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Multidimensional cellular automata and generalization of Fekete's lemma

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Fekete's lemma is a well-known combinatorial result on number sequences: we extend it to functions defined on d-tuples of integers. As an application of the new variant, we show that nonsurjective d-dimensional cellular automata are characterized by loss of arbitrarily much information on finite supports, at a growth rate greater than that of the support's boundary determined by the automaton's neighbourhood index.

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1 Introduction

Let $f : \{1, 2, \ldots\} \to [0, +\infty)$. **Fekete's lemma** [4, 11] states that, if $f(n+k) \leq f(n) + f(k)$ for every n and k, then

$$\lim_{n \to \infty} \frac{f(n)}{n} = \inf_{n \geq 1} \frac{f(n)}{n} \ . \tag{1}$$

The consequences of this simple statement are many and deep, such as the definition of *topological entropy* for dynamical systems [6] and Arratia's bound on the number of permutations avoiding a given pattern [2].

More recently, in a joint work with T. Toffoli and P. Mentrasti [10], we have made use of (1) to prove a result about *cellular automata* (CA). These are presentations of global dynamics in local terms: each global state is a d-dimensional *configuration*, and the global evolution rule changes the state locally at a site by considering only the states of *neighbouring* sites. The task of retrieving *global* properties from such *local* descriptions is in general very difficult, and often leads to undecidability issues. In [10], we were especially concerned with *surjectivity*, a subject on which a vast and fascinating literature exists.

The most straightforward characterization of surjective CA (cf. [8, 9]) follows from the compactness of the space of configurations: the global evolution function of a CA is surjective if and only if every *pattern* (*i.e.*, sub-configuration on a *finite* region of the space) has at least one predecessor according to the evolution rule. From this "simple" argument, and from the properties of the group $\mathbb{Z}^d$, many beautiful theorems follow; of these, *Moore-Myhill's theorem* [8, 9] and *Maruoka-Kimura's balancement theorem* [7] deserve more than a mere citation, and shall be stated later in this paper.

The presence of "Garden-of-Eden patterns" can be restated by saying that nonsurjective CA *lose variety within finite range*, "variety" being the logarithm $V(n)$ of the number of output states on a "patch" (finite connected region) of size n. This quantity is easily checked to be subadditive; we could thus apply (1) and deduce that, in nonsurjective CA, the "loss of variety" $n - V(n)$ must, for n large enough, be larger that δn for some $\delta > 0$, and in particular, larger than the size of the support's *boundary* determined by the neighborhood index. This loss, in turn, allows *compressing* the state of both the support above and its boundary, into as much as it can be encoded by the support alone. Relying on this fact, we have devised a general algorithm [10] to translate from a presentation using an n-inputs, 1-output local map (*i.e.*, CA) to one employing n-inputs, n-outputs events, characteristic of a different class of presentations, specifically, that of *lattice gases* (LG).

In this paper, we state and prove a multivariate version of Fekete's lemma. The motivation for this, is to provide a support to the conjecture that the translation algorithm in [10] could be extended to arbitrary dimension; in particular, we want to prove that the loss of variety of nonsurjective CA grows sufficiently large to encode the boundary, regardless of the dimension of the space. This is not immediate, because for $d > 1$ the boundary grows with the support; but seems feasible, because the boundary grows, in a certain sense, "less" than the support—which, incidentally, is one of the properties of $\mathbb{Z}^d$ that make Moore-Myhill's theorem true. To prove our generalization, we rearrange a proof of (1) so that it works on sequences of integer d-tuples, after a suitable ordering on these is defined. As applications for the newfound formula, we prove two facts. The first one tells that loss of variety for d-dimensional nonsurjective CA is sufficient, on supports large enough, to allow encoding the state of the boundary: which is noteworthy, because for $d > 1$ the size of the boundary increases with that of the region. Incidentally, we get a criterion for CA surjectivity—if the loss of variety is bounded, then the CA is surjective—which, as far as we know, is first stated in this paper. The second fact is that the *Weyl pseudodistance*, introduced by [3] in the study of CA dynamics and defined as an upper limit, is in fact a limit: this property is not shared by its main "competitor", the *Besicovitch pseudodistance*.

