Total domination in K_5- and K_6-covered graphs

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A graph G is K_r-covered if each vertex of G is contained in a K_r-clique. Let $\gamma_t(G)$ denote the total domination number of G. It has been conjectured that every K_r-covered graph of order n with no K_r-component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$. We prove that this conjecture is true for $r = 5$ and 6.

Keywords: total domination, clique cover

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [6] for terminology and notation which are not defined here. The *open neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v. Its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. If S is a set of vertices of G, then $N(S) = \cup_{u \in S} N(u)$ and $N[S] = N(S) \cup S$. For the sake of simplicity we write $N(u, v)$ instead of $N(\{u, v\})$. For each nonempty set S of vertices, the subgraph induced by S is denoted by $G[S]$. For an integer $p \geq 3$, a *multitriangle* of order p is the graph consisting of $p - 2$ triangles sharing one edge. A multitriangle of order 3 is thus a triangle.

A set $D \subseteq V$ is a *total dominating set* if every vertex in V is adjacent to a vertex of D. Obviously every graph without isolated vertices has a total dominating set. The *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a total dominating set. If G has q components G_i, then $\gamma_t(G) = \sum_{i=1}^{q} \gamma_t(G_i)$. As for the domination number, the determination of the total domination number of a graph is NP-hard and it is interesting to determine good upper bounds on $\gamma_t(G)$. Conditions on the density of the graph allow us to lower such bounds. Here we consider the condition that every vertex is contained in a sufficiently large clique.

A K_r-*component* of G is a component isomorphic to a clique K_r. A graph G is K_r-*covered*, $r \geq 2$, if every vertex of G is contained in a clique K_r, and *minimally* K_r-*covered* if it is K_r-covered but $G - e$ is not K_r-covered for any edge of G. These properties were already considered by Henning and Swart in [5] under the terms "with no K_r-isolated vertex" or "Property $C(1, r)$", and "Property $C(2, r)$", respectively.

In a K_r-covered graph, a *good vertex* is a vertex of degree $r-1$ and a *good clique* is a clique K_r containing a good vertex. If z is a good vertex, we denote by $\mathcal{C}_z$ the good clique containing it. The following results, independently proved in [3] and in [4], will be constantly used throughout the paper.

Theorem A *[3, 4] Every edge of a minimally K_r-covered graph is contained in a good clique.*

In 2004, Cockayne, Favaron and Mynhardt [1] conjectured that every K_r-covered graph G of order n with no K_r-component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$. They proved this conjecture for $r = 3, 4$. In this paper we prove it for $r = 5$ and 6.

2 Proof of the conjecture for $r = 5$ and $r = 6$

The proof uses a particular family $\mathcal{F}_r$ of minimally K_r-covered graphs which was already considered in [1, 4]. Recall that the corona of a graph H is obtained from H by adding a pendant edge at each vertex of H and that the middle graph of H is the line graph of the corona of H(see for instance [2]).

Definition 1 *$\mathcal{F}_r$ is the family of middle graphs of $(r-1)-$regular graphs.*

From this definition $\mathcal{F}_r$ is the collection of graphs consisting of edge-disjoint cliques of order r, where each such clique contains exactly one vertex of degree $r-1$ and the remaining $r-1$ vertices have degree $2(r-1)$. Let $\mathcal{S}$ be the set of these edge-disjoint cliques. Then each vertex of G of degree $r-1$ belongs to exactly one K_r in $\mathcal{S}$ and each vertex of degree $2(r-1)$ belongs to exactly two K_r's in $\mathcal{S}$.

The following result is proved in [1].

Theorem B (*See [1]) For $r \geq 3$, every graph of order n of $\mathcal{F}_r$ satisfies $\gamma_t(G) < \frac{2n}{r+1}$.*

We can now prove the conjecture for $r = 5$ and $r = 6$.

