Simultaneous generation for zeta values by the Markov-WZ method

Kh. Hessami Pilehrood and T. Hessami Pilehrood

Mathematics Department, Faculty of Basic Sciences, Shahrekord University, Shahrekord, P.O. Box 115, Iran.


By application of the Markov-WZ method, we prove a more general form of a bivariate generating function identity containing, as particular cases, Koecher’s and Almkvist-Granville’s Apéry-like formulae for odd zeta values. As a consequence, we get a new identity producing Apéry-like series for all \( \zeta(2n + 4m + 3) \), \( n, m \geq 0 \), convergent at the geometric rate with ratio \( 2^{-10} \).

Keywords: Riemann zeta function, Apéry-like series, generating function, convergence acceleration, Markov-Wilf-Zeilberger method, Markov-WZ pair.

1 Introduction

The Riemann zeta function is defined by the series

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } \Re(s) > 1. \tag{1}
\]

Apéry’s irrationality proof of \( \zeta(3) \) \[14\] operates with the faster convergent series

\[
\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \tag{2}
\]

first obtained by A. A. Markov in 1890 \[10\]. The general formula giving analogous series for all \( \zeta(2s+3) \), \( s \geq 0 \), was proved by Koecher \[7\] (and independently in an expanded form by Leshchiner \[9\]). It reads

\[
\sum_{s=0}^{\infty} \zeta(2s+3)x^{2s} = \sum_{k=1}^{\infty} \frac{1}{k(k^2 - x^2)} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \frac{5k^2 - x^2}{k^2 - x^2} \prod_{m=1}^{k-1} \left( 1 - \frac{x^2}{m^2} \right). \tag{3}
\]

A similar identity generating fast convergent series for all \( \zeta(4s+3) \), \( s \geq 0 \), which for \( s > 1 \) is different from Koecher’s result \[5\], was experimentally discovered in \[3\] and proved by G. Almkvist and A. Granville in \[1\]

\[
\sum_{s=0}^{\infty} \zeta(4s+3)x^{4s} = \sum_{k=1}^{\infty} \frac{k}{k^4 - x^4} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^4 \binom{2k}{k}} \frac{k}{k^4 - x^4} \prod_{m=1}^{k-1} \left( \frac{m^4 + 4x^4}{m^4 - x^4} \right). \tag{4}
\]
There exists a bivariate unifying formula for identities (3) and (4),

\[
\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2k}} \frac{5k^2 - x^2}{k^4 - x^2k^2 - y^4} \prod_{m=1}^{k-1} \left( \frac{(m^2 - x^2)^2 + 4y^4}{m^4 - x^2m^2 - y^4} \right),
\]

which was first conjectured by H. Cohen and then proved by D. Bradley [5] and, independently, by T. Rivoal [15]. This identity implies (3) if \( y = 0 \), and gives (4) if \( x = 0 \). The proof of (5) given in [5, 15] relies on Borwein and Bradley’s method [3] and consists of reduction of (5) to a finite non-trivial combinatorial identity which can be proved on the basis of Almkvist and Granville’s work [1].

Recently, in [6] it was shown that Koecher’s formula (3), and similarly Leschiner’s and the identities of Bailey, Borwein and Bradley [9, 4] generating accelerated series for even zeta values \( \zeta(2n + 2) \), can be proved by means of the WZ method.

Formulas (3)-(5) generate accelerated series for odd zeta values and, in particular, series (2) for \( \zeta(3) \) which converge at a geometric rate with ratio \( 1/4 \). Many other more rapidly convergent expressions for \( \zeta(3) \) can be proved on the basis of the WZ method. The following series, for example, convergent at the geometric rate with ratio \( 2^{-10} \),

\[
\zeta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n n!^{10}}{64(2n + 1)!^5} \left( 205n^2 + 250n + 77 \right)
\]

was obtained by T. Amdeberhan and D. Zeilberger [2] by application of WZ-pairs. There are even faster convergent representations for \( \zeta(3) \) with ratios \( 10^{-5}, 10^{-8} \) (see [11]). In [6] it was shown how to get such fast convergent series explicitly for other values \( \zeta(n), n > 3 \). This can be accomplished by applying the WZ method not to the series in (1) itself but to a generating function of a sequence of zeta values.

