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Bounded-Degree Graphs have Arbitrarily Large Queue-Number

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It is proved that there exist graphs of bounded degree with arbitrarily large queue-number. In particular, for all $\Delta \geq 3$ and for all sufficiently large n, there is a simple Δ-regular n-vertex graph with queue-number at least $c\sqrt{\Delta}n^{1/2-1/\Delta}$ for some absolute constant c.

Keywords: graph, queue layout, queue-number, track layout, track-number

Mathematics Subject Classification: 05C62 (graph representations), 05C30 (enumeration of graphs and maps)

1 Introduction

We consider graphs possibly with loops but with no parallel edges. A graph without loops is *simple*. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $S \subseteq E(G)$ then $G[S]$ denotes the spanning subgraph of G with edge set S. We say G is *ordered* if $V(G) = \{1, 2, \dots, |V(G)|\}$. Let G be an ordered graph. Let $\ell(e)$ and $r(e)$ denote the endpoints of each edge $e \in E(G)$ such that $\ell(e) \leq r(e)$. Two edges e and f are *nested* and f is *nested inside* e if $\ell(e) < \ell(f)$ and $r(f) < r(e)$. An ordered graph is a *queue* if no two edges are nested. Observe that the left and right endpoints of the edges in a queue are in first-in-first-out order—hence the name 'queue'. An ordered graph G is a k-*queue* if there is a partition $\{E_1, E_2, \dots, E_k\}$ of $E(G)$ such that each $G[E_i]$ is a queue.

Let G be an (unordered) graph. A k-*queue layout* of G is a k-queue that is isomorphic to G. The *queue-number* of G is the minimum integer k such that G has a k-queue layout. Queue layouts and queue-number were introduced by Heath et al. [15, 16] in 1992, and have applications in sorting permutations [12, 17, 23, 25, 29], parallel process scheduling [3], matrix computations [24], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in [7, 8, 10, 14, 26, 27, 30].

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Prior to this work it was unknown whether graphs of bounded degree have bounded queue-number. The main contribution of this note is to prove that there exist graphs of bounded degree with arbitrarily large queue-number.

Theorem 1 *For all $\Delta \geq 3$ and for all sufficiently large $n > n(\Delta)$, there is a simple Δ-regular n-vertex graph with queue-number at least $c\sqrt{\Delta}n^{1/2-1/\Delta}$ for some absolute constant c.*

The best known upper bound on the queue-number of a graph with maximum degree Δ is $\mathsf{e}(\Delta n/2)^{1/2}$ due to Dujmović and Wood [8] (where e is the base of the natural logarithm). Observe that for large Δ, the lower bound in Theorem 1 tends toward this upper bound. Although for specific values of Δ a gap remains. For example, for $\Delta = 3$ we have an existential lower bound of $\Omega(n^{1/6})$ and a universal upper bound of $\mathcal{O}(n^{1/2})$.

Closely related to a queue layout is the notion of a track layout. Informally speaking, a *track layout* of a graph consists of a proper vertex colouring, and a total order of each colour class, such that between each pair of colour classes, no two edges cross (with respect to the orders of the colour classes that contain the endpoints of the edges). The *track-number* of a graph G, denoted by $\mathsf{tn}(G)$, is the minimum number of colours in a track layout of G; see [4–7, 9–11, 13]. Dujmović et al. [6] proved that $\mathsf{qn}(G) \leq \mathsf{tn}(G) - 1$. Thus a lower bound on the queue-number also provides a lower bound on the track-number[(i)]. In particular, Theorem 1 implies:

Theorem 2 *For all $\Delta \geq 3$ and for all sufficiently large $n > n(\Delta)$, there is a simple Δ-regular n-vertex graph with track-number at least $c\sqrt{\Delta}n^{1/2-1/\Delta}$ for some absolute constant c.*

Note that there is also a $\mathcal{O}(n^{1/2})$ upper bound on the track-number of graphs with bounded degree. The best result is $\mathsf{tn}(G) \leq 5\Delta(G)\sqrt{2n}$, which follows from the result by Dujmović and Wood [11] that $\mathsf{tn}(G) \leq 5d\sqrt{2n}$ for every d-degenerate[(ii)] graph G. Thus for large Δ, the lower bound in Theorem 2 tends toward this upper bound.

2 Proof of Theorem 1

The proof of Theorem 1 is modelled on a similar proof by Barát et al. [1]. Basically, we show that there are more Δ-regular graphs than graphs with bounded queue-number. The following lower bound on the number of Δ-regular graphs is a corollary of more precise bounds due to Bender and Canfield [2], Wormald [31], and McKay [22]; see [1].