Theorem 1 F*or r=5 or 6, every K_r-covered graph G of order n with no K_r-component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$.*

Proof: The proof is by induction on n and the first four claims are established for any value of $r \geq 5$. The statement is obviously true for the smallest possible order $r+1$ since then $\gamma_t(G) = 2$. Suppose the theorem to be true for graphs of order less than n and let G be a K_r-covered graph with no K_r-component of order $n \geq r+2$. Let F be a minimally K_r-covered spanning subgraph of G. Since $\gamma_t(G) \leq \gamma_t(F)$, it is sufficient to prove $\gamma_t(F) \leq \frac{2n}{r+1}$.

If F is not connected but has no K_r-component, then applying the induction hypothesis to each component of F gives the result.

The case where F is not connected and has some K_r-components has already been considered in [1]. The second part of the proof of Theorem 6 in [1] establishes by induction on n that a graph having a certain property $\mathcal{B}(r)$ satisfies $\gamma_t(G) \leq 2n/(r+1)$. In the case where a minimal K_r-covered spanning subgraph F of G has K_r components, the result is proved without using the induction hypothesis $\mathcal{B}(r)$. Hence the same argument holds here. As the proof is rather long, we do not repeat it and refer the reader to [1].

So we suppose now that we are working in a connected minimal K_r-covered graph F of order $n \geq r+2$. Since F is connected, every vertex of F belongs to an edge and thus has a good neighbor by Theorem A. For every pair of adjacent vertices u and v, let

$$P(u,v) = \{x \in N(u,v) \setminus \{u,v\} \mid N(x) \subseteq N(u,v)\}.$$

Claim 1 *If $|P(u,v)| \geq r-1$ for some pair of adjacent non-good vertices u and v, then $\gamma_t(F) \leq \frac{2n}{r+1}$.*

Proof: Let u' be any good neighbor of u. By Theorem A, the edge uu' is contained in a good clique $\mathcal{C}$. The $r-2$ neighbors of u' different from u are vertices of $\mathcal{C}$ and thus are adjacent to u. So every good neighbor of u, and similarly every good neighbor of v, belongs to $P(u,v)$. Note also that if $z \in N(u,v) \setminus (P(u,v) \cup \{u,v\})$, then z has a neighbor $z_1 \notin N(u,v)$, and so z belongs to a good clique of the graph $F' = F[V \setminus (P(u,v) \cup \{u,v\})]$. Hence F' is K_r-covered.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_s$ be the K_r-components of F' if any. Obviously $(N(w) \setminus V(\mathcal{C}_i)) \subseteq P(u,v) \cup \{u,v\}$ for each $w \in V(\mathcal{C}_i)$. Since F is connected, each clique $\mathcal{C}_i$ contains at least one vertex w_i such that $(N(w_i) \setminus V(\mathcal{C}_i)) \cap (P(u,v) \cup \{u,v\}) \neq \emptyset$. From the definition of $P(u,v)$ we have $w_i \in N(u,v)$. Let $X = P(u,v) \cup \{u,v\} \cup (\cup_{i=1}^{s} V(\mathcal{C}_i))$. The graph $F[V \setminus X]$ is still K_r-covered and has no K_r-component. By the induction hypothesis, $\gamma_t(F[V \setminus X]) \leq \frac{2|V \setminus X|}{r+1}$. Moreover $\{u, v, w_1, w_2, \cdots, w_s\}$, or $\{u,v\}$ if $s = 0$, is a total dominating set of order $s+2$ of $F[X]$, and $|X| = |P(u,v)| + sr + 2$ with $s \geq 0$. Hence if $|P(u,v)| \geq r-1$, then $\gamma_t(F[X]) \leq s+2 \leq \frac{2|X|}{r+1}$ and we are done. □

We suppose henceforth $|P(u,v)| \leq r-2$ for every pair of adjacent non-good vertices of F. Recall that all the good neighbors of u or v belong to $P(u,v)$. If G consists of $p \geq 2$ cliques K_r sharing exactly one vertex, then $n = p(r-1)+1$ and $\gamma_t(G) = 2 \leq 2n/(r+1)$. We also suppose in what follows that G has not this structure, which means that every non-good vertex has at least one non-good neighbor.