2 Fekete's lemma, multivariate

A subadditive, nonnegative, univariate function $f(n)$ can be thought of as the *maximum information achievable with n observations* of some given phenomenon. Then Fekete's lemma says that the *average maximum information per observation* converges to its greatest lower bound.

To extend (1) to several variables—which, at the best of our knowledge, has never been done before—we first need to understand what the *meaning* of an inequality $a \leq b$ should be when a and b are vectors; this should reduce to the standard ordering of integers in the case of unidimensional arrays. Next, we have to identify a notion of limit which is in accord with that meaning. Finally, we must write down a version of (1) that keeps into account the number d of components, and reduces to (1) for $d = 1$.

Let $\mathbb{Z}_+ = \mathbb{Z} \cap (0, +\infty)$. Consider the **product ordering** on $\mathbb{Z}_+^d$ defined by $x \leq_\pi y$ iff $x_i \leq y_i$ for every $i \in \{1, \ldots, d\}$. This is the kind of ordering used, *e.g.*, in *linear programming*, by writing $Ax \leq b$ to indicate a set of constraints $a_{1,1}x_1 + \ldots + a_{1,n}x_n \leq b_1, \ldots, a_{m,1}x_1 + \ldots + a_{m,n}x_n \leq b_m$; it is also the *finest* ordering that makes the projections *monotonic*. Observe that $\mathcal{Z}^d = (\mathbb{Z}_+^d, \leq_\pi)$ is a **directed set**, *i.e.*, for any two $x, y \in \mathbb{Z}_+^d$ there exists $z \in \mathbb{Z}_+^d$ such that both $x \leq_\pi z$ and $y \leq_\pi z$. If $\mathcal{X} = (X, \leq)$ is a directed set and $f : X \to \mathbb{R}$, the **lower** and **upper limit** of f in $\mathcal{X}$ are defined as usual, *i.e.*,

$$\liminf_{x \in \mathcal{X}} f(x) = \sup_{x \in X} \inf_{y \geq x} f(y) \quad \text{and} \quad \limsup_{x \in \mathcal{X}} f(x) = \inf_{x \in X} \sup_{y \geq x} f(y) \ ;$$

moreover, f has **limit** $L \in \mathbb{R}$ in $\mathcal{X}$, written $\lim_{x \in \mathcal{X}} f(x) = L$, if for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $|f(x) - L| < \varepsilon$ for every $x \geq x_\varepsilon$. For example, if $r_1, \ldots, r_d \in \mathbb{N} = \mathbb{Z}_+ \cup \{0\}$ are fixed, then

$$\lim_{(x_1, \ldots, x_d) \in \mathcal{Z}^d} \frac{(x_1 + r_1) \cdots (x_d + r_d)}{x_1 \cdots x_d} = 1 \ . \tag{2}$$

It follows from the definitions that $\liminf_{x \in \mathcal{X}} f(x) \leq \limsup_{x \in \mathcal{X}} f(x)$, and that $\lim_{x \in \mathcal{X}} f(x) = L$ iff $\liminf_{x \in \mathcal{X}} f(x) = \limsup_{x \in \mathcal{X}} f(x) = L$. In this case, the limit L can be *recovered on any path going to infinity*. More formally, suppose that a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ satisfies the following property: for every $x \in X$ there exists $n_x \in \mathbb{N}$ such that $x_n \geq x$ for every $n > n_x$. Then the sequence $\{f(x_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ converges to L in the usual sense.

