Discrete Mathematics and Theoretical Computer Science DMTCS vol. **20**:1, 2018, #5

*Hitting minors, subdivisions, and immersions in tournaments**

Jean-Florent Raymond

Technische Universität Berlin, Germany

received 13th Feb. 2017, *revised 20th Dec. 2017*, *accepted 11th Jan. 2018*.

The Erdős–Pósa property relates parameters of covering and packing of combinatorial structures and has been mostly studied in the setting of undirected graphs. In this note, we use results of Chudnovsky, Fradkin, Kim, and Seymour to show that, for every directed graph H (resp. strongly-connected directed graph H), the class of directed graphs that contain H as a strong minor (resp. butterfly minor, topological minor) has the vertex-Erdős–Pósa property in the class of tournaments. We also prove that if H is a strongly-connected directed graph, the class of directed graphs containing H as an immersion has the edge-Erdős–Pósa property in the class of tournaments.

Keywords: directed Erdős–Pósa property, packing and covering, topological minors, immersions, tournaments.

1 Introduction

We are concerned in this note with the Erdős–Pósa property in the setting of directed graphs. This property, which has mostly been studied on undirected graphs, is originated from the following classic result by Erdős and Pósa (1965): there is a function $f\colon \mathbb{N} \to \mathbb{N}$, such that, for every (undirected) graph G and every positive integer k, one of the following holds: (a) G contains k vertex-disjoint cycles; or (b) there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ and such that $G \setminus X$ has no cycle. This theorem expresses a duality between a parameter of *packing*, the maximum number of vertex-disjoint cycles in a graph, and a parameter of *covering*, the minimum number of vertices that intersect all cycles. This initiated a research line aimed at providing conditions for this property to hold, for various combinatorial objects. Formally, we say that a class of graphs $\mathcal{H}$ has the *Erdős–Pósa property* if there is a function $f\colon \mathbb{N} \mapsto \mathbb{N}$ such that, for every positive integer k and every graph G (referred to as the *host graph*) one of the following holds:

- G has k vertex-disjoint subgraphs that are isomorphic to members of $\mathcal{H}$; or
- there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ and such that $G \setminus X$ has no subgraph isomorphic to a member of $\mathcal{H}$.

*This work has been done while the author was affiliated to LIRMM, Montpellier, France and Institute of Computer Science, University of Warsaw, Poland and was supported by the grant PRELUDIUM 2013/11/N/ST6/02706 of the Polish National Science Center (NCN).

ISSN 1365–8050

The Erdős–Pósa Theorem states that the class of cycles has this property. One of the most general extensions of the Erdős–Pósa Theorem is certainly the following byproduct of the Graph Minors series: Robertson and Seymour (1986) proved that the class of graphs that contain H as a minor have the Erdős–Pósa property iff H is planar.

On the other hand, some classes like odd cycles fail to have the Erdős–Pósa property, as proved by Dejter and Neumann-Lara (1987). When this happens, one can consider particular classes of host graphs. In this direction, Reed (1999) proved that odd cycles have the Erdős–Pósa property in planar graphs.

A natural variant of the Erdős–Pósa property is to change, in the definition, vertex-disjoint subgraphs for edge-disjoint ones and sets of vertices for sets of edges. It has been proved that the Erdős–Pósa Theorem also holds in this setting (see (Diestel, 2005, Exercise 5 of Section 9)). Other results have been obtained about this variant, less than on the vertex variant, though. At this point we have to stress that, if the vertex and edge variants of the Erdős–Pósa property have close definitions, one cannot in general deduce one from the other. We refer the reader to the surveys of Reed (1997) and Raymond and Thilikos (2017) for more details about the Erdős–Pósa property.

In the setting of directed graphs however, few results are known. Until recently, the largest class of directed graphs that has been studied under the prism of the Erdős–Pósa property was the class of directed cycles, see Reed et al. (1996); Reed and Shepherd (1996); Guenin and Thomas (2011); Seymour (1996); Havet and Maia (2013). It is worth noting that, besides its combinatorial interest, the Erdős–Pósa property in directed graphs found applications in bioinformatics and in the study of Boolean networks Aracena et al. (2017, 2004). We consider here finite directed graphs (*digraphs*) that may have multiple arcs, but not loops and we respectively denote by $V(G)$ and $E(G)$ the set of vertices and the multiset of arcs of a digraph G. A digraph G is said to be *strongly-connected* if it has at least one vertex and, for every $u, v \in V(G)$, there is a directed path from u to v. In particular the digraph with one vertex is strongly-connected. The most general result about the Erdős–Pósa property in digraphs is certainly the following directed counterpart of the aforementioned results of Robertson and Seymour.