Claim 2 *Each good clique contains at most $r-4$ good vertices.*

Proof: Suppose to the contrary that $\mathcal{C}$ is a good clique ($\neq K_r$) with more than $r-4$ good vertices. Let $z_1, z_2, \cdots, z_s$ with $r-3 \leq s \leq r-1$ be the good vertices and u a non-good vertex of $\mathcal{C}$. If u has a non-good neighbor v not in $\mathcal{C}$, let $\mathcal{C}'$ be a good clique containing uv and $z'_1, \cdots, z'_t$ $(1 \leq t \leq r-2)$ the good vertices of $\mathcal{C}'$. The vertex v belongs to a second good clique $\mathcal{C}'' \neq \mathcal{C}$. Let z'' be a good vertex of $\mathcal{C}''$. Then $\{z_1, \cdots, z_s, z'_1, \cdots, z'_t, z''\}$ is a subset of $P(u,v)$ of order at least $s+2 \geq r-1$, a contradiction to $|P(u,v)| \leq r-2$. Therefore all the non-good neighbors of u belong to $\mathcal{C}$.

Let $u, u_1, \cdots, u_{r-s-1}$ be the non-good vertices of $\mathcal{C}$ with $r-s \geq 2$. Let $\mathcal{C}'$ ($\mathcal{C}'_1$ respectively, possibly equal to $\mathcal{C}'$) be a second good clique containing u (u_1) and let z' (z'_1) be a good vertex of $\mathcal{C}'$ ($\mathcal{C}'_1$). Then $\{z_1, \cdots, z_s, z', z'_1\} \subseteq P(u, u_1)$. Since $|P(u,u_1)| \leq r-2$, $s = r-3$, $z' = z'_1$ and z' is the unique good vertex of the clique $\mathcal{C}' = \mathcal{C}'_1$. Among the $r-1$ non-good vertices of $\mathcal{C}'$, at most three are those of $\mathcal{C}$ and thus at least one, say u_2, is not in $\mathcal{C}$. Let $\mathcal{C}''$ be a second good clique containing u_2 and z'' a good vertex of $\mathcal{C}''$. Then $\{z_1, \cdots, z_{r-3}, z', z''\} \subseteq P(u, u_2)$, a contradiction which completes the proof. □

Claim 3 *No vertex can belong to $r-2$ good cliques K_r.*

Proof: Assume, to the contrary, that a vertex u belongs to $r-2$ good cliques $\mathcal{C}_1, \ldots, \mathcal{C}_{r-2}$, and let x_i be a good vertex of $\mathcal{C}_i$ for $1 \leq i \leq r-2$. Let w and t be two non-good vertices of $\mathcal{C}_1 \setminus \{u, x_1\}$. Since $\{x_1, \cdots, x_{r-2}\} \subseteq P(u,t)$ and $|P(u,t)| \leq r-2$, we have $w \notin P(u,t)$ and w has a good neighbor w' not in $N(u,t)$ and thus distinct from $x_1, \cdots, x_{r-2}$. But then $\{x_1, \cdots, x_{r-2}, w'\} \subseteq P(u,w)$, which is a contradiction. □

Claim 4 *$r-3$ good cliques K_r cannot share more than one vertex.*

Proof: Assume, to the contrary, that $\mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_{r-3}$ are $r-3$ good K_r's all containing the (non-good) vertices u and v, and let x_i be a good vertex of $\mathcal{C}_i$ for $1 \le i \le r-3$. From $|\cup_{i=1}^{r-3} V(\mathcal{C}_i)| \ge (r-1)+(r-3) = 2r-4$, we get $|\cup_{i=1}^{r-3} V(\mathcal{C}_i) \setminus \{x_1, \cdots, x_{r-3}, u, v\}| \ge r-3 \ge 2$ while $|P(u,v) \setminus \{x_1, \cdots, x_{r-3}\}| \le 1$. Therefore at least one vertex z of $\cup_{i=1}^{r-3} V(\mathcal{C}_i) \setminus \{x_1, \cdots, x_{r-3}, u, v\}$ is not in $P(u,v)$. Let z' be a good neighbor of z not in $N(u,v)$. Since $\{x_1, \cdots, x_{r-3}, z'\} \subseteq P(u,z)$ and $|P(u,z)| \le r-2$, v is not in $P(u,z)$ and thus belongs to a $(r-2)$th good clique. This contradicts Claim 3. □

End of the proof of Theorem 1 for r=5

By Claims 2, 3 and 4, each good clique contains exactly one good vertex and four non-good ones, each non-good vertex is contained in exactly two good K_5's, and two good K_5's intersect in at most one vertex. Therefore the graph F belongs to the family $\mathcal{F}_5$ described above and thus $\gamma_t(F) < \frac{2n}{6}$. This completes the proof for $r = 5$.

