Sufficient Conditions for Labelled 0–1 Laws

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If \( F(x) = e^{G(x)} \), where \( F(x) = \sum f(n)x^n \) and \( G(x) = \sum g(n)x^n \), with \( 0 \leq g(n) = O(n^\theta n!/n!) \), \( \theta \in (0, 1) \), and \( \gcd \{ n : g(n) > 0 \} = 1 \), then \( f(n) = o(f(n-1)) \). This gives an answer to Compton’s request in Question 8.3 [Compton 1987] for an “easily verifiable sufficient condition” to show that an adequate class of structures has a labelled first-order 0–1 law, namely it suffices to show that the labelled component count function is \( O(n^\theta n) \) for some \( \theta \in (0, 1) \). It also provides the means to recursively construct an adequate class of structures with a labelled 0–1 law but not an unlabelled 0–1 law, answering Compton’s Question 8.4.

Keywords: ratio test, labelled structure, zero-one law

1 Introduction

Exponentiating a power series can have the effect of smoothing out the behavior of the coefficients. In this paper we look at conditions on the growth of the coefficients of \( G(x) = \sum g(n)x^n \), where \( g(n) \geq 0 \), \( g(d) > 0 \), with \( \gcd \{ j \leq d : g(j) > 0 \} = 1 \), that ensure that \( f(n-1)/f(n) \to \infty \), where \( F(x) = e^{G(x)} \). One application of this result is to 0-1 laws, where we find, see Theorem 7, that if the labelled component count function for an adequate class of structures is \( O(n^\theta n) \) for some \( \theta \in (0, 1) \) then the class has a labelled monadic second-order 0-1 law.

Useful notation will be \( f(n) \prec g(n) \) for \( f(n) \) eventually less than \( g(n) \) and \( f(n) \in \text{RT}_\infty \) for \( f(n-1)/f(n) \to \infty \); the notation RT stands for the ratio test.

2 The Coefficients of \( e^{\text{poly}} \)

Proposition 1 Given

\[
G(x) := g(1)x + \cdots + g(d)x^d, \quad g(i) \geq 0, \quad g(d) > 0, \\
\text{with } \gcd \{ j \leq d : g(j) > 0 \} = 1 \\
F(x) := \sum_{n \geq 0} f(n)x^n = e^{G(x)},
\]

the function \( F(x) \) is Hayman-admissible. Thus

\[
f(n) \sim \frac{F(r_n)}{r_n^n \sqrt{2\pi B(r_n)}}
\]
where \( r_n \) is the unique positive solution to
\[
x \cdot G'(x) = n,
\]
and \( B(x) := x^2G''(x) + xG'(x) \).

**Proof:** Theorem X of Hayman [5] shows that \( F(x) \) is Hayman-admissible. Then the rest of the claim is an immediate consequence of Corollary II of [5] where the saddle-point method is applied to find the asymptotics of the coefficients of an admissible function. \( \square \)

**Corollary 2** For \( F(x), G(x) \) as in the above proposition,
(a) \( f(n) \in RT_\infty \).
(b) \( f(n) = \exp \left( -\frac{n \log n}{d} (1 + o(1)) \right) \).

**Proof:** Item (a) follows immediately from Corollary IV of Hayman [5]. For item (b) one uses \( r_n G'(r_n) = n \) to obtain:
\[
\left( \frac{n}{cdg(d)} \right)^{1/d} \leq r_n \leq \left( \frac{n}{dg(d)} \right)^{1/d} \quad \text{for } c > 1
\]
\[
r_n = (1 + o(1)) \left( \frac{n}{dg(d)} \right)^{1/d}
\]
\[
r_n^n = (1 + o(1)) \left( \frac{n}{dg(d)} \right)^{n/d}
\]
\[
B(r_n) = (1 + o(1)) d^2 g(d) \left( \frac{n}{dg(d)} \right) = (1 + o(1)) dn
\]
\[
G(r_n) = (1 + o(1)) g(d) r_n^d = (1 + o(1)) \frac{n}{d}
\]
\[
F(r_n) = \exp \left( \frac{n}{d} (1 + o(1)) \right).
\]
Apply these results to (1). \( \square \)

## 3 Some Technical Lemmas

Now we drop the assumption that \( G(x) \) is a polynomial, but keep the requirement
\[
gcd \left( n : g(n) > 0 \right) = 1. \tag{2}
\]
This implies that \( f(n) \succ 0 \).