Theorem 1 (Amiri et al. (2016))**.** *Let H be a strongly-connected digraph that is a butterfly minor (resp. topological minor) of a cylindrical grid.*[(i)] *There is a function $f\colon \mathbb{N} \to \mathbb{N}$, such that for every digraph G and every positive integer k, one of the following holds:*

- *G has k vertex-disjoint subdigraphs, each having H as a butterfly minor (resp. topological minor); or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ does not have H as a butterfly minor (resp. topological minor).*

On the other hand, Amiri et al. (2016) proved that Theorem 1 does not hold for the strongly-connected digraphs H that do not satisfy the conditions of its statement. It seems therefore natural to ask under what restrictions on the host digraphs the above result could be true for every strongly-connected digraph, in the same spirit as the aforementioned result of Reed.

The purpose of this note is twofold: obtaining new Erdős–Pósa type results on directed graphs and providing evidence that techniques analogues to those used in the undirected case may be adapted to the directed setting. In particular, we describe conditions on the class of host digraphs so that Theorem 1

[(i)] The notions of butterfly minor and topological minor will be defined in a forthcoming paragraph. We refer the reader to Amiri et al. (2016) for a definition of the cylindrical directed grid.

holds for every strongly-connected digraph H. Before we formally state our results, let us introduce some terminology.

Several directed counterparts of the notion of minor have been introduced in the literature. An arc (u, v) of a digraph is said to be *contractible* if either it is the only arc with head v, or it is the only arc with tail u. Following Johnson et al. (2001) and Kim and Seymour (2015), we say that a digraph H is a *butterfly minor* (resp. *strong minor*) of a digraph G if a digraph isomorphic to H can be obtained from a subdigraph of G by contracting contractible arcs (resp. contracting strongly-connected subdigraphs to single vertices). Notice that these notions are incomparable. A motivation for these definitions is that taking (butterfly or strong) minors does not create directed cycles. Unlike minors, immersions and topological minors are concepts that are easily extended to the setting of directed graphs as they can be defined in terms of paths. We say that a digraph H is a *topological minor* of a digraph G if there is a subdigraph of G that can be obtained from a digraph isomorphic to H by replacing arcs by directed paths (in the same direction) that do not share internal vertices. If we allow these paths to share internal vertices but not arcs, then we say that H is an *immersion* of G. Observe that every topological minor is a butterfly minor. However, as often with the Erdős–Pósa property, this does not allow us in general to deduce an Erdős–Pósa-type result about the one relation from a result about the other one.

Our results hold on superclasses of the extensively studied class of *tournaments*, that are all orientations of undirected complete graphs. For $s \in \mathbb{N}$, a n-vertex digraph is *s-semicomplete* if every vertex v has at least $n - s$ (in- and out-) neighbors. A *semicomplete* digraph is a 0-semicomplete digraph. Note that a semicomplete digraph is not necessarily a tournament as it may have multiple egdes between a pair of vertices. These classes generalize the class of tournaments. Our contributions are the following two theorems.

Theorem 2. *For every digraph (resp. strongly-connected digraph) H and every $s \in \mathbb{N}$, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every s-semicomplete digraph G and every positive integer k, one of the following holds:*

- *G has k vertex-disjoint subdigraphs, each containing H as a strong minor (resp. butterfly minor, topological minor); or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ does not contain H as a strong minor (resp. butterfly minor, topological minor).*

Theorem 3. *For every strongly-connected digraph H on at least two vertices, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every semicomplete digraph G and every positive integer k, one of the following holds:*

- *G has k arc-disjoint subdigraphs, each containing H as an immersion; or*
- *there is a set $X \subseteq E(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ does not contain H as an immersion.*

Theorem 2 and Theorem 3 can be easily extended to finite families of graphs H, as noted in their proofs. These theorems deal with the two variants of the Erdős–Pósa property: the first one is related to vertex-disjoint subdigraphs and sets of vertices (vertex version), whereas the second one is concerned with arc-disjoint subdigraphs and sets of arcs (arc version). In Theorem 3, the requirement on the order of H is necessary as we cannot cover an arcless subdigraph (as the one-vertex digraph) with arcs. Our proofs rely on exclusion results for the parameters of cutwidth and pathwidth, that are stated in the sections where they are used.