End of the proof of Theorem 1 for r=6

Henceforth, each good clique is a K_6 containing at most two good vertices.

Claim 5 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two good K_6's such that $|V(\mathcal{C}_1) \cap V(\mathcal{C}_2)| \ge 2$. Then*

1. *$|V(\mathcal{C}) \cap V(\mathcal{C}_i)| \le 1$ for each other good clique $\mathcal{C}$ and $i = 1, 2$;*
2. *the clique $\mathcal{C}_i$ contains exactly one good vertex for $i = 1, 2$;*
3. *each other good clique $\mathcal{C}$ intersecting $\mathcal{C}_1$ or $\mathcal{C}_2$, contains exactly one good vertex.*

Proof:

(1) Let $u, v \in V(\mathcal{C}_1) \cap V(\mathcal{C}_2)$ and let $\mathcal{C}$ be another good clique in G. Assume, to the contrary, $|V(\mathcal{C}_1) \cap V(\mathcal{C})| \ge 2$. Let x, x_1, x_2 be good vertices of $\mathcal{C}, \mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Suppose $u \in \mathcal{C}$. By Claim 4, $v \notin \mathcal{C}$. Let $w \in V(\mathcal{C}) \cap V(\mathcal{C}_1)$ and $w \ne u$. Since $\{x, x_1, x_2\} \subseteq P(u,v)$ and $|P(u,v)| \le 4$, at least one vertex t of $V(\mathcal{C}_1) \setminus \{u, v, x_1, w\}$ is not in $P(u,v)$. Let t' be a good neighbor of t not in $N(u,v)$. Now we have $\{x, x_1, x_2, t'\} \subseteq P(u,t)$. Since $|P(u,t)| \le 4$, w is not in $P(u,t)$ and has a good neighbor w' not in $N(u,t)$. Then $\{x, x_1, x_2, w'\} \subseteq P(u,w)$. Thus v is not in $P(u,w)$ and has a good neighbor v' not in $N(u,w)$. This implies $\{x, x_1, x_2, w', v'\} \subseteq P(v,w)$ which is a contradiction. Thus $u, v \notin V(\mathcal{C})$. Let $w_1, w_2 \in V(\mathcal{C}) \cap V(\mathcal{C}_1)$. Since $\{x, x_1, x_2\} \subseteq P(u, w_1)$ and $|P(u,w_1)| \le 4$, $v \notin P(u,w_1)$ or $w_2 \notin P(u,w_1)$.

First let $v \notin P(u,w_1)$. Let v' be a good neighbor of v not in $N(u,w_1)$. Now we have $\{x, x_1, x_2, v'\} \subseteq P(v,w_1)$. Since $|P(v,w_1)| \le 4$, w_2 is not in $P(v,w_1)$ and has a good neighbor w_2' not in $N(v,w_1)$. Now we have $\{x, x_1, x_2, v', w_2'\} \subseteq P(v,w_2)$ which is a contradiction.

Now let $w_2 \notin P(u,w_1)$. Let w_2' be a good neighbor of w_2 not in $N(u,w_1)$. Now we have $\{x, x_1, x_2, w_2'\} \subseteq P(u,w_2)$. Since $|P(u,w_2)| \le 4$, v is not in $P(u,w_2)$ and has a good neighbor v' not in $N(u,w_2)$. This implies that $|P(v,w_2)| \ge 5$ which is a contradiction.