Theorem 1 *Let* $f : \mathbb{Z}_+^d \to [0, +\infty)$ *satisfy*

$$f(x_1, \ldots, x_j + y_j, \ldots, x_d) \leq f(x_1, \ldots, x_j, \ldots, x_d) + f(x_1, \ldots, y_j, \ldots, x_d) \tag{3}$$

for every $x_1, \ldots, x_n, y_j \in \mathbb{Z}_+$, $j \in \{1, \ldots, d\}$. *Then*

$$L = \lim_{(x_1, \ldots, x_d) \in \mathcal{Z}^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \tag{4}$$

exists, and equals

$$\inf_{x_1, \ldots, x_d \in \mathbb{Z}_+} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \ . \tag{5}$$

Proof: Because of (3), for every $j \in \{1, \ldots, d\}$, $x_1, \ldots, x_d \in \mathbb{Z}_+$, if $x_j = qt + r$ with $q \in \mathbb{N}$ and $r \in \mathbb{Z}_+$, then

$$f(x_1, \ldots, x_j, \ldots, x_d) \leq q \cdot f(x_1, \ldots, t, \ldots, x_d) + f(x_1, \ldots, r, \ldots, x_d) \ . \tag{6}$$

Fix $t_1, \ldots, t_d \in \mathbb{Z}_+$. For each $(x_1, \ldots, x_d) \in \mathbb{Z}_+^d$, d pairs $(q_j, r_j) \in \mathbb{N} \times \mathbb{Z}_+$ are uniquely determined by $x_j = q_j t_j + r_j$ and $1 \leq r_j \leq t_j$. By repeatedly applying (6) to all of the x_j's, for $y_j^{(0)} = r_j$ and $y_j^{(1)} = t_j$ we find

$$f(x_1, \ldots, x_d) \leq \sum_{\alpha \in \{0,1\}^d} q_1^{\alpha_1} \cdots q_d^{\alpha_d} \cdot f\left(y_1^{(\alpha_1)}, \ldots, y_d^{(\alpha_d)}\right) \ . \tag{7}$$

Note how, on the right-hand side of (7), each occurrence of f has k arguments chosen from the t_j's and $d - k$ chosen from the r_j's, is multiplied by the q_j's corresponding to the t_j's, and is bounded from above by the constant $M = t_1 \cdots t_d \cdot f(1, \ldots, 1)$. By dividing both sides of (7) by $x_1 \cdots x_d$ we get

$$\frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \frac{q_1 \cdots q_d}{x_1 \cdots x_d} f(t_1, \ldots, t_d) + M \cdot \sum_{\alpha \in \{0,1\}^d \setminus \{1^d\}} \frac{q_1^{\alpha_1} \cdots q_d^{\alpha_d}}{x_1 \cdots x_d} \ . \tag{8}$$

Now, by construction, $\lim_{x_j \to \infty} q_j / x_j = 1/t_j$. If all the x_j's are large enough, the first summand in (8) becomes very close to $f(t_1, \ldots, t_d)/t_1 \cdots t_d$, and the other ones become very small; in other words, for every $\varepsilon > 0$, there exist $x_1^{(\varepsilon)}, \ldots, x_d^{(\varepsilon)} \in \mathbb{Z}_+$ such that, if $x_i \geq x_i^{(\varepsilon)}$ for each $i = 1, \ldots, d$, then

$$\frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} < \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} + \varepsilon \; .$$

From this and the definition of $\leq_\pi$ follows

$$\limsup_{(x_1, \ldots, x_d) \in \mathcal{Z}^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} \; ;$$

this is true whatever the t_j's are, hence

$$\limsup_{(x_1, \ldots, x_d) \in \mathcal{Z}^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \inf_{t_1, \ldots, t_d \in \mathbb{Z}_+} \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} \; .$$

Equality between (4) and (5) follows then from

$$\inf_{t_1, \ldots, t_d \in \mathbb{Z}_+} \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} \leq \liminf_{(x_1, \ldots, x_d) \in \mathcal{Z}^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \; .$$

□

We propose the following interpretation of Theorem 1. For a function f satisfying (3), the quantity $f(x_1, \ldots, x_j, \ldots, x_d)$ can be thought of as the maximum *joint* information obtainable by the observations of d *independent* observers, when the j-th of them has performed x_j observation. Theorem 1 then says that the *average maximum joint information per observation per observer* converges to its greatest lower bound.