The techniques that we use are originated from the undirected setting, where they have been repeatedly applied (see for instance (Robertson and Seymour, 1986, (8.8)) and (Fiorini et al., 2013, Lemma 2.3)). They deal with structural decompositions like tree decompositions or tree-cut decompositions and their associated widths and can be informally described as follows. If the considered host graph has large width, then, using a structural result, we can immediately conclude that it contains several disjoint subgraphs of the desired type. Otherwise, the graph admits a structural decomposition of small width, that can be used to find a small set of vertices/edges covering all such subgraphs (see (Raymond and Thilikos, 2017, Theorem 3.1) for an unified presentation in undirected graphs). Similar ideas have been used in the context of directed graphs in the proof of Theorem 1. With this note, we provide more examples of cases where the techniques used in the undirected setting appear useful when dealing with digraphs.

2 Hitting minors and subdivisions

This section is devoted to the proof of Theorem 2. For every $k \in \mathbb{N}$ and every graph H, we denote by $k \cdot H$ the disjoint union of k copies of H. The structural decompositions that we use in this section are path decompositions. Formally, a *path-decomposition* of a digraph G is a sequence $(X_1, \dots, X_r)$ of subsets of $V(G)$ satisfying the following properties:

(i) $V(G) = \bigcup_{i=1}^{r} X_i$

(ii) for every arc $(u, v) \in E(G)$, there are integers i and j with $1 \leq j \leq i \leq r$ and such that $u \in X_i$ and $v \in X_j$;

(iii) for every $i, j \in \{1, \dots, r\}$, if a vertex $u \in V(G)$ belongs to X_i and X_j, then it also belongs to X_k for every $k \in \{i, \dots, j\}$.

The sets $\{X_i\}_{i \in \{1,\dots,r\}}$ are called *bags* of the path-decomposition. Intuitively, item (ii) asks that every arc of G either have its endpoints in some bag, or is oriented "backwards". The *width* of the above path-decomposition is defined as $\max_{i \in \{1,\dots,r\}} |X_i| - 1$. The *pathwidth* of G is the minimum width over all path-decompositions of G. The following properties of pathwidth are crucial in our proofs.

Theorem 4 ((Kim, 2013, Theorem 2.2.7), (Fradkin and Seymour, 2013, (1.1)), (Kim and Seymour, 2015, (1.4)))**.** *For every digraph H, there is a positive integer w such that every semicomplete G that has pathwidth more than w contains H as a strong minor, butterfly minor and topological minor.*

Theorem 5 ((Kitsunai et al., 2015, Theorem 2))**.** *For every $s, w \in \mathbb{N}$, there is a positive integer w' such that every s-semicomplete digraph with pathwidth at least w' has a subdigraph that is semicomplete and is of pathwidth at least w.*

Corollary 1. *For every $s \in \mathbb{N}$ and every digraph H, there is a constant $\zeta_{s,H}$ such that every s-semicomplete digraph that has pathwidth at least $\zeta_{s,H}$ contains H as a strong minor, butterfly minor and topological minor.*

A classic result states that if a collection of subpaths of a path does not contain more than k vertex-disjoint elements, then there is a set of k vertices meeting all the subpaths (see Gyárfás and Lehel (1969)). We use here the following generalization of the above statement, due to Alon.

Lemma 1 (Alon (1998))**.** *Let P be a path (undirected) and let $\mathcal{P}$ be a collection of subgraphs of P that does not contain $k+1$ pairwise vertex-disjoint members. Then there is a set of $2p^2k$ vertices of P meeting every element of $\mathcal{P}$, where p is the maximal number of connected components of a graph in $\mathcal{P}$.*

A *strongly-connected component* of a digraph is a maximal subdigraph that is strongly-connected. Observe that a single vertex may be a strongly-connected component. If a subdigraph of a digraph G is isomorphic to some member of a digraph class $\mathcal{H}$, we call it an $\mathcal{H}$*-subdigraph* of G.