In this note, we prove a more general form of the bivariate identity (5) by application of the Markov-WZ method. We show that identity (5) and the series (6) of Amdeberhan and Zeilberger can be proved with the help of the same Markov-WZ pair, but using different summation formulas. Moreover, we get a new identity generating accelerated series for all \( \zeta(2n + 4m + 3), n, m \geq 0 \), convergent at a geometric rate with ratio \( 2^{-10} \).

2 Statement of the main results

We start by giving several definitions, and by reviewing known facts related to the Markov-Wilf-Zeilberger theory (see [8, 10, 11, 12]).

A function \( H(n, k) \), in the integer variables \( n \) and \( k \), is called hypergeometric or closed form (CF) if the quotients

\[
\frac{H(n+1, k)}{H(n, k)} \quad \text{and} \quad \frac{H(n, k+1)}{H(n, k)}
\]

are both rational functions of \( n \) and \( k \). A hypergeometric function that can be written as a ratio of products of factorials is called pure-hypergeometric. A pair of CF functions \( F(n, k) \) and \( G(n, k) \) is called a WZ-pair if

\[
F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k).
\]
A P-recursive function is a function that satisfies a linear recurrence relation with polynomial coefficients. If for a given hypergeometric function \( H(n, k) \), there exists a polynomial \( P(n, k) \) in \( k \) of the form

\[
P(n, k) = a_0(n) + a_1(n)k + \cdots + a_L(n)k^L,
\]

for some non-negative integer \( L \), and P-recursive functions \( a_0(n), \ldots, a_L(n) \) such that \( F(n, k) := H(n, k)P(n, k) \) satisfies (7) with some function \( H \) and \( \nu \in \mathbb{N} \), then a pair \( (F, G) \) is called a Markov-WZ pair associated with the kernel \( H(n, k) \) (MWZ-pair for short). We call \( G(n, k) \) an MWZ mate of \( F(n, k) \).

In 2005, M. Mohammed [11] showed that for any pure-hypergeometric kernel \( H(n, k) \), there exists a non-negative integer \( L \) and a polynomial \( P(n, k) \) as above such that \( F(n, k) = H(n, k)P(n, k) \) has an MWZ mate \( G(n, k) = F(n, k)Q(n, k) \), where \( Q(n, k) \) is a ratio of two P-recursive functions.

From relation (7) we get the following summation formulas.

**Proposition 1 (\[11\] Theorem 2(b))** Let \( (F, G) \) be an MWZ-pair. If \( \lim_{n \to \infty} F(n, k) = 0 \) for every \( k \geq 0 \), then

\[
\sum_{k=0}^{\infty} F(0, k) - \lim_{k \to \infty} \sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} G(n, 0),
\]

whenever both sides converge.

**Proposition 2 (\[11\] Cor. 2)** Let \( (F, G) \) be an MWZ-pair. If \( \lim_{k \to \infty} \sum_{n=0}^{\infty} G(n, k) = 0 \), then

\[
\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} (F(n, n) + G(n, n + 1)),
\]

whenever both sides converge.

Formulas (8), (9) with an appropriate choice of MWZ-pairs can be used to convert a given hypergeometric series into a different rapidly converging one.

Let \( (\lambda)_\nu \) be the Pochhammer symbol (or the shifted factorial) defined by

\[
(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 
1, & \nu = 0; \\
\lambda(\lambda + 1)\cdots(\lambda + \nu - 1), & \nu \in \mathbb{N}.
\end{cases}
\]

Let \( a, b \) be complex numbers such that \( |a| < 1, |b| < 1 \). In Section 3, we construct a Markov-WZ pair associated with the kernel

\[
H(n, k) = \frac{(1 + a)_k(1 - a)_k(1 + b)_k(1 - b)_k}{(1 + a)_n + k + 1(1 - a)_n + k + 1(1 + b)_n + k + 1(1 - b)_n + k + 1}
\]

and then apply Propositions 1, 2 to get the following two theorems.