Choose a positive integer \( L \geq 2 \) sufficiently large so
\[
n > L \Rightarrow [x^n] \exp \left( g(1)x + \cdots + g(L)x^L \right) > 0. \tag{3}
\]
Given \( \ell > L \) with \( g(\ell) > 0 \) let

\[
G_0(x) := \sum_{n \geq 1} g_0(n)x^n := \sum_{1 \leq n \leq \ell} g(n)x^n \\
F_0(x) := \sum_{n \geq 0} f_0(n)x^n := \exp(G_0(x)) \\
G_1(x) := \sum_{n \geq 1} g_1(n)x^n := \sum_{n \geq \ell + 1} g(n)x^n \\
F_1(x) := \sum_{n \geq 0} f_1(n)x^n := \exp(G_1(x)).
\] (4)

**Lemma 3** Suppose \( r \geq -1 \) is such that

\[
ng(n) = O(f_0(n + r)).
\] (5)

Then

\[
nf_1(n) = O(f(n + r)).
\]

**Proof:** In view of (3) and (5) we can choose \( C_r \) such that

\[
ng(n) \leq C_r f_0(n + r) \quad \text{for } n + r \geq L + 1.
\] (6)

Differentiating (4) gives

\[
nf_1(n) = \sum_{j=\ell+1}^{n} jg(j) \cdot f_1(n-j) \\
\leq C_r \sum_{j=\ell+1}^{n} f_0(j + r) \cdot f_1(n-j) \quad \text{by (6)} \\
\leq C_r \sum_{j=0}^{n+r} f_0(j) \cdot f_1(n + r - j) \\
= C_r f(n + r),
\]

the last line following from \( F(x) = F_0(x) \cdot F_1(x) \).

**Lemma 4** Suppose for every integer \( r \geq -1 \)

\[
ng(n) = O(f_0(n + r)).
\]

Then \( f(n-1)/f(n) \to \infty \).

**Proof:** Since \( f_0(n) \in \text{RT}_\infty \) by Corollary 2 there is a monotone decreasing function \( \varepsilon(n) \) such that for any sufficiently large \( M \) we have \( \varepsilon(n) > f_0(n)/f_0(n-1) \) for \( n \geq M \), and \( \varepsilon(n) \to 0 \) as \( n \to \infty \).
Thus
\[ f(n) = \sum_{0 \leq j \leq n} f_0(j) f_1(n-j) \]
\[ = \sum_{0 \leq j \leq M-1} f_0(j) f_1(n-j) + \sum_{M \leq j \leq n} f_0(j) f_1(n-j) \]
\[ \leq o(f(n-1)) + \varepsilon(M) \sum_{M \leq j \leq n} f_0(j) f_1(n-j) \]
by Lemma 3 and the choice of \( \varepsilon \)
\[ \leq o(f(n-1)) + \varepsilon(M) f(n-1). \]

Thus
\[ \limsup_{n \to \infty} \frac{f(n)}{f(n-1)} \leq \varepsilon(M), \]
and as \( M \) can be arbitrarily large it follows that
\[ \lim_{n \to \infty} \frac{f(n)}{f(n-1)} = 0. \]

\section{Main Result}
We are now in a position to prove the main result, making use of
\[ n! = \exp(n \log n \cdot (1 + o(1))), \]
which follows from Stirling’s result.