Note that, for $d = 1$, Theorem 1 is the same as Fekete's lemma.

Corollary 2 *Let f and L be as in Theorem 1. If $\lim_{n \to \infty} x_{j,n} = +\infty$ for every $j \in \{1, \ldots, d\}$, then*

$$\lim_{n \to \infty} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}} = \inf_{n \geq 1} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}} = L \; . \tag{9}$$

In particular, $\lim_{n \to \infty} f(n, \ldots, n)/n^d = \inf_{n \geq 1} f(n, \ldots, n)/n^d = L$.

Proof: Straightforward from Theorem 1, the observations above, and

$$\inf_{x_1, \ldots, x_d \in \mathbb{Z}_+} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \inf_{n \geq 1} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}} \; ,$$

which follows from $A \supseteq B$ implying $\inf A \leq \inf B$. □

3 An application to cellular automata

A **cellular automaton** (briefly, CA) is a quadruple $\mathcal{A} = \langle d, Q, \mathcal{N}, f\rangle$ where the **dimension** $d > 0$ is an integer, the **set of states** Q is finite and has at least two distinct elements, the **neighbourhood index** $\mathcal{N} = \{\nu_1, \ldots, \nu_n\}$ is a finite subset of $\mathbb{Z}^d$, and the **local evolution function** f maps Q^n into Q. A **global evolution function** F is induced by f of the space $Q^{\mathbb{Z}^d}$ of d-dimensional **configurations** by

$$F(c)(x) = f\left(c(x+\nu_1), \ldots, c(x+\nu_n)\right) . \tag{10}$$

$\mathcal{A}$ is said to be surjective if F is. For example, if $d = 1$, $Q = \{0,1\}$, $\mathcal{N} = \{1\}$, $f(x) = x$, then $\langle d, Q, \mathcal{N}, f\rangle$ is the *shift cellular automaton* and $F(c)(x) = c(x+1)$ is the *shift map*, which is surjective; on the other hand, for same d and Q, $\mathcal{N} = \{0,1\}$, and $f(a,b) = a \cdot b$, we get a nonsurjective CA, because if $\overline{c}(x)$ is 0 for $x = 0$ and 1 otherwise, then $F(c) \neq \overline{c}$ for any c.

For every finite $E \subseteq \mathbb{Z}^d$, calling $E + \mathcal{N} = \{x + \nu \mid x \in E, \nu \in \mathcal{N}\}$, a function $F_E : Q^{E+\mathcal{N}} \to Q^E$ is induced by f, again by applying (10). Observe that the number $|F_E(Q^{E+\mathcal{N}})|$ of **patterns** over E obtainable by applying (10) does not depend on the *displacement* of E along $\mathbb{Z}^d$, *i.e.*, if $x + E = \{x + y \mid y \in E\}$, then $|F_{x+E}(Q^{x+E+\mathcal{N}})| = |F_E(Q^{E+\mathcal{N}})|$.

Put $E(x_1, \ldots, x_d) = \{z \in \mathbb{Z}^d \mid 0 \leq z_i < x_i \, \forall i\}$. Call **right d-polytope** any set of the form $H = y + E(x_1, \ldots, x_d)$ for some $y \in \mathbb{Z}^d$. (Here, "right" has the same meaning as in "right triangle".) If $\mathcal{N}$ is contained in a right d-polytope of sides $r_1, \ldots, r_d$, then $E(x_1, \ldots, x_d) + \mathcal{N}$ is contained in a right d-polytope of sides $x_1 + r_1, \ldots, x_d + r_d$, which is the disjoint union of $E(x_1, \ldots, x_d)$ and a **boundary**. It is well-known [8, 9] that $\mathcal{A}$ is surjective iff F_E is surjective for every right d-polytope E. From this characterization many more follow, two of which at least deserve some words.

