permutations from C, the LR-inflation $\pi\langle\sigma_1, \ldots, \sigma_k\rangle$ belongs to C. The *closure of C under LR-inflations*, denoted C^{LR}, is the smallest class which contains C and is closed under LR-inflations.

Recall that one can characterize inflation-closed classes by a basis that consists of simple permutations. We can derive a similar characterization in the case of classes closed under LR-inflations. We say that a permutation is ***LR-simple*** if it cannot be obtained by LR-inflations except for the trivial ones. Using the same arguments, it is easy to see that a permutation class is closed under LR-inflations if and only if every permutation in its basis is LR-simple.

4.1 LR-splittability

We aim to define a stronger version of splittability that would help us connect the properties of permutation classes and their LR-closures. A natural way to do that is to consider an operation similar to the regular merge, with LR-minima being shared between both parts.

Definition 4.3. We say that a permutation π is a ***LR-merge*** of permutations τ and σ, if its non LR-minimal elements can be partitioned into two disjoint sequences, such that one of them is, together with the sequence of LR-minima of π, an occurrence of τ, and the other is, together with the sequence of LR-minima of π, an occurrence of σ. For two permutation classes A and B, we write $A \odot_{\mathrm{LR}} B$ for the class of all LR-merges of a permutation from A with a compatible permutation from B. Trivially, $A \odot_{\mathrm{LR}} B$ is again a permutation class.

Note that we can also look at LR-merges as a special red-blue colorings of permutations in which the LR-minima are both blue and red at the same time. Naturally we can use this definition of LR-merge to define LR-splittability in the same way that the concept of regular merge gives rise to the definition of splittability.

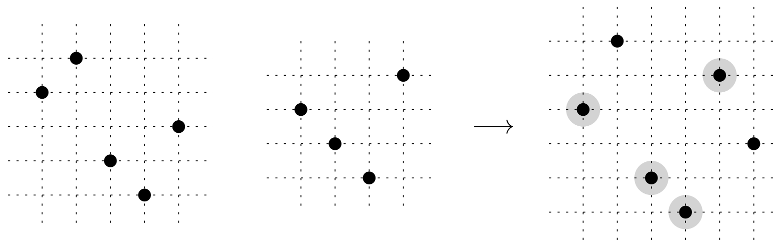

Figure 4: For example one possible LR-merge of 45213 and 3214 is the permutation 462153. The corresponding embedding of 3214 is indicated.

Definition 4.4. We say that a multiset of permutation classes $\{P_1, \dots, P_m\}$ forms a *LR-splitting* of a permutation class C if $C \subseteq P_1 \odot_{\text{LR}} \cdots \odot_{\text{LR}} P_m$. We call P_i the *parts* of the LR-splitting. The LR-splitting is *nontrivial* if none of its parts is a superset of C, and the LR-splitting is *irredundant* if no proper submultiset of $\{P_1, \dots, P_m\}$ forms an LR-splitting of C. A permutation class C is then *LR-splittable* if C admits a nontrivial LR-splitting.

Clearly, every LR-splittable class is splittable. Moreover, some properties of LR-splittability are analogous to the properties of splittability, as shown by the following lemma. We omit the proof as it uses the very same (and easy) arguments as the proof of Lemma 3.1.

Lemma 4.5. *For a class C of permutations, the following properties are equivalent:*

(a) C is LR-splittable.

(b) C has a nontrivial LR-splitting into two parts.

(c) C has an LR-splitting into two parts, in which each part is a proper subclass of C.

(d) C has a nontrivial LR-splitting into two parts, in which each part is a principal class.

Now we can state some of the results connecting splittability and LR-splittability of permutation classes and their LR-closures.