Lemma 2. *Let $\mathcal{H}$ be a (possibly infinite) class of digraphs with at most p strongly-connected components. For every digraph G and a positive integer k, one of the following holds: (a) G contains k pairwise vertex-disjoint $\mathcal{H}$-subdigraphs; or (b) there is a set $X \subseteq V(G)$ with $|X| \leq 2p^2(k-1)(\mathbf{pw}(G)+1)$ such that $G \setminus X$ has no $\mathcal{H}$-subdigraph.*

Proof: We proceed by induction on $k \in \mathbb{N}$. The base case $k = 1$ is trivial. Let us prove the statement of the lemma for $k > 1$ assuming that it holds for all lower values of k (induction step). For this we consider a digraph G such that (a) does not hold. Let $(X_1, \dots, X_l)$ be a path decomposition of G of minimum width. Let P be the undirected path on vertices $v_1, \dots, v_l$, in this order. For every subdigraph F of G, we set:

$$\begin{aligned} A_F &= \{i \in \{1, \dots, l\},\ V(F) \cap X_i \neq \emptyset\} \quad \text{and} \\ P_F &= P\left[\{v_i,\ i \in A_F\}\right]. \end{aligned}$$

In other words, A_F is the set of indices of the bags met by F and P_F is the subgraph of P induced by the vertices with these indices. For every $\mathcal{H}$-subdigraph H of G, we consider the subgraph P_H of P. Let us denote by $\mathcal{P}$ the class of all such graphs (which is finite). Notice that for every pair F, F' of subdigraphs of G, if P_F and $P_{F'}$ are vertex-disjoint, then so are F and F'. Using our initial assumption on G, we deduce that $\mathcal{P}$ does not contain k pairwise vertex-disjoint members. Besides, if F is strongly-connected, then P_F is connected. Moreover, if F has at most p strongly-connected components, then P_F has at most p connected components. Hence, every member of $\mathcal{P}$ has at most p connected components.

By the virtue of Lemma 1, there is a set Q of $2p^2(k-1)$ vertices of P such that $P \setminus Q$ does not contain a subgraph of $\mathcal{P}$. Let $X = \bigcup_{i \in \{j \in \{1,\dots,l\},\ v_j \in Q\}} X_i$. Let us show that X satisfies the requirements of (b). By contradiction, we assume that $G \setminus X$ has an $\mathcal{H}$-subdigraph H. Then $P_H \in \mathcal{P}$. Let v_i be a vertex of $V(P_H) \cap Q$, which, by definition of Q, is not empty. Then $X_i \subseteq X$ and $V(H) \cap X_i \neq \emptyset$. This contradicts the fact that H is a subdigraph of $G \setminus X$. Consequently, X is as required. As it is the union of $2p^2(k-1)$ bags of an optimal path decomposition of G, we have $|X| \leq 2p^2(k-1)(\mathbf{pw}(G)+1)$. This concludes the proof. □

We would like to mention that a weaker form of Lemma 2 where $\mathcal{H}$ consists of digraphs whose connected components are strongly-connected can be obtained by adapting the ideas used by (Robertson and Seymour, 1986, (8.8)), with a dependency in p that is linear instead of quadratic.

Lemma 3. *Let G be a digraph and let H be the digraph obtained by contracting one strongly-connected subdigraph S of G to one single vertex v_S. Then H and G have the same number of strongly-connected components.*

Proof: Let f be the map such that, for every $C \subseteq V(H)$ that induces a strongly-connected component,

$$f(C) = \begin{cases} C & \text{if } v_S \notin C \\ (C \setminus \{v_S\}) \cup S & \text{otherwise} \end{cases}$$

Let C be a subset of $V(H)$ that induces a strongly-connected component and let us show that $f(C)$ induces a strongly-connected subdigraph of G. For this, we show that, for any $x, y \in f(C)$, there is a directed path from x to y. If $x, y \in S$, this is true since in this case, $S \subseteq f(C)$ and $G[S]$ is strongly-connected. If none of x, y belongs to S, they are both vertices of C as well. Let $v_0 \dots v_l$ be a directed path from $x = v_0$ to $y = v_l$ in H. Notice that $v_0, \dots, v_l \in C$. If v_S does not belong to this path, then this is a path of G as well and we are done. Otherwise, let i be such that $v_S = v_i$. By definition of H, there are arcs (v_{i-1}, u) and (u', v_{i+1}) in G, for some $u, u' \in S$. As $G[S]$ is strongly-connected, it contains a directed path Q from u to u'. Therefore, concatenating $v_0 \dots v_{i-1}u$, Q, and $u'v_{i+1} \dots v_l$ yields a path from x to y. The case where exactly one of x, y belongs to S is similar. Consequently, $f(C)$ induces a strongly-connected subdigraph in G.