Moore-Myhill's theorem [8, 9] states that a CA is surjective if and only if for no two *different* patterns p_i on the *same* d-hypercube E may happen that $c_i|_E = p_i$, $c_1|_{\mathcal{C}\setminus E} = c_2|_{\mathcal{C}\setminus E}$, and $F(c_1) = F(c_2)$. This has the important consequence that *injective CA are surjective*. Note that a key part of the proof relies on the fact that any boundary of fixed "range" has a lower growth rate than the right d-polytope it surrounds.

Maruoka-Kimura's balancement theorem [7] states that in a surjective CA each pattern on a *given* rectangular support has the *same* number of predecessors. Note that, in the proof of this theorem, the previous one is used. If $\mathcal{N}$ is contained in a right d-polytope of sides $r_1, \ldots, r_d$ (which we may call $\mathcal{N}'$) we may identify the original CA with one having neighbourhood index $\mathcal{N}'$; then the CA is surjective iff for all $x_1, \ldots, x_d \in \mathbb{Z}_+$, $p \in Q^{E(x_1, \ldots, x_d)}$

$$\left| F^{-1}_{E(x_1, \ldots, x_d)}(p) \right| = |Q|^{(x_1+r_1)\cdots(x_d+r_d) - x_1 \cdots x_d} .$$

Let $\mathcal{A} = \langle d, Q, \mathcal{N}, f\rangle$ be a CA. If $\mathcal{A}$ is nonsurjective, then there must exist a support of suitable size where not every possible pattern is reachable, *i.e.*, a part of the state is lost. In the 1D case [10], such lost state is proved to be ultimately as much as the boundary can transport; which allowed devising a CA-to-LG conversion algorithm. If the technique employed there is to be extended to higher dimension, then we must determine whether such large a loss can still be achieved.

Call **output size** of f on a right d-polytope of sides $x_1, \ldots, x_d$ the quantity

$$\mathrm{Out}_f(x_1, \ldots, x_d) = \left| F_{E(x_1, \ldots, x_d)}\left(Q^{E(x_1, \ldots, x_d)+\mathcal{N}}\right) \right| .$$

Then $\mathcal{A}$ is surjective iff $\mathrm{Out}_f(x_1, \ldots, x_d) = |Q|^{x_1 \cdots x_d}$ for every $x_1, \ldots, x_d \in \mathbb{Z}_+$. By switching to a logarithmic measure unit, we can associate to $\mathcal{A}$ a **loss of variety** on a right d-polytope of sides $x_1, \ldots, x_d$

defined as

$$\Lambda_{\mathcal{A}}(x_1, \ldots, x_d) = x_1 \cdots x_d - \log_{|Q|} \mathrm{Out}_f(x_1, \ldots, x_d)\ . \tag{11}$$

Observe how such loss is measured in *qits* (with $q = |Q|$), a $q$it being the amount of information carried by a q-states device; n $q$its correspond to $n \log_2 q$ bits.

Theorem 3 *Let $\mathcal{A} = \langle d, Q, \mathcal{N}, f \rangle$ be a CA. Define $\Lambda_{\mathcal{A}}$ as by (11). Then*

1. *either $\mathcal{A}$ is surjective and $\Lambda_{\mathcal{A}}$ is identically zero,*
2. *or $\mathcal{A}$ is nonsurjective and for every $K \geq 0,\ r_1, \ldots, r_d \in \mathbb{N}$, there exist $t_1, \ldots, t_d \in \mathbb{Z}_+$ such that, if $x_j \geq t_j$ for every $j \in \{1, \ldots, d\}$, then*

$$\Lambda_{\mathcal{A}}(x_1, \ldots, x_d) \geq (x_1 + r_1) \cdots (x_d + r_d) - x_1 \cdots x_d + K\ . \tag{12}$$