**Theorem 5** Suppose \( F(x) = \exp(G(x)) \) with \( F(x) = \sum_{n \geq 0} f(n)x^n, G(x) = \sum_{n \geq 1} g(n)x^n, \) and \( f(n), g(n) \geq 0. \) Suppose also that \( \gcd(n : g(n) > 0) = 1 \) and that for some \( \theta \in (0,1) \)
\[ g(n) = O(n^{\theta n}/n!). \]
Then
\[ f(n) \in \mathbb{R}T_\infty. \]

**Proof:** From Corollary 2 for any integer \( r \geq -1 \) and any \( \theta \in (0,1), \) by choosing \( \ell > L \) such that \( 1/\ell < 1 - \theta, \) we have
\[ f_0(n+r) = \exp \left( -\frac{(n+r) \log(n+r)}{\ell} (1 + o(1)) \right) \]
\[ = \exp \left( -\frac{n \log n}{\ell} (1 + o(1)) \right) \]
\[ \asymp \frac{n^{\theta n}}{(n-1)!}. \]
Thus \( ng(n) = O(f_0(n+r)). \) The Theorem then follows from Lemma 4. \( \square \)
5 Best Possible Result

The main result is in a natural sense the best possible.

**Proposition 6** Suppose \( t(n) \geq 0 \) with \( \gcd(n : t(n) > 0) = 1 \) is such that for every \( \theta \in (0, 1) \)

\[
t(n) \neq O(n^{\theta n}/n!).
\]

Then there is a sequence \( g(n) \geq 0 \) with \( \gcd(n : g(n) > 0) = 1 \) and \( g(n) \leq t(n) \) but \( f(n) \notin RT_{\infty} \), where one has \( F(x) = \exp(G(x)) \).

**Proof:** For \( \theta \in (0, 1) \) let

\[
S(\theta) = \{ n \geq 1 : t(n) > n^{\theta n}/n! \}.
\]

Then \( S(\theta) \) is an infinite set.

Let \( M \) be such that \( \gcd(n \leq M : t(n) > 0) = 1 \), and let

\[
g_1(n) := \begin{cases} t(n) & \text{if } n \leq M \\ 0 & \text{if } n > M \end{cases}
\]

\[
G_1(x) := \sum g_1(n) x^n \\
F_1(x) := e^{\sum G_1(n) x^n},
\]

For \( m \geq 2 \) we give a recursive procedure to define polynomials \( G_m(x) \); then letting

\[
d_m := \deg(G_m(x)) \\
F_m(x) := e^{G_m(x)},
\]

by Proposition[1] \( f_m(n) = \exp \left( -\frac{n \log n}{d_m} (1 + o(1)) \right) \).

To define \( G_{m+1}(x) \), having defined \( G_m(x) \), let

\[
h_m(n) := \frac{1}{n!} n^{(1-1/2d_m)n}.
\]

Then

\[
\frac{h_m(n)}{f_m(n-1)} \to \infty \quad \text{as } n \to \infty.
\]

Thus we can choose an integer \( d_{m+1} \geq d_m \) such that

\[
d_{m+1} \in S \left( 1 - \frac{1}{2d_m} \right) \\
h_m(d_{m+1}) > f_m(d_{m+1} - 1).
\]
This ensures that $h_m(d_{m+1}) \leq t(d_{m+1})$. Let

$$G_{m+1} := G_m(x) + h_m(d_{m+1})x^{d_{m+1}}.$$ 

Then

$$\frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1} - 1)} \geq \frac{h_m(d_{m+1})}{f_m(d_{m+1} - 1)} > 1.$$

Now let $G(x)$ be the nonnegative power series defined by the sequence of polynomials $G_m(x)$; and let $F(x) = e^{G(x)}$. Then $g(n) \leq t(n)$ but $f(n) \notin RT_\infty$ as

$$\frac{f(d_{m+1})}{f(d_{m+1} - 1)} = \frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1} - 1)} > 1.$$

\[\square\]

6 Application to 0–1 laws

A class $\mathcal{K}$ of finite relational structures is adequate if it is closed under disjoint union and the extraction of components. One can view the structures as being unlabelled with the component count function $p_U(n)$ and the total count function $a_U(n)$, both counting up to isomorphism. The corresponding ordinary generating series are

$$P_U(x) := \sum_{n \geq 1} p_U(n)x^n, \quad A_U(x) := \sum_{n \geq 0} a_U(n)x^n$$

connected by the fundamental equation

$$A_U(x) = \prod_{j \geq 1} (1 - x^j)^{-p_U(j)}.$$ \hfill (7)