Lemma 1 ([2, 22, 31]) *For all integers $\Delta \geq 1$ and for sufficiently large $n \geq n(\Delta)$, the number of labelled simple Δ-regular n-vertex graphs is at least*

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2}$$

It remains to count the graphs with bounded queue-number. We will need the following lemma, which was previously known only for loopless graphs.

Lemma 2 ([8]) *Every n-vertex queue has at most $2n - 1$ edges.*

(i) A converse result in fact holds. Dujmović et al. [7] proved that track-number is bounded by a function of queue-number. In particular, $\mathsf{tn}(G) \leq 4\,\mathsf{qn}(G) \cdot 4^{\mathsf{qn}(G)(2\,\mathsf{qn}(G)-1)(4\,\mathsf{qn}(G)-1)}$ for every graph G.

(ii) A graph G is d-degenerate if every subgraph of G has a vertex of degree at most d.

Proof: If $v + w = x + y$ for two distinct edges vw and xy, then vw and xy are nested. No two edges are nested in a queue. The result follows since $2 \leq v + w \leq 2n$. $\square$

Let $g(n)$ be the number of queues on n vertices. For our purposes it suffices to show that $g(n) \leq c^n$. While such a bound can be concluded from the work of Klazar [19], who described a generating function for $g(n)$ with rough asymptotic analysis, we include a different and simpler proof for the sake of completeness. It is similar to a proof of a more general result by Klazar [20]; also see [21, 28] for other related and more general results.

Lemma 3 *$g(n) \leq c^n$ for some absolute constant c.*

Proof: Say G is an n-vertex queue. Let G' be an ordered $2n$-vertex graph obtained by the following *doubling* operation. For every edge vw of G, add to G' a non-empty set of edges between $\{2v-1, 2v\}$ and $\{2w-1, 2w\}$, no pair of which are nested. If $v \neq w$ then there are 11 possible ways to do this, as illustrated in Figure 1, and if $v = w$ then there are 7 possible ways to do this, as illustrated in Figure 2. Now G has at most $2n-1$ edges by Lemma 2. Thus at most 11^{2n-1} queues on $2n$ vertices can be obtained from G by doubling. On the other hand, every $2n$-vertex queue can be obtained from some n-vertex queue by doubling. To see this, merge every second pair of vertices, introduce a loop between merged vertices that are adjacent, and replace any resulting parallel edges by a single edge. No two edges are nested in the obtained graph. Hence $g(2n) \leq 11^{2n-1} \cdot g(n)$. It follows that $g(n) \leq 11^{2n}$. $\square$

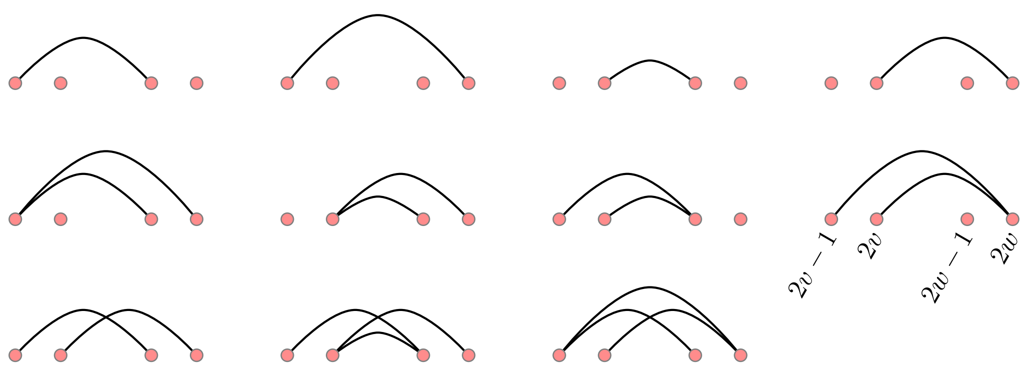

Fig. 1: The 11 possible ways to add a non-empty set of non-nested edges between $\{2v-1, 2v\}$ and $\{2w-1, 2w\}$.

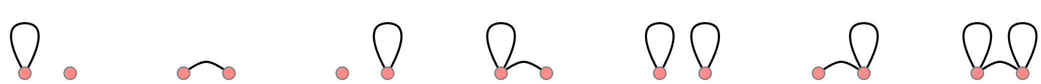

Fig. 2: The 7 possible ways to add a non-empty set of edges on $2v-1$ and $2v$.

Lemma 3 implies:

Corollary 1 *The number of k-queues on n vertices is at most c^{kn} for some absolute constant c.*

It is easily seen that Lemma 1 and Corollary 1 imply a lower bound of $c(\Delta/2-1)\log n$ on the queue-number of some Δ-regular n-vertex graph. To improve this logarithmic bound to polynomial, we now give a more precise analysis of the number of k-queues that also accounts for the number of edges in the graph. Let $g(n,m)$ be the number of queues on n vertices and m edges.

