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On the multipacking number of grid graphs

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In 2001, Erwin introduced *broadcast domination* in graphs. It is a variant of classical domination where selected vertices may have different domination powers. The minimum cost of a dominating broadcast in a graph G is denoted $\gamma_b(G)$. The dual of this problem is called *multipacking*: a multipacking is a set $M \subseteq V(G)$ such that for any vertex v and any positive integer r, the ball of radius r around v contains at most r vertices of M. The maximum size of a multipacking in a graph G is denoted $\mathrm{mp}(G)$. Naturally $\mathrm{mp}(G) \leq \gamma_b(G)$. Earlier results by Farber and by Lubiw show that broadcast and multipacking numbers are equal for strongly chordal graphs.

In this paper, we show that all large grids (height at least 4 and width at least 7), which are far from being chordal, have their broadcast and multipacking numbers equal.

Keywords: grid graph, broadcast number, multipacking number

Introduction

Given a graph G with vertex set V and edge set E, a *dominating broadcast* of G is a function f from V to $\mathbb{N}$ such that for any vertex u in V, there is a vertex v in V with $f(v)$ positive and greater than the distance from u to v. Define the *ball of radius r around v* by $N_r(v) = \{u : d(u, v) \leq r\}$. Thus a dominating broadcast is a cover of the graph with balls of several positive radii. The *cost* of a dominating broadcast f is $\sum_{v \in V} f(v)$ and the minimum cost of a dominating broadcast in G, its *broadcast number*, is denoted $\gamma_b(G)$.

Remark. One may consider the cost to be any function of the powers (for example the sum of the squares), see e.g. [10]. We shall stick to the classical convention of linear cost.

The dual problem of broadcast domination is *multipacking*. A multipacking in a graph G is a subset M of its vertices such that for any positive integer r and any vertex v in V, the ball of radius r centred at v contains at most r vertices of M. The maximum size of a multipacking of G, its *multipacking number*, is denoted $\mathrm{mp}(G)$. We may write γ_b and mp when the graph in question is clear from context or unimportant.

Broadcast domination was introduced by Erwin [7, 8] in his doctoral thesis in 2001. Multipacking was then defined in Teshima's Master's Thesis [13] in 2012, see also [1]. However, this work fits into the

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general study of coverings and packings, which has a rich history in Graph Theory, see for example the monograph by Cornuéjols [3].

Since minimum dominating broadcast and multipacking are dual problems, we know that for any graph G,

$$\mathrm{mp}(G) \leq \gamma_b(G).$$

A natural question comes to mind. Under which conditions are they equal? For example, it is known that strongly chordal graphs have their broadcast and multipacking numbers equal. This follows from a primal-dual algorithm of Farber [9] applied to Γ-free matrices, used to solve the (weighted) dominating set problem for strongly chordal graphs. The work of Lubiw [11, 12] shows the vertex-neighbourhood ball incidence matrix is Γ-free for strongly chordal graphs, and hence the primal-dual algorithm can also be used to solve the broadcast domination problem for strongly chordal graphs. For trees, direct proofs of $\mathrm{mp}(T) = \gamma_b(T)$ and linear-time algorithms to find $\mathrm{mp}(T)$ appear in [1, 2] (see also [4, 5]). For strongly chordal graphs, Farber's algorithm runs in $O(n^3)$ time. The general broadcast domination problem can be solved in $O(n^6)$ time [10]. In this paper we study grid graphs which are far from being strongly chordal (or even chordal). We show the following theorem.

Theorem 1. *For any pair of integers $n \geq 4$ and $m \geq 4$,*

$$\mathrm{mp}(P_n \square P_m) = \gamma_b(P_n \square P_m).$$

with the exception of $P_4 \square P_6$ where $\mathrm{mp}(P_4 \square P_6) = 4$ and $\gamma_b(P_4 \square P_6) = 5$.

This gives an infinite family of non-chordal graphs for which $\mathrm{mp} = \gamma_b$. Another such family is the cycles of length 0 modulo 3. It is trivial to verify that $\mathrm{mp}(C_{3k}) = \gamma_b(C_{3k}) = k$.

Dunbar et al. [6] gave the exact value of the broadcast number for grids.

Theorem 2 (Dunbar et al. [6, Th. 28])**.** *For any pair of positive integers n and m,*

$$\gamma_b(P_n \square P_m) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor.$$

Remark. The value of $\gamma_b(P_n \square P_m)$ given by Theorem 2 is the radius of the grid. Since there is always a dominating broadcast with cost $\mathrm{rad}(G)$ [6, 8], and our proof of Theorem 1 yields a multipacking of size $\mathrm{rad}(G)$, this paper gives an alternative proof of Theorem 2.

1 Preliminaries and small grids

We use standard notation throughout the paper. Specific to our work is the following: the grid $P_n \square P_m$ has n rows and m columns. We may also say the grid has height n and length m. The vertex in row i and column j is denoted $v_{i,j}$. As a convention, the vertex $v_{0,0}$ is the bottom, left corner of the grid. The integers between k and ℓ inclusive are denoted $[\![k, \ell]\!]$.