(2) Suppose $\mathcal{C}_1$ contains two good vertices x_1 and x_1'. Since $|P(u,v)| \le 4$, $V(\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \{u,v\}$ has a non-good vertex, say w, not in $P(u,v)$. Let w' be a good neighbor of w not in $N(u,v)$. Then $\{x_1, x_1', x_2, w'\} \subseteq P(u,w)$. Hence v is not in $P(u,w)$ and has a good neighbor $v' \notin N(u,w)$, which implies $|P(v,w)| \ge 5$, a contradiction.

(3) Suppose $\mathcal{C}$ contains two good vertices y and y'. If $\mathcal{C}$ intersects $\mathcal{C}_1 \cup \mathcal{C}_2$ in u, then $\{x_1, x_2, y, y'\} \subseteq P(u, v)$ and there exists a vertex t of $V(\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \{u, v\}$ with a good neighbor t' not in $N(u, v)$. Then $\{x_1, x_2, y, y', t'\} \subseteq P(u, t)$, a contradiction. If $\mathcal{C}$ intersects $\mathcal{C}_1 \cup \mathcal{C}_2$ in w different from u and v, then, since $\{x_1, x_2, y, y'\} \subseteq P(u, w)$, v has a good neighbor v' not belonging to $N(u, w)$. Hence $\{x_1, x_2, y, y', v'\} \subseteq P(v, w)$, a contradiction.

□

Claim 6 *Let three good cliques $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ share one vertex u. Then*

1. $|V(\mathcal{C}) \cap V(\mathcal{C}_i)| \leq 1$ *for each other good clique $\mathcal{C}$ and $i = 1, 2, 3$;*
2. *for $i = 1, 2, 3$, each non-good vertex of $\mathcal{C}_i \setminus \{u\}$ belongs to exactly two good cliques;*
3. *for $i = 1, 2, 3$, each clique $\mathcal{C}_i$ and each good clique $\mathcal{C}$ intersecting one of the $\mathcal{C}_i$'s contains exactly one good vertex.*

Proof:

(1) Suppose that $\mathcal{C}$ is a good clique such that $|V(\mathcal{C}) \cap V(\mathcal{C}_1)| \geq 2$. By claim 3, $u \notin V(\mathcal{C})$. Let $v, w \in V(\mathcal{C}) \cap V(\mathcal{C}_1)$. Let x, x_1, x_2, x_3 be good vertices of $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$, respectively. We have $\{x, x_1, x_2, x_3\} \subseteq P(u, v)$ and so w is not in $P(u, v)$ and has a good neighbor w' not in $N(u, v)$. Then we have $|P(u, w)| \geq 5$ which is a contradiction.

(2) If a vertex $v \neq u$ of some $\mathcal{C}_i$ belongs to two other good cliques, let v' and v'' two good neighbors of v respectively belonging to these two cliques. Then $\{x_1, x_2, x_3, v', v''\} \subseteq P(u, v)$, a contradiction.

(3) If, say, $\mathcal{C}_1$ has a second good vertex x'_1, then $\mathcal{C}_2$ and $\mathcal{C}_3$ have one good vertex each, for otherwise $|P(u, v)| \geq 5$ for any neighbor v of u. Hence there exists at least one non-good vertex v belonging to exactly one of the $\mathcal{C}_i$'s. This vertex v has a good neighbor $v' \notin \{x_1, x'_1, x_2, x_3\}$ and $|P(u, v)| \geq 5$, a contradiction. If a good clique $\mathcal{C}$ intersecting one of the $\mathcal{C}_i$'s in one vertex v (necessarily different from u) contains two good neighbors x and x', then $\{x_1, x_2, x_3, x, x'\} \subseteq P(u, v)$, a contradiction.

□

Claim 7 *Let $\mathcal{C}$ be a good clique containing two good vertices z_1, z_2. Then*

1. *each good clique intersects $\mathcal{C}$ in at most one vertex;*
2. *each non-good vertex of $\mathcal{C}$ belongs to exactly two good cliques;*
3. *if $\mathcal{C}'$ is a good clique intersecting $\mathcal{C}$ in u, then $\mathcal{C}'$ contains exactly one good vertex, each non-good vertex of $\mathcal{C}'$ belongs to exactly two good cliques, $|V(\mathcal{C}') \cap V(\mathcal{C}_1)| \leq 1$ for each good clique $\mathcal{C}_1$ and if $|V(\mathcal{C}') \cap V(\mathcal{C}_1)| = 1$, then $\mathcal{C}_1$ contains exactly one good vertex.*

Proof: (1) and (2) are consequences of Claim 5 (2) and 6 (3).