Let us now show that $f(C)$ is a strongly-connected component. By contradiction, let us assume that there is in G a directed walk $u_0 \dots u_l$ with $l > 1$ such that $\{u_0 \dots u_l\} \cap f(C) = \{u_0, u_l\}$. If $S \cap (f(C) \cup \{u_0 \dots u_l\}) = \emptyset$ then C does not induce a strongly-connected component of H, a contradiction. If $S \cap f(C) = \emptyset$, then $f(C) = C$ and then G has a path (that is a subpath of $u_0 \dots u_l$) from a vertex of $f(C)$ to one of S and vice-versa. Therefore, there is in H a path from a vertex of C to v_S and vice-versa, which contradicts the definition of C. Thus S intersects $f(C)$: by definition of f we have $S \subseteq f(C)$ and $v_S \in C$. We deduce that $P = u_0 \dots u_l$ (or $P = v_S, u_1 \dots u_l$, resp. $P = u_0 \dots u_{l-1}, v_S$ if $u_0 \in S$, resp. $u_l \in S$) is an oriented walk of H on at least 3 vertices that has its endpoints in C and contains vertices that do not belong to C. This is not possible since C induces a strongly-connected component of H. We deduce that $f(C)$ is a strongly-connected component of G.

The function f is clearly injective. Let us show that it is surjective. Let now $C \subseteq V(G)$ be a strongly-connected component of G. If C contains a vertex of S, then $S \subseteq C$ as C is a maximal strongly-connected subdigraph and S is strongly-connected. In this case observe that $f(C \setminus S) \cup \{v_S\} = C$. Otherwise, $C \cap S = \emptyset$ and $f(C) = C$. Strongly-connected components of G are in bijection with those of H, hence they are equally many. $\square$

Corollary 2. *Let H be a digraph and let G be a subdigraph-minimal digraph containing H as a strong minor. Then H and G have the same number of strongly-connected components.*

Lemma 4. *For every (possibly infinite) family $\mathcal{H}$ of digraphs with bounded number of strongly-connected components and every $s \in \mathbb{N}$, there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that, for every s-semicomplete digraph G and every positive integer k, one of the following holds: (a) G contains k vertex-disjoint subdigraphs, each having a digraph of $\mathcal{H}$ as a strong minor; or (b) there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ contains no digraph of $\mathcal{H}$ as a strong minor.*

Proof: We prove the lemma for $f(k) = 2p^2(k-1)\zeta_{s,k\cdot H}$, where p denotes the maximum number of strongly-connected components of a digraph in $\mathcal{H}$. Let us assume that (a) does not hold (otherwise we are done). Let $H \in \mathcal{H}$. According to Corollary 1, we have $\mathbf{pw}(G) < \zeta_{s,k\cdot H}$. Let $\hat{\mathcal{H}}$ be the class of all subdigraph-minimal digraphs containing a digraph of $\mathcal{H}$ as a strong minor. Observe that G has a digraph of $\mathcal{H}$ as a strong minor iff it has a subgraph isomorphic to a digraph in $\hat{\mathcal{H}}$. Also, according to Corollary 2, the digraphs in $\hat{\mathcal{H}}$ have at most p strongly-connected components. We can now apply Lemma 2 and obtain a set X of at most $2p^2(k-1)\zeta_{s,k\cdot H}$ vertices such that $G \setminus X$ contains no digraph of $\mathcal{H}$ as a strong minor, that is, item (b). This concludes the proof. $\square$

In general, subdigraph-minimal digraphs containing a digraph H as a butterfly minor (resp. topological minor) may have more strongly-connected components than H. Therefore we focus on strongly-connected digraphs where the following result plays the role of Lemma 3.