Proof: Put $q = |Q|$. Since a pattern over $E(x_1, \ldots, x_j + y_j, \ldots, x_d)$ can always be seen as the *joining* of a pattern over $E(x_1, \ldots, x_j, \ldots, x_d)$ and another one over $E(x_1, \ldots, y_j, \ldots, x_d)$, there cannot be more patterns obtainable over the former than pairs of patterns obtainable over the latter, *i.e.*,

$$\mathrm{Out}_f(x_1, \ldots, x_j + y_j, \ldots, x_d) \leq \mathrm{Out}_f(x_1, \ldots, x_j, \ldots, x_d) \cdot \mathrm{Out}_f(x_1, \ldots, y_j, \ldots, x_d)$$

whatever $x_1, \ldots, x_n, y_j \in \mathbb{Z}_+,\ j \in \{1, \ldots, d\}$ are; consequently, $\log_q \mathrm{Out}_f$ is subadditive in each of its arguments (and nonnegative). Let

$$\lambda_f = \lim_{(x_1, \ldots, x_d) \in \mathcal{Z}^d} \frac{\log_q \mathrm{Out}_f(x_1, \ldots, x_d)}{x_1 \cdots x_d}\ , \tag{13}$$

whose existence and value are given by Theorem 1; observe that $\lambda_f \leq 1$, and $\mathcal{A}$ is surjective iff $\lambda_f = 1$. Suppose $\mathcal{A}$ is nonsurjective. Let $\delta \in (\lambda_f, 1)$. Choose $t_1, \ldots, t_d \in \mathbb{Z}_+$ so that, for every $x_1 \geq t_1, \ldots, x_d \geq t_d$, both

$$\frac{\log_q \mathrm{Out}_f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \delta \tag{14}$$

and

$$\frac{(x_1 + r_1) \cdots (x_d + r_d) - x_1 \cdots x_d + K}{x_1 \cdots x_d} \leq 1 - \delta \tag{15}$$

are satisfied, the latter following from (2). Then, for such $x_1, \ldots, x_d$,

$$\begin{array}{rcl} x_1 \cdots x_d - \log_q \mathrm{Out}_f(x_1, \ldots, x_d) & \geq & (x_1 \cdots x_d)(1 - \delta) \\ & \geq & (x_1 + r_1) \cdots (x_d + r_d) - x_1 \cdots x_d + K\ . \end{array}$$

□

Corollary 4 *Let $\mathcal{A}$ be as in Theorem 3.*

1. *If $\Lambda_{\mathcal{A}}$ is bounded, then $\mathcal{A}$ is surjective.*
2. *If $\mathcal{A}$ is nonsurjective, then there exist $k_1, \ldots, k_d$ such that, if $x_i \geq k_i$ for all i, then $\Lambda_{\mathcal{A}}(x_1, \ldots, x_d) \geq |E(x_1, \ldots, x_d) + \mathcal{N}| - |E(x_1, \ldots, x_d)|$.*

Proof: Point 1 follows from point 2 of Theorem 3. For point 2, put $K = 0$ and choose $r_1, \dots, r_d$ so that $\mathcal{N} \subseteq z + E(r_1, \dots, r_d)$ for some $z \in \mathbb{Z}^d$. $\square$

The unbounded growth of $\Lambda_{\mathcal{A}}$ for nonsurjective $\mathcal{A}$ could also be proved via the following argument, akin to the one used in the proof of Moore-Myhill's theorem. Let π be a GoE pattern with support $E(k_1, \dots, k_d)$: then $\mathrm{Out}_f(k_1, \dots, k_d) \leq q^k - 1$, where $k = k_1 \cdots k_d$. For every $n \geq 1$, none of the n^d portions of the image via (10) of a pattern over $E(nk_1, \dots, nk_d) + \mathcal{N}$ can equal π, so in fact

$$\mathrm{Out}_f(nk_1, \dots, nk_d) \leq (q^k - 1)^{n^d} \quad \forall n \geq 1 \; ; \tag{16}$$

this implies

$$\Lambda_{\mathcal{A}}(nk_1, \dots, nk_d) \geq n^d(k - \log_q(q^k - 1)) \quad \forall n \geq 1 \; ,$$