(3) Let u' be a good vertex of $\mathcal{C}'$, w be a non-good vertex in $V(\mathcal{C}') \setminus \{u\}$ and w' a good neighbor of w not in $N(u)$. If w has another good neighbor w'', which can be either a second good vertex of $\mathcal{C}'$ or of a second clique $\mathcal{C}_1$ containing w, or a good vertex of a third good clique containing w, then $\{z_1, z_2, u', w', w''\} \subseteq P(u, w)$, a contradiction. If a good clique $\mathcal{C}_1$ intersects $\mathcal{C}'$ in v and w (both different from u by (2)), then $v \notin P(u, w)$ since $\{z_1, z_2, u', w'\} \subseteq P(u, w)$. Therefore v has another good neighbor $v' \notin N(u, w)$ and $\{z_1, z_2, u', w', v'\} \subseteq P(u, v)$, a contradiction. □

Let $V_i = \{u \in V(F) \mid u \text{ belongs to exactly } \ i \text{ good cliques}\}, \quad i = 1, 2, 3$. By Claim 2, V_1, V_2, V_3 partition $V(F)$. Obviously V_1 consists of all good vertices of F. Let t be the number of good cliques that contain two good vertices. Counting the number of edges of F with one endpoint in V_1 and another in $V_2 \cup V_3$, implies by Claim 7 that

$$5|V_1| - 2t = 2|V_2| + 4t + 3|V_3|.$$

On the other hand, we have

$$2n = 2(|V_1| + |V_2| + |V_3|).$$

It follows from the last two equations

$$|V_1| = \frac{2n}{7} + \frac{|V_3| + 6t}{7}\ . \tag{1}$$

The following claim gives the structure of the subgraph induced by V_3 in F.

Claim 8 *$F[V_3]$ is a disjoint union of s cliques with $s \geq |V_3|/5$.*

Proof: Let u and v be two adjacent vertices in V_3. If the edge uv belongs to only one good clique $\mathcal{C}_z$, let $\mathcal{C}_{u'}$ and $\mathcal{C}_{u''}$ (respectively $\mathcal{C}_{v'}$ and $\mathcal{C}_{v''}$) be the other two good cliques containing u (respectively v). Then $\{u', u'', v', v'', z\}$ is a set of five vertices contained in $P(u, v)$, a contradiction. Therefore every edge joining two vertices in V_3 is contained in exactly (by Claim 5) two good cliques. Let now uvw be a path of $F[V_3]$. Among the three good cliques containing v, two contain uv and two contain vw. Hence one of them, say $\mathcal{C}_z$, contains $\{u, v, w\}$ and u and w are adjacent. Moreover, the second good cliques respectively containing uv and vw are the same by Claim 5. This implies that $\{u, v, w\}$ is contained in exactly two good cliques $\mathcal{C}_z$ and $\mathcal{C}_{z'}$. The preceding arguments show that $F[V_3]$ is a disjoint union of s cliques Q_i. Each Q_i is a part of the intersection of two good cliques $\mathcal{C}_z$ and $\mathcal{C}_{z'}$, thus implying $|Q_i| \leq 5$, and each vertex u of Q_i belongs to a third clique intersecting $\mathcal{C}_z$ and $\mathcal{C}_{z'}$ exactly in u. Finally, since $|Q_i| \leq 5$, $s \geq |V_3|/5$. □

We define now the graph F^* with vertex set $\{z \in V(F) \mid z \text{ is a good vertex in F}\}$ and two vertices of F^* are adjacent if and only if they belong to the same clique or their corresponding good cliques have a common vertex. Since F is connected and each edge of F belongs to a good clique, the graph F^* is connected.