**Theorem 1** Let \( a, b \) be complex numbers, with \( |a| < 1, |b| < 1 \). Then for arbitrary complex numbers \( A_0, B_0, C_0 \) we have

\[
\sum_{k=1}^{\infty} \frac{A_0 + B_0k + C_0k^2}{(k^2 - a^2)(k^2 - b^2)} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^{n}(m^2 - a^2)(m^2 - b^2)}.
\]
with

\[
d_n = \frac{(-1)^{n-1} B_0 (5n^2 - a^2 - b^2)}{2n \binom{2n}{n}} \prod_{m=1}^{n-1} \left( \left( m^2 - a^2 - b^2 \right)^2 - 4a^2 b^2 \right)
\]

\[
+ \frac{20n + 5}{2(5n^2 - 2a^2 - 2b^2)} L_n + \frac{35n^5 - 35n^3(a^2 + b^2) + 4n(3a^4 + 3b^4 - 4a^2 b^2)}{4(5n^2 - 2a^2 - 2b^2)} L_{n-1},
\]

where \( L_n \) is a solution of the second order difference equation

\[
4(4n + 3)(4n + 5)(5n^2 - 2a^2 - 2b^2)L_{n+1} + 2(n + 1)p(n)L_n
\]

\[
- n(n + 1)(5(n + 1)^2 - 2a^2 - 2b^2)q(n)L_{n-1} = 0, \quad n = 1, 2, \ldots
\]

with initial conditions \( L_0 = C_0 \),

\[
L_1 = \left( \frac{1}{3} - \frac{2}{15} (a^2 + b^2) \right) A_0 + \left( \frac{1}{6} (a^2 + b^2) - \frac{2}{15} (a^4 + b^4 - 4a^2 b^2) - \frac{1}{30} \right) C_0,
\]

whose growth is described by the inequality

\[
\lim_{n \to \infty} \left( \frac{|L_n|}{n^4} \right)^{\frac{1}{n}} \leq \frac{1}{4},
\]

and

\[
p(n) = 30n^7 + 105n^6 + n^5(145 - 52(a^2 + b^2)) + n^4(100 - 130(a^2 + b^2))
\]

\[
+ n^3(35 - 124(a^2 + b^2) + 56(a^4 + b^4) - 208a^2 b^2) + n^2(5 - 56(a^2 + b^2)
\]

\[
+ 84(a^4 + b^4) - 312a^2 b^2) + n(80a^2 b^2(a^2 + b^2) - 16(a^6 + b^6) + 48(a^4 + b^4 - 3a^2 b^2)
\]

\[
- 14(a^2 + b^2)) + (10(a^2 - b^2)^2 - 2(a^2 + b^2) + 40a^2 b^2(a^2 + b^2) - 8(a^6 + b^6)),
\]

\[
q(n) = n^8 - 6n^6(a^2 + b^2) + n^4(9(a^4 + b^4) + 30a^2 b^2)
\]

\[
- n^2(28a^2 b^2(a^2 + b^2) + 4(a^6 + b^6)) + 16a^2 b^2(a^2 - b^2)^2.
\]

If in Theorem [1] we take \( B_0 = 1, A_0 = C_0 = 0 \), then \( L_n = 0 \) for all \( n \geq 0 \) and we get

\[
\sum_{k=1}^{\infty} \frac{k}{(k^2 - a^2)(k^2 - b^2)}
\]

\[
= \sum_{n=1}^{\infty} \frac{(5n^2 - a^2 - b^2)(1 + a + b)n^{-1}(1 + a - b)n^{-1}(1 - a - b)n^{-1}}{2(-1)^{n-1} n \binom{2n}{n} (1 + a)_n (1 - a)_n (1 + b)_n (1 - b)_n}.
\]

If we now put

\[
a^2 = \frac{x^2 + \sqrt{x^4 + 4y^4}}{2}, \quad b^2 = \frac{x^2 - \sqrt{x^4 + 4y^4}}{2},
\]

(13)
we get the bivariate identity (5) conjectured by H. Cohen.

If \( A_0 = 1, B_0 = C_0 = a = b = 0 \), we get the following series for \( \zeta(4) \) mentioned by Markov in [10, p.18]:
\[
\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} \left( \frac{4n + 1}{2n^2} L_n + \frac{7n^3}{4} L_{n-1} \right),
\]
where \( L_0 = 0, L_1 = 1/3, \) and
\[
4(4n + 3)(4n + 5)L_{n+1} + 2(n + 1)^3(6n^3 + 9n^2 + 5n + 1)L_n - n^7(n + 1)^3L_{n-1} = 0, \quad n \geq 1.
\]