One can also view the structures as being labelled (in all possible ways) with the count functions $p_L(n)$ for the connected members of $\mathcal{K}$, and $a_L(n)$ for all members of $\mathcal{K}$. The corresponding exponential generating series are

$$P_L(x) := \sum_{n \geq 1} p_L(n)x^n/n!, \quad A_L(x) := \sum_{n \geq 0} a_L(n)x^n/n!$$

connected by the fundamental equation

$$A_L(x) = e^{P_L(x)}.$$ \hfill (8)

All references to Compton in this section are to the two papers [3] and [4].
6.1 Unlabelled 0–1 Laws for Adequate Classes

Let $\mathcal{K}$ be an adequate class with unlabelled count functions and ordinary generating functions as described above. Compton showed that if the radius of convergence $\rho_U$ of $A_U(x)$ is positive then $\mathcal{K}$ has an unlabelled 0–1 law \(^{(iii)}\) if $a_U(n) \in \text{RT}_1$, that is,

$$\frac{a_U(n - 1)}{a_U(n)} \to 1 \text{ as } n \to \infty.$$  

$\mathcal{K}$ is finitely generated if $r = \sum p_U(n) < \infty$, that is, there are only finitely many connected structures in $\mathcal{K}$. In the finitely generated case the asymptotics for the coefficients $a_U(n)$ have long been known to have the simple polynomial form \(^{(vii)}\)

$$a_U(n) \sim C n^{-r-1}$$  

provided $\gcd(n : p_U(n) > 0) = 1$. Item \(^{(iv)}\) leads to the fact that $a_U(n) \in \text{RT}_1$, and hence to an unlabelled 0–1 law. In addition to using this result, Compton notes that the work of Bateman and Erdős \(^{(i)}\) shows that if $p_U(n) \in \{0, 1\}$, for all $n$, then one has $a_U(n) \in \text{RT}_1$.

Both of these results were subsumed in the powerful result of Bell \(^{(2)}\) which says that if $p_U(n)$ is polynomially bounded, that is, there is a $c$ such that $p_U(n) = O(n^c)$, then $a_U(n) \in \text{RT}_1$.

6.2 Labelled 0–1 Laws

Compton shows that if $\rho_L$, the radius of convergence of $A_L(x)$, is positive, then $\mathcal{K}$ has a labelled 0–1 law iff

$$\frac{a_L(n - k)/(n - k)!}{a_L(n)/n!} \to \infty \text{ whenever } p_L(k) > 0.$$  

In particular it suffices to show that $a_L(n)/n! \in \text{RT}_\infty$.

Compton’s method to show that a given adequate class of finite relational structures $\mathcal{K}$ has a labelled 0–1 law is to show that its exponential generating function $A_L(x) = \sum a_L(n)x^n/n!$ is Hayman-admissible with an infinite radius of convergence. This guarantees that $a_L(n)/n! \in \text{RT}_\infty$ \(^{(v)}\). However, as Compton notes, showing that $A_L(x)$ is Hayman-admissible can be quite a challenge.

Question 8.3 of \(^{(3)}\) first asks if, in the unlabelled case, the result of Bateman and Erdős, namely $p_U(n) \in \{0, 1\}$ implies $a_U(n) \in \text{RT}_1$, can be extended to the much more general statement that $p_U(n) = O(n^k)$ implies $a_U(n) \in \text{RT}_1$, yielding an unlabelled 0–1 law. As mentioned earlier, this was proved to be true by Bell. The second part of Question 8.3 asks if there is a simple sufficient condition along similar lines for the labelled case. We can now answer this in the affirmative with a result that is an excellent parallel to Bell’s result for unlabelled structures.

**Theorem 7** If $\mathcal{K}$ is an adequate class of structures with

$$p_L(n) = O \left( n^{\theta n} \right) \text{ for some } \theta \in (0, 1)$$

then $a_L(n)/n! \in \text{RT}_\infty$, and consequently $\mathcal{K}$ has a labelled monadic second-order 0–1 law.