Proposition 4.6. *Let C be a permutation class that is closed under LR inflations. Then C is splittable if and only if C is LR-splittable.*

Proof: Trivially, LR-splittability implies splittability since we can take the corresponding red-blue coloring and simply assign an arbitrary color to each of the LR-minima. Now suppose that C admits splitting $\{D, E\}$ for some proper subclasses D and E. We aim to prove that also $C \subseteq D \odot_{\text{LR}} E$. Let us first show that C contains a permutation τ that belongs neither to D nor to E. From the definition of splittability, there are permutations $\tau_D \in C \setminus D$ and $\tau_E \in C \setminus E$. Define τ as the LR-inflation of τ_D with τ_E, which clearly lies outside both subclasses D and E.

Let us suppose that there is some $\pi \in C$ not belonging to $D \odot_{\text{LR}} E$, i.e., there is no red-blue coloring of π which proves it is an LR-merge of a permutation $\alpha \in D$ and a permutation $\beta \in E$. Let π' be the permutation created by inflating each LR-minimum of π with τ. Since π' belongs to C, it has a regular red-blue coloring with the permutation corresponding to the red elements $\pi'_R \in D$ and the permutation corresponding to the blue elements $\pi'_B \in E$. However there must be both colors in each block created by inflating a LR-minimum of π with τ, and therefore there is a valid red-blue coloring of π that assigns both colors to the LR-minima. □

Finally, we want to show that, under modest assumptions, the LR-splittability of a permutation class implies the LR-splittability (and thus the splittability) of its LR-closure.

Proposition 4.7. *If C, D and E are permutation classes satisfying $C \subseteq D \odot_{\mathrm{LR}} E$, then $C^{\mathrm{LR}} \subseteq D^{\mathrm{LR}} \odot_{\mathrm{LR}} E^{\mathrm{LR}}$. Consequently, if neither D^{LR} nor E^{LR} contain the whole class C, then its closure C^{LR} is LR-splittable into parts D^{LR} and E^{LR}.*

Proof: We will inductively construct a valid red-blue coloring which proves that $C^{\mathrm{LR}} \subseteq D^{\mathrm{LR}} \odot_{\mathrm{LR}} E^{\mathrm{LR}}$. First, any permutation in C^{LR} that cannot be obtained from shorter permutations using LR-inflations must belong to C and we simply use the red-blue coloring that witnesses the inclusion $C \subseteq D \odot_{\mathrm{LR}} E$.

Now take $\pi \in C^{\mathrm{LR}}$ that can be obtained by LR-inflation from shorter permutations as $\pi = \alpha\langle\beta_1, \ldots, \beta_k\rangle$. We can already color the permutation α and all the permutations β_i and we construct a coloring of π in the following way: color the inflated blocks β_i according to the coloring of β_i and the remaining uninflated elements of α get the color according to the coloring of α. It remains to show that the permutation π_R corresponding to the red elements of π belongs to D^{LR} and the permutation π_B corresponding to the blue elements of π belongs to E^{LR}. Since the LR-minima of α are both red and blue, the permutation π_R is an LR-inflation of the red elements of α by the red elements of the permutations β_i. All these permutations belong to D^{LR} and thus their LR-inflation also belongs to D^{LR}. Using the very same argument we can show that π_B belongs to E^{LR}.

It remains to show that the splitting of C^{LR} into D^{LR} and E^{LR} is nontrivial. However that follows from the assumption that neither D^{LR} nor E^{LR} contain the whole class C. □

4.2 LR-amalgamability

Similarly to the situation with LR-splittability we want to describe a property of permutation classes which would imply 1-amalgamability of their respective LR-closures.

Definition 4.8. We say that a permutation class C is *LR-amalgamable* if for any two permutations $\tau_1, \tau_2 \in C$ and any two mappings f_1 and f_2, where f_i is an embedding of the singleton permutation into τ_i and its image is not an LR-minimum of τ_i, there is a permutation $\sigma \in C$ and two mappings g_1 and g_2 such that g_i is an embedding of τ_i into σ, $g_1 \circ f_1 = g_2 \circ f_2$ and moreover g_i preserves the property of being a LR-minimum.