The proof of Theorem 1 is technical. In order to ease the process, we start with an easy counting lemma.

Lemma 3. *Let G be a graph, k be a positive integer and $u_0, \ldots, u_{3k}$ be an isometric path in G. Let $P = \{u_{3i} : i \in [\![0, k]\!]\}$ be the set of every third vertex on this path. Then, for any positive integer r and any ball B of radius r in G,*

$$|B \cap P| \leq \left\lceil \frac{2r+1}{3} \right\rceil.$$

Proof: Let B be a ball of radius r in G, then any two vertices in B are at distance at most $2r$. Since the path $(u_0, \ldots, u_{3k})$ is isometric the intersection of the path and B is included in a subpath of length $2r$. This subpath contains at most $2r + 1$ vertices and only one third of those vertices can be in P. □

For the sake of completeness, we also determine the multipacking numbers of grids with height 2 and 3.

Proposition 4. *Let n be a positive integer. Then*

$$\mathrm{mp}(P_n \square P_2) = \left\lceil \frac{2n}{5} \right\rceil$$

Proof: Let P be a maximum multipacking of $P_n \square P_2$. We claim that no five consecutive columns contain three members of P. Suppose to the contrary that columns i to $i + 4$ contain three members of P. No two consecutive columns each contain a member of P, as any pair of vertices in $P_2 \square P_2$ are at distance at most 2 apart (and thus in a ball of radius 1). Hence, the three elements are without loss of generality $\{v_{i,0}, v_{i+2,1}, v_{i+4,0}\}$. However, this implies $|N_2[v_{i+2,0}] \cap P| = 3$, a contradiction.

Writing $n = 5q + r, 0 \leq r \leq 4$, we conclude that the first $5q$ columns of the grid contain at most $2q$ elements of P. Next, it is easy to verify that $\mathrm{mp}(P_1 \square P_2) = \mathrm{mp}(P_2 \square P_2) = 1$, and $\mathrm{mp}(P_3 \square P_2) = \mathrm{mp}(P_4 \square P_2) = 2$. Let s be the number of elements of P in the final r columns of the grid. Then, $s = 0$ if $r = 0$, $s \leq 1$ if $r = 1, 2$ and $s \leq 2$ if $r = 3, 4$. Thus, $|P| \leq 2q + \lceil 2r/5 \rceil$. Equivalently, $|P| \leq \lceil \frac{2n}{5} \rceil$.

On the other hand, consider the set P defined as follows.

$$\begin{aligned} v_{i,0} \in P \quad &\text{for} \quad i \equiv 0 \pmod 5 \\ v_{i,1} \in P \quad &\text{for} \quad i \equiv 2 \pmod 5 \end{aligned}$$

Consider a ball B of radius $r \geq 2$. It contains vertices from at most $2r + 1$ consecutive columns of $P_n \square P_2$. By construction, every five consecutive columns contain at most 2 elements of P.

$$|B \cap P| \leq 2 \left\lceil \frac{(2r+1)}{5} \right\rceil$$

It is straightforward to check that, $2\lceil (2r + 1)/5 \rceil \leq r$ for $r \neq 1, 3, 5$. (For $r < 10$, simply evaluate $2\lceil (2r + 1)/5 \rceil$. For $r \geq 10$, $2\lceil (2r + 1)/5 \rceil \leq 2(2r/5 + 1) \leq r$.) It is easy to check that each ball of radius 1 contains at most one element of P. When $r = 3$, B contains vertices from 7 columns of which at most 3 columns may contain packing vertices. Similarly, when $r = 5$, we observe that any 11 consecutive columns contain at most 5 packing vertices. □

We now turn to the special case when $m = 3$. The following result gives $\mathrm{mp}(P_n \square P_3)$. Since $\gamma_b(P_n \square P_3) = \lfloor \frac{n}{2} \rfloor + 1$, we note that $\mathrm{mp} = \gamma_b$ for $n \not\equiv 0 \pmod 4$.

Proposition 5. *Let n be a positive integer. Then*

$$\mathrm{mp}(P_n \square P_3) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 0 \pmod 4 \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{if } n \equiv 1, 2, 3 \pmod 4 \end{cases}$$

Proof: Since $\mathrm{rad}(P_n \square P_3) = \lfloor n/2 \rfloor + \lfloor 3/2 \rfloor$, we know that $\mathrm{mp}(P_n \square P_3) \leq \lfloor n/2 \rfloor + 1$. Given a multipacking P of $P_n \square P_3$ and any four consecutive columns say $i, i+1, i+2, i+3$, if P has three members in these columns, then without loss of generality they belong to columns $i, i+2, i+3$. Moreover, we can assume that the two in columns $i+2, i+3$ are $v_{i+2,2}$ and $v_{i+3,0}$. The only vertices that are not within distance 2 of either of these two packing vertices are $v_{i,0}$ and $v_{i,1}$. However, all three of these vertices are in a ball of radius 2 centred at $v_{i+2,0}$ in the former case and $v_{i+2,1}$ in the latter, a contradiction. Thus, the four columns contain at most 2 packing vertices. Specifically, in the case $n = 4q$, $\mathrm{mp}(P_n \square P_3) \leq 2q = \lfloor n/2 \rfloor$.