Lemma 5. *Let G be a strongly-connected digraph and let H be the digraph obtained by contracting a contractible arc (s,t) of G. Then H is strongly-connected.*

Proof: In the case where H is a single vertex, it is strongly-connected and we are done. So we now assume that H has at least two vertices. Towards a contradiction, let us assume that there are two vertices $x, y \in V(H)$ such that there is a directed path $v_1 \dots v_l$ from $x = v_1$ to $y = v_l$ in G but not in H. As G and H differ only by the contraction of (s,t), there are distinct $i, j \in \{1, \dots, l\}$ such that $s = v_i$ and $t = v_j$. If $i < j$, then $v_1 \dots v_{i-1} v_j \dots v_l$ is a directed path from x to y in H, a contradiction. Let us now assume that $i > j$. In order to handle the case where $i = l$ or $j = 0$, we observe that since G is strongly-connected, there are arcs (v_0, v_1) and (v_l, v_{l+1}) (for some vertices v_0, v_{l+1} that may belong to the path we consider). Now, v_i is the tail of the two arcs (v_i, v_j) and (v_i, v_{i+1}) and v_j is the head of the two arcs (v_i, v_j) and (v_{j-1}, v_j), which contradicts the contractibility of (s,t). Therefore, H is strongly-connected. □

Corollary 3. *Let H be a digraph whose connected components are strongly-connected and let G be a subdigraph-minimal digraph containing H as a butterfly minor (resp. topological minor). Then H and G have the same number of strongly-connected components.*

Lemma 6. *For every finite family $\mathcal{H}$ of digraphs whose connected components are strongly-connected and every $s \in \mathbb{N}$, there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that, for every s-semicomplete digraph G and every positive integer k, one of the following holds: (a) G contains k vertex-disjoint subdigraphs, each having a digraph of $\mathcal{H}$ as a butterfly minor (resp. topological minor); or (b) there is a set $X \subseteq V(G)$ with $|X| \le f(k)$ such that $G \setminus X$ contains no digraph of $\mathcal{H}$ as a butterfly minor (resp. topological minor).*

Proof: We prove the lemma for $f(k) = 2p^2(k-1)\zeta_{s,k \cdot H}$. This proof is similar to that of Lemma 4. Again, we can assume that (a) does not hold and deduce $\mathbf{pw}(G) < \zeta_{s,k \cdot H}$ from Corollary 1, for some $H \in \mathcal{H}$. We denote by p the maximum number of connected components of a digraph in $\mathcal{H}$ and by $\hat{\mathcal{H}}$ the class of all subdigraph-minimal digraphs containing a digraph of $\mathcal{H}$ as a butterfly minor (resp. topological minor). The digraphs in $\hat{\mathcal{H}}$ have at most p strongly-connected components, according to Corollary 3. We now apply Lemma 2 and obtain a set X of at most $2p^2(k-1)\zeta_{s,k \cdot H}$ vertices satisfying item (b). □

The part of Theorem 2 related to strong minors is a consequence of Lemma 4 and Corollary 2. The part related to butterfly minors and topological minors follows from Lemma 6.

3 Hitting immersions

This section is devoted to the proof of Theorem 3. For every two subsets $X, Y \subseteq V(G)$, we denote by $E_G(X,Y)$ the set of arcs of G of the form (x,y) with $x \in X$ and $y \in Y$. Recall that an *$\mathcal{H}$-subdigraph* of a digraph G is any subdigraph of G that is isomorphic to some digraph in $\mathcal{H}$. The parameter that plays a major role in this section is cutwidth. If G is a digraph on n vertices, the *width* of an ordering $v_1 \dots, v_n$ of its vertices is defined as

$$\max_{i \in \{2,\dots,n\}} \left| E_G(\{v_1, \dots, v_{i-1}\}, \{v_i, \dots, v_n\}) \right|.$$

The *cutwidth* of G, that we write $\mathbf{ctw}(G)$, is the minimum width over all orderings $V(G)$. Intuitively, a digraph that has small cutwidth has an ordering where the number of "left-to-right" arcs is small. The following result plays a similar role as Theorem 4 in the previous section.

Theorem 6 ((Chudnovsky et al., 2012, (1.2))). *For every digraph H, there is a positive integer η_H such that every semicomplete digraph G that has cutwidth at least η_H contains H as an immersion.*

Proof Proof of Theorem 3.: Let $\mathcal{H}$ be the class of all subdigraph-minimal digraphs containing H as an immersion and observe that these digraphs are strongly-connected. Again, G has contains H as an immersion iff it has an $\mathcal{H}$-subdigraph.

According to Theorem 6, we are done if $\mathbf{ctw}(G) \geq \eta_{k \cdot H}$. Therefore we now consider digraphs of cutwidth at most $\eta_{k \cdot H}$.