Observe that LR-amalgamability does not imply 1-amalgamability since it does not guarantee 1-amalgamation over LR-minima and conversely, 1-amalgamability does not imply LR-amalgamability because it may not preserve the property of being an LR-minimum. However, we can at least prove that LR-amalgamability implies 1-amalgamability for classes that are closed under LR-inflations. Recall that we actually derived equivalence between LR-splittability and splittability in Proposition 4.6.

Lemma 4.9. *Let C be a permutation class that is closed under LR inflations. If C is LR-amalgamable then C is also 1-amalgamable.*

Proof: Let π_1 and π_2 be arbitrary permutations from C and f_1, f_2 embeddings of the singleton permutation into π_1 and π_2 respectively. If neither of the images of f_1 and f_2 is an LR-minimum of the respective permutation we obtain their 1-amalgamation directly since C is LR-amalgamable.

Now we can assume without loss of generality that the single element in the image of f_1 is a LR-minimum of π_1. We can create the resulting 1-amalgamation by simply inflating this LR-minimum by the permutation π_2. It is then easy to derive the mappings g_1 and g_2 that show it is the desired 1-amalgamation. □

We conclude this section by relating LR-amalgamability of a permutation class and 1-amalgamability of its LR-closure.

Proposition 4.10. *If a permutation class C is LR-amalgamable then its LR-closure C^{LR} is LR-amalgamable and thus also 1-amalgamable.*

Proof: Let $\pi_1, \pi_2 \in C^{\text{LR}}$ be permutations and f_1, f_2 embeddings of the singleton permutation, f_i into π_i such that the image of f_i avoids the LR-minima of π_i. We aim to prove by induction on the length of π_1 and π_2 that there is a corresponding LR-amalgamation of π_1 and π_2. Consider two cases. If neither of the two permutations π_1 and π_2 can be obtained as an LR-inflation of a shorter permutation then they both belong to C. And since C itself is LR-amalgamable they have a desired LR-amalgamation that belongs to C.

Without loss of generality we can now assume that π_1 can be obtained by LR-inflations as $\pi_1 = \alpha\langle\beta_1, \dots, \beta_k\rangle$ where the permutations $\alpha, \beta_1, \dots, \beta_k$ are all strictly shorter than π_1. Again we consider two separate cases. First, assume that the image of the embedding f_1 lies inside the block corresponding to the j-th inflated LR-minimum of α, which is order-isomorphic to β_j. From induction we get a LR-amalgamation σ of β_j and π_2 for the embeddings f_1' and f_2, where f_1' is the embedding f_1 restricted to the inflated block of β_j. Observe that the permutation $\alpha\langle\beta_1, \dots, \beta_{j-1}, \sigma, \beta_{j+1}, \dots, \beta_k\rangle$ is precisely the LR-amalgamation of π_1 and π_2 we were looking for.

Finally we have to deal with the situation when the image of the embedding f_1 lies outside of the blocks corresponding to the inflated LR-minima of π_1. We can obtain from induction a LR-amalgamation σ of α and π_2 for the embeddings f_1'' and f_2, where f_1'' is the embedding f_1 restricted to the permutation α. Let g_1 be the corresponding embedding of α into σ that preserves the LR-minima. We construct the desired LR-amalgamation of π_1 and π_2 in the following way: take σ and for every LR-minimum of α inflate its image under g_1 with the corresponding permutation β_i. The resulting permutation is clearly a 1-amalgamation of π_1 and π_2, and it also preserves the LR-minima.

Lemma 4.9 implies that C^{LR} is also 1-amalgamable. □

5 Main result

Now we are ready to prove that 1-amalgamability and unsplittability are not equivalent by exhibiting as a counterexample the LR-closure of Av(123). First, let us show that this class actually has a nice basis consisting of only two patterns.