**Theorem 2** Let \( x, y \) be complex numbers such that \(|x|^2 + |y|^4 < 1\). Then
\[
\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} r(n) \prod_{m=1}^{n-1} ((m^2 - x^2)^2 + 4y^4) \prod_{m=n}^{n-1} (m^4 - x^2m^2 - y^4), \tag{14}
\]
where
\[
r(n) = 205n^6 - 160n^5 + (32 - 62x^2)n^4 + 40x^2n^3 + (x^4 - 8x^2 - 25y^4)n^2 + 10y^4n + y^4(x^2 - 2).
\]

Since
\[
\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{n + m}{n} \right) \zeta(2n + 4m + 3)x^{2n}y^{4m},
\]
the formula (14) generates Apéry-like series for all \( \zeta(2n+4m+3) \), \( n, m \geq 0 \), convergent at the geometric rate with ratio \( 2^{-10} \). So, for example, if \( x = y = 0 \), we get Amdeberhan and Zeilberger’s series (6) for \( \zeta(3) \):
\[
\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1}(205n^2 - 160n + 32) \frac{1}{n^5(2n^2)^5}.
\]

If \( y = 0 \), we recover Theorem 4 from [6]. If \( x = 0 \), we find, in particular, the following expression for \( \zeta(7) \):
\[
\zeta(7) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n(25n^2 - 10n + 2)}{n^9(2n^2)^5} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n(205n^2 - 160n + 32)}{n^5(2n^2)^5} \left( \sum_{m=1}^{2n} \frac{1}{m^4} + \sum_{m=1}^{n-1} \frac{3}{m^4} \right).
\]

### 3 Proof of Theorem [11]

Let \( a, b \) be complex numbers such that \(|a| < 1, |b| < 1\). Let
\[
H(n,k) = \frac{(1 + a)k(1 - a)k(1 + b)k(1 - b)k}{(1 + a)n+k+1(1 - a)n+k+1(1 + b)n+k+1(1 - b)n+k+1}.
\]
We are interested in finding a Markov-WZ pair associated with \( H(n, k) \). For this purpose, we define the function \( F(n, k) = H(n, k)P(n, k) \), where \( P(n, k) \) is a polynomial in \( k \) of degree \( L_1 \) with unknown coefficients as functions of \( n \). Then

\[
F(n+1, k) - F(n, k) = \frac{(1 + a)_k(1 - a)_k(1 + b)_k(1 - b)_k}{(1 + a)_{n+k+2}(1 - a)_{n+k+2}(1 + b)_{n+k+2}(1 - b)_{n+k+2}} P_1(n, k),
\]

where \( P_1(n, k) \) is a polynomial in \( k \) of degree \( L_1 + 4 \). From (15) it follows that we can determine a MWZ mate of \( F(n, k) \) in the form \( G(n, k) = H(n, k)Q(n, k) \), where \( Q(n, k) \) is a polynomial in \( k \) of degree \( L_2 \) with unknown coefficients as functions of \( n \). Indeed, for such a choice we have

\[
G(n+1, k) - G(n, k) = \frac{(1 + a)_k(1 - a)_k(1 + b)_k(1 - b)_k}{(1 + a)_{n+k+2}(1 - a)_{n+k+2}(1 + b)_{n+k+2}(1 - b)_{n+k+2}} Q_1(n, k),
\]

where \( Q_1(n, k) \) is a polynomial in \( k \) of degree \( L_2 + 3 \). Therefore, \((F, G)\) is a Markov-WZ pair if and only if

\[
P_1(n, k) = Q_1(n, k) \quad \text{identically for all } n, k.
\]  

(16)

This implies that \( L_2 = L_1 + 1 \). On the other hand, equating coefficients of powers of \( k \) on both sides of (16), we get a system of \( L_1 + 5 \) linear homogeneous equations with \( L_1 + L_2 + 2 = 2L_1 + 3 \) unknowns. In order to guarantee a solution, we should at least have that \( 2L_1 + 3 \geq L_1 + 5 \) and hence \( L_1 \geq 2, L_2 \geq 3 \).