and the factor in parentheses is a positive constant. However, the proof based on Fekete's lemma gives us more *comprehensive* information about the behaviour of $\mathcal{A}$. In fact, if we only rely on the existence of a GoE pattern, we only get a *family* of values on which $\Lambda_{\mathcal{A}}$ grows arbitrarily large; instead, Fekete's lemma immediately tells us that $\Lambda_{\mathcal{A}}$ grows *ultimately larger* than any given value. To get such thing from (16), we must first rewrite each x_i as $nk_i + r_i$, then employ

$$\mathrm{Out}_f(x_1, \dots, x_d) \leq (q^k - 1)^{n^d} \cdot q^{x_1 \cdots x_d - n^d \cdot k} \; .$$

Point 2 of Corollary 4 says that, for $(x_1, \dots, x_d)$ satisfying both (14) and (15), *the loss of variety is at least the size of the boundary*: this is precisely the fact used in [10], and supports the conjecture that a similar construction can be carried out in dimension $d > 1$. On the other hand, since surjectivity of d-dimensional CA is only decidable when $d = 1$ [1, 5], no algorithm exists to determine, given an arbitrary multidimensional CA, whether its loss of variety (11) is bounded.

4 A consideration on the Weyl pseudodistance

The **Weyl pseudodistance** on d-dimensional configurations is defined as

$$d_W(c_1, c_2) = \limsup_{(x_1, \dots, x_d) \in \mathcal{Z}^d} \left(\max_{y \in \mathbb{Z}^d} \frac{|\{z \in y + E(x_1, \dots, x_d) \mid c_1(z) \neq c_2(z)\}|}{x_1 \cdots x_d} \right) \; . \tag{17}$$

This is an extension of the definition given in [3] for one-dimensional configurations. Essentially, $d_W(c_1, c_2)$ is the upper limit of the maximum *probability* of getting different values for the c_i's when choosing at random a point in a given $(x_1, \dots, x_d)$-hypercube. Note that d_W is *translation invariant*, i.e., $d_W(c_1^x, c_2^x) = d_W(c_1, c_2)$ for every $x \in \mathbb{Z}^d$, $c_1, c_2 \in Q^{\mathbb{Z}^d}$, where $c^x(z) = c(x + z)$ for all $z \in \mathbb{Z}^d$: this is impossible for any distance inducing the product topology (cf. [3]). Moreover, CA induce continuous transformations of the quotient space obtained by identifying configurations having Weyl distance zero, and information about the behaviour of the CA can be inferred from that of the induced function.

The Weyl pseudodistance has been neglected in favor of the *Besicovitch pseudodistance* defined, for $d = 1$ (cf. [3]), as

$$d_B(c_1, c_2) = \limsup_{n \geq 1} \frac{|\{x \in [-n, \dots, n] \mid c_1(x) \neq c_2(x)\}|}{2n + 1} \; ,$$

which is also translation invariant and induces a quotient space with better topological properties, on which CA also induce continuous transformations whose properties may reflect those of the original. However, there is a feature distinctive of d_W which is not shared by d_B.

Theorem 5 *For any $c_1, c_2 \in Q^{\mathbb{Z}^d}$,*

$$d_W(c_1, c_2) = \lim_{(x_1,\dots,x_d)\in\mathcal{Z}^d} \left(\max_{y\in\mathbb{Z}^d} \frac{|\{z \in y + E(x_1,\dots,x_d) \mid c_1(z) \neq c_2(z)\}|}{x_1 \cdots x_d} \right) . \tag{18}$$

Moreover,

$$d_W(c_1, c_2) = \inf_{x_1,\dots,x_d\in\mathbb{Z}_+} \left(\max_{y\in\mathbb{Z}^d} \frac{|\{z \in y + E(x_1,\dots,x_d) \mid c_1(z) \neq c_2(z)\}|}{x_1 \cdots x_d} \right) . \tag{19}$$