Proposition 5.1. *The class Av$(1423, 1342)$ is the closure of Av(123) under LR-inflation.*

Proof: First, let us show that any permutation from the LR-closure of Av(123) avoids both 1423, 1342. Because both of these patterns contain 123, they would have to be created by the LR-inflations. However,

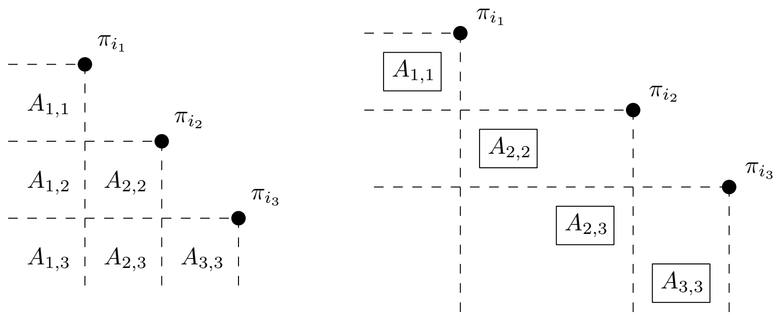

Figure 5: Partition of a general permutation with 3 RL-maxima into the sets $A_{j,k}$ and an example how the non-empty sets might look for some $\pi \in \text{Av}(1423, 1342)$.

that is not possible since there is no nontrivial interval in either 1423 or 1342 which contains the minimum element.

Now, let π be a permutation from Av$(1423, 1342)$. We will show by induction that this permutation can be obtained by a repeated LR-inflation of permutations from Av(123). If π does not contain 123 the statement is trivially true. Otherwise, consider the set of the right-to-left maxima of π. We want to show that the remaining elements of π can be split into a descending sequence of intervals. If this holds then we can get π as an LR-inflation of a 123-avoiding permutation by permutations order-isomorphic to the intervals. And by induction these shorter permutations can be obtained as repeated LR-inflations of 123-avoiding permutations.

Let us show that there is no occurrence of the pattern 132 that maps only the letter 2 on an RL-maximum. For a contradiction suppose we have such an occurrence and a corresponding embedding f of 132 into π. Then there must be an element covered by $\pi_{f(3)}$ since it is not an RL-maximum, i.e., an element π_k such that $k > f(3)$ and $\pi_k > \pi_{f(3)}$. However, π restricted to these four indices would form the pattern 1342. Using the same argument, we can also show that there is no occurrence of the pattern 132 which maps only the letter 3 on an RL-maximum as we would get an occurrence of the pattern 1423 together with the RL-maximum covered by the image of 2.

And finally, we conclude by showing that the elements of π that are not RL-maxima can indeed be split into a descending sequence of intervals. Let $I = \{i_1, \ldots, i_m\}$ be the index set of the RL-maxima of π and furthermore define $i_0 = 0$ and $\pi_0 = n + 1$. Let us represent the remaining elements of π as a set A of $n - m$ points on a plane

$$A = \{(i, \pi_i) \mid \pi_i \text{ is not an RL-maximum of } \pi\}.$$

We define a partition of A into sets $A_{j,k}$ for any $1 \leq j < k \leq m$

$$A_{j,k} = \{(x, y) \mid (x, y) \in A \text{ and } i_{j-1} < x < i_j \text{ and } \pi_{i_k} < y < \pi_{i_{k-1}}\}.$$

For any j, k and l, every element of $A_{j,k}$ is larger than all the elements of $A_{j+1,l}$ in the second coordinate since otherwise we would get a 132 occurrence with the letter 3 mapped to π_{i_j}. Similarly for any j, k and l, every element of $A_{j,k}$ is to the left of all the elements of $A_{l,k+1}$ as otherwise we would get a 132 occurrence with the letter 2 mapped to π_{i_k}. This transitively implies that all non-empty sets $A_{j,k}$ correspond to a sequence of descending intervals. □

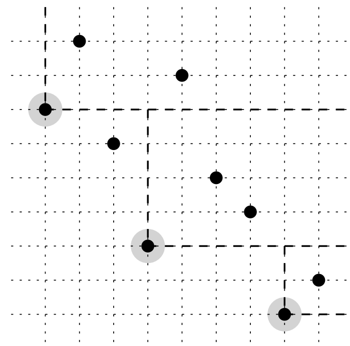

Figure 6: For example the 123-avoiding permutation 796385412 with the non-minimal elements split into three different runs.