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\(^{(i)}\) Given a logic $\mathcal{L}$, $\mathcal{K}$ has an unlabelled $\mathcal{L}$ 0–1 law means that for any $\mathcal{L}$ sentence $\phi$, the probability that $\phi$ holds in $\mathcal{K}$ will be either 0 or 1. In \(^{(3)}\) Compton worked with first-order logic, in \(^{(4)}\) with monadic second-order logic. In both papers he simply used the phrases “unlabeled 0–1 law” and “labeled 0–1 law”.

\(^{(ii)}\) This result is usually known as Schur’s Theorem \(^{(6)}\) 3.15.2. One can easily find the asymptotics \(^{(v)}\) using a partial fraction decomposition of the right side of \(^{(7)}\). The labelled case with finitely many components is more difficult—we needed to invoke Hayman’s treatise \(^{(5)}\) just to obtain the asymptotics for $\log a_L(n)/n!$ (see Corollary \(^{(2)}\)).
Proof: This is an immediate consequence of Theorem 5 and Compton’s proof that $a_L(n)/n! \in \mathbb{R}T_{\infty}$ guarantees such a 0–1 law.

Now we list the examples of classes $\mathcal{K}$ which Compton shows have a labelled 0–1 law, giving $p_L(n)$ in each case. It is trivial to check in each case that $p_L(n) = O\left(\frac{n^{n/2}}{n}\right)$; thus the 0–1 law in each case follows from our Theorem 7.

(a) 7.1 Unary Predicates $p_L(n) = 0$ for $n > 1$.

(b) 7.12 Forests of Rooted Trees of Height 1 $p_L(n) = n$.

(c) 7.15 Only Finitely Many Components $p_L(n)$ is eventually 0.

(d) 7.16 Equivalence Relations $p_L(n) = 1$.

(e) 7.17 Partitions with a Selection Subset $p_L(n) = 2^n - 1$.

We can now augment this list by, in each case, coloring the members of $\mathcal{K}$ by a fixed set of $r$ colors in all possible ways. This will increase the original $p_L(n)$ by a factor of at most $r^n$. This will still give $p_L(n) = O\left(\frac{n^{n/2}}{n}\right)$. Furthermore, in each of these colored cases let $\mathcal{P}$ be any subset of the connected members, and let $\mathcal{K}$ be the closure of $\mathcal{P}$ under disjoint union. Each such $\mathcal{K}$ has a labelled 0–1 law.

Another application of Theorem 7 is to answer Question 4 of [3] by exhibiting an adequate class $\mathcal{K}$ such that $p_L(n) = O\left(\frac{n^{3n/4}}{n}\right)$, hence there is a labelled 0–1 law for $\mathcal{K}$; but also such that $\rho_U \in (0, 1)$, so $\mathcal{K}$ does not have an unlabelled 0–1 law.

Let the components of $\mathcal{K}$ be the one-element tree $T_1$ along with rooted trees $T_{3n}$ of size $3n$ and height $n$ consisting of a chain $C_n$ of $n$ nodes, with an antichain $L_{2n}$ of $2n$ nodes (the leaves of the tree) below the least member of the chain; and the chain $C_n$ is two-colored while the remaining nodes are uncolored. One can visualize these as brooms with 2-colored handles, see Figure 1.

The number of unlabelled components is given by $p_U(1) = 1$, $p_U(3n) = 2^n$. Thus the radius of convergence of the ordinary generating function of $\mathcal{K}$ is $\rho_U = \frac{3}{\sqrt{2}}$. Since this is positive and not 1 it follows from Theorem 5.9(ii) of [3] that $\mathcal{K}$ does not have an unlabelled 0–1 law.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{brooms.png}
\caption{Brooms with two-colored handles}
\end{figure}
For the number $p_L(3n)$ of labelled components of size $3n$:

$$
p_L(3n) \leq 2^n \binom{3n}{n} n! \leq 2^n (3n)^n \exp(n \log n \cdot (1 + o(1))) = \exp(2n \log n \cdot (1 + o(1))) = (3n)^{(2/3)(3n)}(1+o(1)) = O\left(\frac{1}{3n^{3/4}}\right).
$$

Thus $p_L(n) = O(n^{3n/4})$, so $a_L(n)/n! \in \text{RT}_\infty$ by Theorem 7, showing that $\mathcal{K}$ has a labelled 0–1 law.
References