Three good vertices z_1, z_2, z_3 form a triangle in F^* if and only if

1. the good cliques $\mathcal{C}_{z_1}$, $\mathcal{C}_{z_2}$ and $\mathcal{C}_{z_3}$ are different and share one vertex,

2. or, say, $\mathcal{C}_{z_1} = \mathcal{C}_{z_2}$ and $\mathcal{C}_{z_1} \cap \mathcal{C}_{z_3} \neq \emptyset$,

3. or the three cliques are pairwise intersecting but $\mathcal{C}_{z_1} \cap \mathcal{C}_{z_2} \cap \mathcal{C}_{z_3} = \emptyset$.

We only consider the triangles of the first two types and call them respectively 1-triangles and 2-triangles.

A 1-triangle of F^* comes from a vertex of V_3. From Claim 6 (2), if two 1-triangles $z_1z_2z_3$ and $z'_1z'_2z'_3$ are not disjoint, then they share one edge, say, $z_1 = z'_1$ and $z_2 = z'_2$. From Claim 8, $|V(\mathcal{C}_{z_1}) \cap V(\mathcal{C}_{z_2})| \geq 2$ and each good clique $\mathcal{C}_{z_3}$ and $\mathcal{C}_{z'_3}$ shares one vertex with $\mathcal{C}_{z_1}$ and $\mathcal{C}_{z_2}$. Since $|V(\mathcal{C}_{z_1}) \cap V(\mathcal{C}_{z_2})| \leq 5$, at most five 1-triangles share a common edge. Hence the 1-triangles of F^* form multitriangles MT_i of respective order p_i with $3 \leq p_i \leq 7$. We call them multitriangles of type 1 and we associate to each of them the clique $Q_i \subseteq V(\mathcal{C}_{z_1}) \cap V(\mathcal{C}_{z_2})$ of order $p_i - 2 \leq 5$ as described in Claim 8. Therefore there are $s \geq |V_3|/5$ multitriangles of type 1 and all of them are disjoint.

A 2-triangle of F^* comes from a good clique $\mathcal{C}$ of F with two good vertices z_1 and z_2. By Claim 7, the four other vertices of $\mathcal{C}$ belong to exactly one other good clique and these four good cliques are different. Hence the edge z_1z_2 belongs to exactly four 2-triangles forming a multitriangle of order $p_i = 6$, called multitriangle of type 2. To each multitriangle MT_i of type 2 we associate the clique Q_i of order $p_i - 2 = 4$ of F formed by the non-good vertices of $\mathcal{C}$. There are t multitriangles of type 2, the number of good cliques with two good vertices. By Claim 7, they are pairwise disjoint and disjoint from the multitriangles of type 1.

Let F^{**} be a spanning subgraph of F^* containing all the edges of the multitriangles but no other cycle (the edges of F^{**} not in multitriangles form a spanning tree of the graph of order $|V_1| - \sum_{i=1}^{s+t}(p_i - 1)$ obtained from F^* by contracting each multitriangle into one vertex). We form a subset D of vertices of F as follows. For each multitriangle MT_i of order p_i, $0 \leq i \leq s+t$, put in D the $p_i - 2$ vertices of its associated clique Q_i. For each edge z_iz_j of F^{**} not in a multitriangle, put in D one vertex of $\mathcal{C}_{z_i} \cap \mathcal{C}_{z_j}$. The induced subgraph $F[D]$ is connected since F^{**} is connected and can be seen as the graph representative of the 1- and 2-triangles and of the cutting edges of F^{**}. The set D contains a vertex in each good clique and thus dominates F. Hence $\gamma_t(F) \leq |D|$. Since F^{**} contains $|V_1| - \sum_{i=1}^{s+t}(p_i - 1) - 1$ cutting edges,

$$|D| = \sum_{i=1}^{s+t}(p_i - 2) + |V_1| - \sum_{i=1}^{s+t}(p_i - 1) - 1 = |V_1| - s - t - 1$$

with $s \geq |V_3|/5$. By (1) we get

$$|D| \leq \frac{2n}{7} + \frac{|V_3|}{7} + \frac{6t}{7} - \frac{|V_3|}{5} - t - 1 < \frac{2n}{7}\ .$$

This completes the proof of Theorem 1 for $r = 6$. □

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