We will prove the statement on digraphs of cutwidth at most t by induction on k with $f\colon k \mapsto k \cdot t$. The case $k = 0$ is trivial, therefore we assume $k > 0$ and that the result holds for every $k' < k$. We also assume that G does not contain k arc-disjoint $\mathcal{H}$-subdigraphs, otherwise we are done. Let $v_1, \dots, v_n$ be an ordering of the vertices of G of minimum width. Let $i \in \mathbb{N}$ be the minimum integer such that $G[v_1, \dots, v_i]$ has an $\mathcal{H}$-subdigraph, that we call J. Notice that $i > 1$ as we assume that H has at least two vertices. We set $Y = E_G(\{v_1, \dots, v_{i-1}\}, \{v_i, \dots, v_n\})$. Notice that $|Y| \leq t$. As the digraphs in $\mathcal{H}$ are strongly-connected, any $\mathcal{H}$-subdigraph of $G \setminus Y$ belongs to exactly one of $\{v_1, \dots, v_{i-1}\}, \{v_i, \dots, v_n\}$. By definition of i, every such subdigraph belongs to $\{v_i, \dots, v_n\}$. Notice that every subdigraph of $\{v_i, \dots, v_n\}$ is arc-disjoint with J. Therefore $G[\{v_i, \dots, v_n\}]$ does not contains $k-1$ arc-disjoint $\mathcal{H}$-subdigraphs. It is clear that this subdigraph has cutwidth at most t. By induction hypothesis, there is a set $Y' \subseteq E(G[\{v_{i+1}, \dots, v_n\}])$ such that $G[\{v_{i+1}, \dots, v_n\}] \setminus Y'$ has no $\mathcal{H}$-subdigraph and $|Y'| \leq (k-1) \cdot t$. We deduce that $G \setminus (Y \cup Y')$ has no $\mathcal{H}$-subdigraph and that $|Y \cup Y'| \leq k \cdot t$, as required. This concludes the induction. We saw above that we only need to consider digraphs of cutwidth at most $\eta_{k \cdot H}$ and we just proved that in this case there is a suitable set of arcs of size at most $k \cdot \eta_{k \cdot H}$. This concludes the proof. Observe that Theorem 3 also holds when considering a finite family of strongly-connected graphs $\mathcal{F}$ instead of H. For this $\mathcal{H}$ should be defined as the subdigraph-minimal digraphs containing a digraph of $\mathcal{F}$ as an immersion, and H as any digraph of $\mathcal{F}$. The proof then follows the exact same lines. □

4 Discussion

In this note we obtained new Erdős–Pósa type results about classes defined by the relations of strong minors, butterfly minors, topological minors and immersions. The restriction of the host class to tournaments (or slightly larger classes) allowed us to obtain results for every strongly-connected pattern H. In particular, we provided conditions on the host class where Theorem 1 holds for every strongly-connected digraph H, which is not the case in general. Our proofs support the claim that techniques analogue to those used in the undirected case may be adapted to the directed setting. Let us now highlight two directions for future research.

Optimization of the gap.

The bounds on the function f in our results (gap of the Erdős–Pósa property) depend on the exclusion bounds of Theorem 4 and Theorem 6. Therefore, any improvement of these bounds yields an improvement of f. The upper bound on η_H of Theorem 6 that can be obtained from the proof of Chudnovsky et al. (2012) is $72 \cdot 2^{2h(h+2)} + 8 \cdot 2^{h(h+2)}$, where $h = |V(H)| + 2|E(H)|$. As a consequence, we have $f(k) = 2^{O(k^2 h^2)}$ in Theorem 3. It would be interesting to know whether a gap that is polynomial in k can be obtained. The same question can be asked for Theorem 2, however the upper bound in Theorem 4 that we can compute from the proof of Fradkin and Seymour (2013) is large (triply exponential).

Generalization.

The results presented in this note were related to (generalizations of) semicomplete digraphs. One direction for future research would be to extend them to wider classes of hosts. On the other hand, in Theorem 2, we require the guest digraph to be strongly connected when dealing with butterfly and topological minors. It is natural to ask if we can drop this condition. This would require a different proof as ours draws upon this condition.

References

N. Alon. Piercing d-intervals. *Discrete & Computational Geometry*, 19(3):333–334, 1998. ISSN 1432-0444. doi:10.1007/PL00009349.

S. A. Amiri, K.-I. Kawarabayashi, S. Kreutzer, and P. Wollan. The Erdős–Pósa property for directed graphs. *Preprint*, 2016. arXiv:1603.02504.