In order to show that $\mathrm{Av}(1423, 1342)$ is splittable, we shall first prove the LR-splittability of $\mathrm{Av}(123)$ and then apply the results we have obtained in Subsection 4.1.

Lemma 5.2. *The class Av*(123) *is LR-splittable, and more precisely, it satisfies*

$$Av(123) \subseteq Av(463152) \odot_{\mathrm{LR}} Av(463152).$$

Proof: Let π be a permutation from $\mathrm{Av}(123)$. Clearly π is a merge of two descending sequences, its LR-minima and the remaining elements. The idea is to decompose the non-minimal elements into runs such that for every run there is a specific LR-minimum covering each element of the run but covering none from the following run. This can be done easily by the following greedy algorithm. In one step of the algorithm, let π_i be the first non-minimal element which was not used yet and let j be the maximum integer such that π_j is an LR-minimum covering π_i. The next run then consists of all non-minimal elements starting from π_i that are covered by π_j.

We color each run blue or red such that adjacent runs have different colors. We obtained a red-blue coloring of the non-minimal elements and it only remains to check whether the monochromatic permutations form a proper subclass of $\mathrm{Av}(123)$. Observe that the first elements of two adjacent runs cannot be covered by a single LR-minimum, which implies that two elements from different non-adjacent runs cannot be covered by a single LR-minimum. By this observation, in the monochromatic permutations π_B and π_R any two elements covered by the same LR-minimum must belong to the same run.

We claim that a monochromatic copy of the pattern $463152 \in \mathrm{Av}(123)$ can never be created this way. Assume for contradiction that there is a permutation $\pi \in \mathrm{Av}(123)$ on which the algorithm creates a monochromatic copy of 463152 and let f be the corresponding embedding of 463152 into π. Observe that every LR-minimum of 463152 is covering some other element and therefore f must preserve the property of being an LR-minimum, otherwise we would get an occurrence of the pattern 123. Following our earlier observations, the elements $\pi_{f(6)}$, $\pi_{f(5)}$ and $\pi_{f(2)}$ must fall into the same run since $\pi_{f(5)}$ shares LR-minima with both of the other two elements. And because elements of the same run are covered by a single LR-minimum, there is an LR-minimum π_i covering $\pi_{f(6)}$ and $\pi_{f(2)}$. However, π_i must then also cover $\pi_{f(3)}$ which contradicts the fact that $\pi_{f(3)}$ itself is an LR-minimum of π. □

Corollary 5.3. *The class Av*$(1423, 1342)$ *is splittable.*

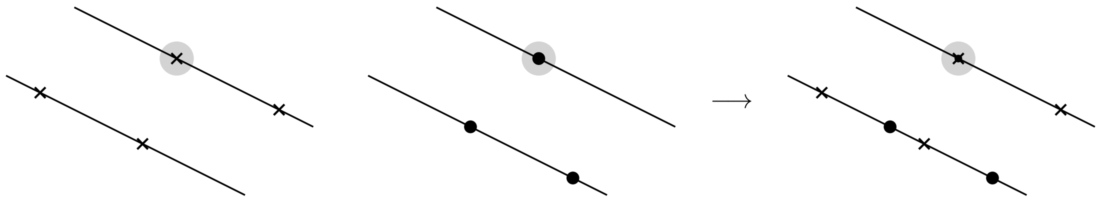

Figure 7: Example of two permutations 3142 and 231 drawn from two parallel lines with highlighted embeddings of the singleton permutation and their LR-amalgamation 532614.