Proof: For $y \in \mathbb{Z}^d$, $x_1, \dots, x_d \in \mathbb{Z}_+$, put

$$\Phi_y(x_1,\dots,x_d) = |\{z \in y + E(x_1,\dots,x_d) \mid c_1(z) \neq c_2(z)\}|$$

and

$$f(x_1,\dots,x_d) = \max_{y\in\mathbb{Z}} \Phi_y(x_1,\dots,x_d) .$$

Then

$$d_W(c_1, c_2) = \limsup_{(x_1,\dots,x)\, d\in\mathcal{Z}^d} \frac{f(x_1,\dots,x_d)}{x_1 \cdots x_d} .$$

Moreover, if $(z^{(j)})_i = x_j \cdot \delta_i^j$ (δ_i^j being the Kronecker symbol) then

$$\Phi_y(x_1,\dots,x_j + x'_j,\dots,x_d) = \Phi_y(x_1,\dots,x_j,\dots,x_d) + \Phi_{y+z^{(j)}}(x_1,\dots,x'_j,\dots,x_d) ,$$

so that

$$f(x_1,\dots,x_j + x'_j,\dots,x_d) \leq f(x_1,\dots,x_j,\dots,x_d) + f(x_1,\dots,x'_j,\dots,x_d) ,$$

because the maximum over y is a subadditive function and is invariant by translations of y by a fixed value. By Theorem 1,

$$\inf_{x_1,\dots,x_d\in\mathbb{Z}_+} \frac{f(x_1,\dots,x_d)}{x_1 \cdots x_d} = \lim_{(x_1,\dots,x_d)\in\mathcal{Z}^d} \frac{f(x_1,\dots,x_d)}{x_1 \cdots x_d} = \limsup_{(x_1,\dots,x_d)\in\mathcal{Z}^d} \frac{f(x_1,\dots,x_d)}{x_1 \cdots x_d} ,$$

which yields (18) and (19). □

We are now going to check that d_B is not, in general, a limit. Let $x_n = \sum_{k=1}^{n} 2^k$; let $c_1(x) = 0$ for every $x \in \mathbb{Z}$ and let

$$c_2(x) = \begin{cases} 0 & \text{if } x_{2k} < x \leq x_{2k+1} , \\ 1 & \text{if } x_{2k-1} < x \leq x_{2k} , \\ 1 & \text{if } x = 0 , \\ c_2(-x) & \text{if } x < 0 . \end{cases}$$

For n odd, the interval $[0, x_n]$ contains twice as many points where $c_2(x) = 0$ than points where $c_2(x) = 1$; for n even, the interval $[1, x_n]$ contains twice as many points where $c_2(x) = 1$ than points where

$c_2(x) = 0$. From this follows that $d_B(c_1, c_2) \geq 2/3$ but $\liminf_{n \in \mathbb{N}} |\{|x| \leq n, c_1(x) \neq c_2(x)\}|/(2n + 1) \leq 1/3$. (With a bit more patience one can see that equalities actually hold.)

As a consequence of Theorem 5 and Corollary 2, the Weyl pseudodistance can be obtained as a limit on any sequence of right d-polytopes which "grow infinitely large in all directions".

Corollary 6 *Let* $\lim_{n \to \infty} x_{i,n} = +\infty$ *for each* $i \in \{1, \dots, d\}$. *Then*

$$\lim_{n \to \infty} \max_{y \in \mathbb{Z}^d} \frac{|\{z \in y + E(x_{1,n}, \dots, x_{d,n}) \mid c_1(z) \neq c_2(z)\}|}{x_{1,n} \cdots x_{d,n}} = d_W(c_1, c_2) \ .$$

In particular,

$$\lim_{n \to \infty} \max_{y \in \mathbb{Z}^d} \frac{|\{z \in y + E(n, \dots, n) \mid c_1(z) \neq c_2(z)\}|}{n^d} = d_W(c_1, c_2) \ .$$

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