We now show that there is a non-zero solution of (16) with the optimal choice \( L_1 = 2, L_2 = 3 \). To see this, define two functions

\[
F(n, k) = H(n, k)(A(n) + B(n)(k + 1) + C(n)(k + 1)^2),
\]

\[
G(n, k) = H(n, k)(D(n) + E(n)k + K(n)k^2 + L(n)k^3),
\]

with 7 unknown coefficients \( A(n), B(n), C(n), D(n), E(n), K(n), L(n) \) as functions of \( n \). We require that

\[
F(n+1, k) - F(n, k) = G(n+1, k) - G(n, k).
\]

(17)

Substituting \( F, G \) into (17) and cancelling common factors we get that (17) is equivalent to the following equation of degree 6 in the variable \( k \) :

\[
((n + k + 2)^2 - a^2)((n + k + 2)^2 - b^2)(A(n) + B(n)(k + 1) + C(n)(k + 1)^2)
\]

\[
- A(n + 1) - B(n + 1)(k + 1) - C(n + 1)(k + 1)^2 = ((n + k + 2)^2 - a^2)
\]

\[
\times ((n + k + 2)^2 - b^2)(D(n) + E(n)k + K(n)k^2 + L(n)k^3) - ((k + 1)^2 - a^2)
\]

\[
\times ((k + 1)^2 - b^2)(D(n) + E(n)(k + 1) + K(n)(k + 1)^2 + L(n)(k + 1)^3).
\]

To satisfy condition (17), all the coefficients of the powers of \( (k + 1) \) in the equation (18) must be identically zero. This leads to a system of first order linear recurrence equations with polynomial coefficients for \( A(n), B(n), C(n), D(n), E(n), K(n), L(n) \)

\[
C(n) = (4n_1 - 3)L(n), \quad B(n) = (4n_1 - 2)K(n) - (10n_1^2 - 3)L(n), \quad n_1 = n + 1,
\]

(19)

\[
A(n) = (4n_1 - 1)E(n) - (10n_1^2 - 1)K(n) + (20n_1^3 + 2n_1(a^2 + b^2) - 1)L(n),
\]

(20)
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\[ 4D(n) = 10n_1E(n) - (20n_1^2 + 2a^2 + 2b^2)K(n) + (35n_1^3 + 11n_1(a^2 + b^2))L(n), \quad (21) \]

\[ 2(4n_1 + 1)L(n + 1) = 2n_1(5n_1^2 - 2a^2 - 2b^2)E(n) - 2n_1^2(15n_1^2 - 6(a^2 + b^2))K(n) + n_1(63n_1^4 - 17n_1^2(a^2 + b^2) - 4(a^4 + b^4))L(n), \quad (22) \]

\[ 2(4n_1 + 2)K(n + 1) = 2(10n_1^2 + 20n_1 + 7)L(n + 1) \]
\[ = 2n_1^2(5n_1^2 - 2(a^2 + b^2))E(n) - 2(16n_1^2 - 8n_1(a^2 + b^2) + n_1(a^2 - b^2)^2)K(n) + (70n_1^6 - 31n_1^4(a^2 + b^2) + n_1^2(3a^4 + 3b^4 - 14a^2b^2))L(n), \quad (23) \]

\[ 4(20n_1 + 1)^3 + 2(n_1 + 1)(a^2 + b^2) - 1)L(n + 1) - 4(10n_1^2 + 20n_1 + 9)K(n + 1) + 4(4n_1 + 3)E(n + 1) = (6n_1^5 - 6n_1^3(a^2 + b^2) + 16a^2b^2n_1)E(n) - (20n_1^4 - 22n_1^2(a^2 + b^2) + 2n_1^2(a^4 + b^4 + 22a^2b^2))K(n) + (45n_1^7 - 48n_1^5(a^2 + b^2) + n_1^3(3a^4 + 3b^4 + 86a^2b^2) + 8a^2b^2n_1(a^2 + b^2))L(n). \quad (24) \]

Now multiplying equation (22) by \( n_1 \) and subtracting from (23), we get

\[ 2K(n + 1) - 7(n_1 + 1)L(n + 1) = -\frac{n_1((n_1^2 - a^2 - b^2)^2 - 4a^2b^2)}{2n_1 + 1}(2K(n) - 7n_1L(n)), \]

which yields

\[ K(n) - \frac{7}{2}n_1L(n) = \frac{(-1)^n(2K(0) - 7L(0))n!}{2^{n_1}} \prod_{m=1}^{n} \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \]

From (19) it follows that \( 2K(0) = B(0) + 7L(0) \), and therefore we have

\[ K(n) = \frac{7}{2}n_1L(n) + \frac{(-1)^nB(0)n!}{2^{n_1}} \prod_{m=1}^{n} \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \quad (25) \]