Lemma 4

$$g(n,m) \leq \begin{cases} \binom{n}{2m} \cdot c^{2m} & \text{, if } m \leq \frac{n}{2} \\ c^n & \text{, if } m > \frac{n}{2}, \end{cases}$$

for some absolute constant c.

Proof: By Lemma 3, we have the upper bound of c^n regardless of m. Suppose that $m \leq \frac{n}{2}$. An m-edge graph has at most $2m$ vertices of non-zero degree. Thus every n-vertex m-edge queue is obtained by first choosing a set S of $2m$ vertices, and then choosing a queue with $|S|$ vertices. The result follows. □

Let $g(n,m,k)$ be the number of k-queues on n vertices and m edges.

Lemma 5 *For all integers k such that $\frac{2m}{n} \leq k \leq m$,*

$$g(n,m,k) \leq \left(\frac{ckn}{m}\right)^{2m}$$

for some absolute constant c.

Proof: Fix non-negative integers $m_1 \leq m_2 \leq \cdots \leq m_k$ such that $\sum_i m_i = m$. Let $g(n; m_1, m_2, \dots, m_k)$ be the number of k-queues G on n vertices such that there is a partition $\{E_1, E_2, \dots, E_k\}$ of $E(G)$, and each $G[E_i]$ is a queue with $|E_i| = m_i$. Then

$$g(n; m_1, m_2, \dots, m_k) \leq \prod_{i=1}^{k} g(n, m_i).$$

Now $m_1 \leq \frac{n}{2}$, as otherwise $m > \frac{kn}{2} \geq m$. Let j be the maximum index such that $m_j \leq \frac{n}{2}$. By Lemma 4,

$$g(n; m_1, m_2, \dots, m_k) \leq \left(\prod_{i=1}^{j} \binom{n}{2m_i} c^{2m_i}\right) (c^n)^{k-j}.$$

Now $\sum_{i=1}^{j} m_i \leq m - \frac{1}{2}(k-j)n$. Thus

$$\begin{aligned} g(n; m_1, m_2, \dots, m_k) &\leq \left(\prod_{i=1}^{j} \binom{n}{2m_i}\right) \left(c^{2m-(k-j)n}\right) \left(c^{(k-j)n}\right) \\ &\leq c^{2m} \prod_{i=1}^{k} \binom{n}{2m_i}. \end{aligned}$$

We can suppose that k divides $2m$. It follows (see [1]) that

$$g(n;m_1,m_2,\dots,m_k) \le c^{2m}\binom{n}{2m/k}^k.$$

It is well known [18, Proposition 1.3] that $\binom{n}{t} < (\mathsf{e}n/t)^t$. Thus with $t = 2m/k$ we have

$$g(n;m_1,m_2,\dots,m_k) < \left(\frac{c\,\mathsf{e}\,kn}{2m}\right)^{2m}.$$

Clearly

$$g(n,m,k) \le \sum_{m_1,\dots,m_k} g(n;m_1,m_2,\dots,m_k),$$

where the sum is taken over all non-negative integers $m_1 \le m_2 \le \dots \le m_k$ such that $\sum_i m_i = m$. The number of such partitions [18, Proposition 1.4] is at most

$$\binom{k+m-1}{m} < \binom{2m}{m} < 2^{2m}.$$

Hence

$$g(n,m,k) \le 2^{2m}\left(\frac{c\,\mathsf{e}\,kn}{2m}\right)^{2m}.$$

□

Every ordered graph on n vertices is isomorphic to at most $n!$ labelled graphs on n vertices. Thus Lemma 5 has the following corollary.

Corollary 2 *For all integers k such that $\frac{2m}{n} \le k \le m$, the number of labelled n-vertex m-edge graphs with queue-number at most k is at most*

$$\left(\frac{ckn}{m}\right)^{2m} n!,$$

for some absolute constant c. □

Proof of Theorem 1: Let k be the minimum integer such that every simple Δ-regular graph with n vertices has queue-number at most k. Thus the number of labelled simple Δ-regular graphs on n vertices is at most the number of labelled graphs with n vertices, $\frac{1}{2}\Delta n$ edges, and queue-number at most k. By Lemma 1 and Corollary 2,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \le \left(\frac{ck}{\Delta}\right)^{\Delta n} n! \le \left(\frac{ck}{\Delta}\right)^{\Delta n} n^n.$$

Hence $k \ge \sqrt{\Delta}n^{1/2-1/\Delta}/(\sqrt{3}c)$. □

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