Proof: In the previous Lemma 5.2 we showed that $\mathrm{Av}(123)$ is LR-splittable, more precisely that $\mathrm{Av}(123) \subseteq \mathrm{Av}(463152) \odot_{\mathrm{LR}} \mathrm{Av}(463152)$. Since the permutation 463152 is LR-simple, we get the splittability of $\mathrm{Av}(123)^{\mathrm{LR}}$ from Proposition 4.7. Finally, owing to Proposition 5.1, we know that $\mathrm{Av}(123)^{\mathrm{LR}}$ and $\mathrm{Av}(1423, 1342)$ are in fact identical. □

Our final task is to show that $\mathrm{Av}(1423, 1342)$ is 1-amalgamable by proving the LR-amalgamability of $\mathrm{Av}(123)$. In order to do that we will use the following result which is due to Waton [6]. Note that Waton in fact proved the equivalent claim for parallel lines of positive slope and the permutation class $\mathrm{Av}(321)$.

Proposition 5.4 (Waton [6])**.** *The class of permutations that can be drawn on any two parallel lines of negative slope is Av*(123).

Lemma 5.5. *The class Av*(123) *is LR-amalgamable.*

Proof: Fix arbitrary two parallel lines of negative slope in the plane. Let π_1 and π_2 be permutations avoiding 123 and f_1 and f_2 be mappings where f_i is an embedding of the singleton permutation into π_i and its image is not an LR-minimum of π_i. According to Proposition 5.4 both π_1 and π_2 can be drawn from our fixed parallel lines. Fix sets of points A_1 and A_2 which lie on these lines whose corresponding respective permutations are π_1 and π_2. Moreover, we can choose the sets such that the elements in the images of f_1 and f_2 share the same coordinates. Otherwise we could translate one of the sets in the direction of the lines to align these two points. Finally, if a point $x \in A_1$ and a point $y \in A_2$ share one identical coordinate we can move x a little bit in the direction of the lines without changing the permutation corresponding to the set A_1.

We may easily see that the permutation corresponding to the union $A_1 \cup A_2$ with the natural mappings of π_1 and π_2 is the desired LR-amalgamation of π_1 and π_2. □

Applying Proposition 4.10, we get the desired result that the class $Av(1423, 1342)$ is indeed 1-amalgamable.

Corollary 5.6. *The class Av*$(1423, 1342)$ *is 1-amalgamable.*

6 Further directions

Using our results about LR-inflations, we proved that a single class $\mathrm{Av}(1423, 1342)$ is both 1-amalgamable and splittable. Naturally, the same holds for its three symmetrical classes, i.e. $\mathrm{Av}(3241, 2431)$, $\mathrm{Av}(4132, 4213)$ and $\mathrm{Av}(2314, 3124)$, since both splittability and 1-amalgamability is preserved when looking at the reverses or complements of the permutations. However, the question remains whether these results can be used to find more classes that are both 1-amalgamable and splittable or even infinitely many such classes. It would be particularly interesting to find other such classes with small basis.

Our method of obtaining a splittable 1-amalgamable class was based on the notion of LR-inflations, and the related concepts of LR-amalgamations and LR-splittings. These notions can be generalized to a more abstract setting as follows: suppose that we partition every permutation π into 'inflatable' and 'non-inflatable' elements, in such a way that for any embedding of a permutation σ into π, the non-inflatable elements of σ are mapped to non-inflatable elements of π. We might then consider admissible inflations of π (in which only the inflatable elements can be inflated), admissible splittings of π (which are based on two-colorings in which each inflatable element receives both colors), as well as admissible amalgamations (where we amalgamate by identifying non-inflatable elements, and the amalgamation must preserve the inflatable elements of the two amalgamated permutations). In this paper, we only considered the special case when the inflatable elements are the LR-minima; however, the main properties of LR-inflations, LR-splittings and LR-amalgamations extend directly to the more abstract setting.

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