Substitution of (25) into (22) yields the formula

\[ E(n) = \frac{4n_1 + 1}{n_1(5n_1^2 - 2a^2 - 2b^2)}L(n + 1) + \frac{42n_1^4 - 25n_1^2(a^2 + b^2) + 4(a^4 + b^4)}{2(5n_1^2 - 2a^2 - 2b^2)}L(n) \]
\[ + \frac{3B(0)(-1)^nn_1!}{2^{n_1}} \prod_{m=1}^{n} \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \quad (26) \]

Substitution of (25) and (26) into (21) gives the formula

\[ D(n) = \frac{(40n_1 + 10)L(n + 1) + (35n_1^3 - 35n_1^2(a^2 + b^2) + 4n_1^3(3a^4 + 3b^4 - 4a^2b^2))L(n)}{4(5n_1^2 - 2a^2 - 2b^2)} \]
\[ + \frac{(-1)^nB(0)n!(5n_1^2 - a^2 - b^2)}{2^{n_1 + 1}} \prod_{m=1}^{n} \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \quad (27) \]
Finally, substitution of (25), (26) into (24) gives the second-order difference equation
\[
4(4n + 3)(4n + 5)(5n^2 - 2a^2 - 2b^2)L(n + 1) + 2(n + 1)p(n)L(n)
- n(n + 1)(5(n + 1)^2 - 2a^2 - 2b^2)q(n)L(n - 1) = 0, \quad n = 1, 2, \ldots,
\]
with initial conditions \(L(0) = C(0),\)
\[
L(1) = \left(\frac{1}{3} - \frac{2}{15}(a^2 + b^2)\right)A(0) + \left(\frac{1}{6}(a^2 + b^2) - \frac{2}{15}(a^4 + b^4 - 4a^2b^2) - \frac{1}{30}\right)C(0),
\]
derived from (19), (20), (22), and polynomials \(p(n), q(n)\) defined in (10), (11).
If we put \(l(n) = L(n)/(n!)^4, n = 0, 1, 2, \ldots,\) then it is easily seen that the sequence \(l(n)\) satisfies the following recurrence equation:
\[
4(4n + 3)(4n + 5)(5n^2 - 2a^2 - 2b^2)n^3(n + 1)^3l(n + 1) + 2n^3p(n)l(n)
- (5(n + 1)^2 - 2a^2 - 2b^2)q(n)l(n - 1) = 0, \quad n = 1, 2, \ldots
\] (28)

Its characteristic polynomial \(64\lambda^2 + 12\lambda - 1 = 0\) has two different zeros \(\lambda_1 = -1/4, \lambda_2 = 1/16.\)
Then, by Poincaré’s theorem [13], for each solution \(l(n), n = 0, 1, 2, \ldots,\) of (28), either \(l(n) = 0\) for all sufficiently large \(n \geq n_0,\) or the limit \(\lim_{n \to \infty} l(n + 1)/l(n)\) exists and equals one of the roots of the characteristic polynomial. Therefore, in both cases we get that the limit \(\lim_{n \to \infty} |l(n)|^{1/n}\) exists and does not exceed 1/4 or
\[
\lim_{n \to \infty} \left(\frac{|L(n)|}{(n!)^4}\right)^{1/n} \leq \frac{1}{4},
\] (29)
The limit inequality (29) implies
\[
\lim_{k \to \infty} \sum_{n=0}^{\infty} G(n, k) = 0 \quad \text{and} \quad \lim_{n \to \infty} F(n, k) = 0 \quad \text{for every} \quad k \geq 0,
\]
and therefore, by Proposition 1, we have
\[
\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} G(n, 0),
\]
yielding the theorem with \(d_n = D(n - 1), L_n = L(n), A_0 = A(0), B_0 = B(0), C_0 = C(0).\)

4 Proof of Theorem 2

To deduce (14) from (17), take \(A(0) = C(0) = 0, B(0) = 1,\) and apply Proposition 2 to obtain
\[
\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} (F(n, n) + G(n, n + 1)).
\] (30)
Since in this case \(L(n) = 0\) for all \(n \geq 0,\) an easy computation of the right-hand side of (30) by (19), (20), (25)–(27), and substitution (13) lead to the desired conclusion.
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References