J. Aracena, J. Demongeot, and E. Goles. Positive and negative circuits in discrete neural networks. *Neural Networks, IEEE Transactions on*, 15(1):77–83, Jan 2004. ISSN 1045-9227. doi:10.1109/TNN.2003.821555.

J. Aracena, A. Richard, and L. Salinas. Number of fixed points and disjoint cycles in monotone boolean networks. *SIAM Journal on Discrete Mathematics*, 31(3):1702–1725, 2017. doi:10.1137/16M1060868.

M. Chudnovsky, A. Fradkin, and P. Seymour. Tournament immersion and cutwidth. *Journal of Combinatorial Theory, Series B*, 102(1):93 – 101, 2012. ISSN 0095-8956. doi:10.1016/j.jctb.2011.05.001.

I. J. Dejter and V. Neumann-Lara. Unboundedness for generalized odd cyclic transversality. In *Combinatorics (Eger, 1987), Colloquia Mathematica Societatis János Bolyai*, volume 52, pages 195–203, 1987.

R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, third edition, 2005.

P. Erdős and L. Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965. doi:10.4153/CJM-1965-035-8.

S. Fiorini, G. Joret, and D. R. Wood. Excluded forest minors and the Erdős-Pósa property. *Combinatorics, Probability & Computing*, 22(5):700–721, 2013. doi:10.1017/S0963548313000266. arXiv:1204.5192.

A. Fradkin and P. Seymour. Tournament pathwidth and topological containment. *Journal of Combinatorial Theory, Series B*, 103(3):374 – 384, 2013. ISSN 0095-8956. doi:10.1016/j.jctb.2013.03.001.

B. Guenin and R. Thomas. Packing directed circuits exactly. *Combinatorica*, 31(4):397–421, 2011. doi:10.1007/s00493-011-1687-5. arXiv:1012.2749.

A. Gyárfás and J. Lehel. A Helly-type problem in trees. In *Combinatorial Theory and its applications*, pages 571–584. Colloquia Mathematica Societatis János Bolyai, 1969.

F. Havet and A. K. Maia. On disjoint directed cycles with prescribed minimum lengths. Research Report RR-8286, INRIA, Apr. 2013. URL `https://hal.inria.fr/hal-00816135`.

T. Johnson, N. Robertson, P. Seymour, and R. Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138 – 154, 2001. ISSN 0095-8956. doi:10.1006/jctb.2000.2031.

I. Kim. *On containment relations in directed graphs*. PhD thesis, Princeton University, 2013.

I. Kim and P. Seymour. Tournament minors. *Journal of Combinatorial Theory, Series B*, 112:138 – 153, 2015. ISSN 0095-8956. doi:10.1016/j.jctb.2014.12.005. arXiv:1206.3135.

K. Kitsunai, Y. Kobayashi, and H. Tamaki. On the pathwidth of almost semicomplete digraphs. In *Algorithms - ESA 2015: 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015*, pages 816–827, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg. ISBN 978-3-662-48350-3. doi:10.1007/978-3-662-48350-3_68. arXiv:1507.01934.

J.-F. Raymond and D. M. Thilikos. Recent techniques and results on the Erdős–Pósa property. *Discrete Applied Mathematics*, 231:25–43, 2017. ISSN 0166-218X. doi:10.1016/j.dam.2016.12.025. arXiv:1603.04615.

B. Reed. Mangoes and blueberries. *Combinatorica*, 19(2):267–296, 1999. doi:10.1007/s004930050056.

B. Reed and B. Shepherd. The Gallai-Younger conjecture for planar graphs. *Combinatorica*, 16(4): 555–566, 1996. ISSN 1439-6912. doi:10.1007/BF01271273.

B. A. Reed. *Tree width and tangles: A new connectivity measure and some applications*, pages 87–162. Cambridge University Press, 1997. doi:10.1017/CBO9780511662119.006.

B. A. Reed, N. Robertson, P. D. Seymour, and R. Thomas. Packing directed circuits. *Combinatorica*, 16 (4):535–554, 1996. doi:10.1007/BF01271272.

N. Robertson and P. D. Seymour. Graph Minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(2):92–114, 1986. doi:10.1016/0095-8956(86)90030-4.

P. D. Seymour. Packing circuits in Eulerian digraphs. *Combinatorica*, 16(2):223–231, 1996. ISSN 1439-6912. doi:10.1007/BF01844848.