On the other hand, consider the set P defined as follows.

$$v_{i,0} \in P \quad \text{for} \quad i \equiv 0 \pmod 4$$
$$v_{i,2} \in P \quad \text{for} \quad i \equiv 1 \pmod 4$$

As the minimum distance between vertices in P is 3, no ball of radius 1 contains more than one element of P. Consider a ball B of radius $r \geq 2$. The ball contains vertices from at most $2r+1$ consecutive columns. We need to confirm that the ball has at most r elements of P. First, suppose that $r = 2t$. By symmetry, we may assume that the left most column of B is in $\{0, 1, 2, 3\}$. If the left most column is 2 or 3, then B contains vertices from columns $\{4, 5, \ldots, 4t+2, 4t+3\}$. Each contiguous block of four columns contains two members of P, giving B has a total of at most $2t = r$ vertices of P. If the left most column of B is 0 or 1, then B covers columns $0, 1, \ldots, 4t$ or $1, \ldots, 4t, 4t+1$. In both cases, B has exactly $2t+1$ columns with a vertex of P. However, in both cases $v_{1,2}$ and $v_{4t,0}$ are at distance $4t+1 = 2r+1$ apart and thus, at most one belongs to B. In all cases, $|B \cap P| \leq r$. If $r = 2t+1$, the analysis is similar. Either the $4t+3$ columns of B contain at most $2t+1 = r$ vertices of P, or the ball B has $2t+2 = r+1$ columns containing vertices of P, but there is a pair (for example $\{v_{0,0}, v_{4t+1,2}\}$) at distance $2r+1$, in which case B itself contains at most r vertices of P. □

2 Multipacking number for large grids

In this section, we prove Theorem 1. The radius of a grid graph $P_n \square P_m$ is $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. Since the broadcast number of a graph is at most its radius, it is sufficient to find a multipacking of size $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. We now proceed with the construction of such multipackings.

2.1 Restriction to even sizes

First, we shall prove that we can restrict ourselves to cases when n and m are both even numbers. Because of the singularity for the grid of size 4×6, we need to check the grids of sizes 5×6 and 4×7 by hand (see Figure 1). Now, suppose that n is odd. Then, $n-1$ is even and is at least 4. Moreover the $n-1 \times m$ grid is not 4×6 since we ruled out the 5×6 and 4×7 cases. Thus, if we know that the grid of size $n-1 \times m$ has a multipacking of size $\frac{n-1}{2} + \lfloor \frac{m}{2} \rfloor$, which is equal to $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$, we can add an empty column in the middle of this grid. We obtain a multipacking of the desired size for our grid. Indeed, given a vertex v from the smaller grid, the ball of radius r with centre v in the larger grid only contains vertices of the packing which were at distance at most r from v in the smaller grid. A ball of radius r centred at a vertex of the new column only contains vertices of the packing which are within distance r of both its neighbours from the former grid. Thus, these balls cannot contain more than r elements of the packing which satisfies our claim. The same reasoning works for m.

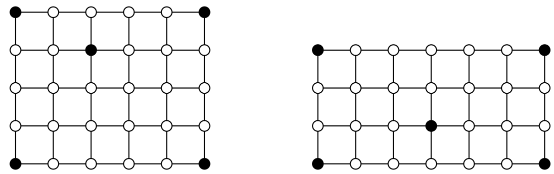

Fig. 1: Multipackings of order 5 for grids of size 5×6 and 4×7

The remainder of the proof is concerned with grids with even dimensions. Small cases require some specific care so that we will treat them after the general case. In all cases, we shall use a systematic way of selecting vertices along the sides of the grid. We describe them in the following paragraph.

2.2 The i-pattern

Fix an integer i. Given a path $v_0 v_1 \dots v_{z-1}$ of order z greater than or equal to $3i$, the *i-pattern* on this path consists in selecting every third vertex from v_0 to $v_{3(i-1)}$ and then every fourth vertex starting from v_{3i} (if it exists). Note that the i-pattern on a path of order z selects exactly i vertices from the beginning and one fourth (rounded up) of the rest. This amounts to $i + \lceil \frac{z-3i}{4} \rceil$ which can be simplified.

$$\text{The } i\text{-pattern on a path of order } z \text{ selects exactly } \left\lceil \frac{z+i}{4} \right\rceil \text{ vertices.} \tag{1}$$

Moreover, the density of the i-pattern is bounded above by a function of i. By this, we mean that a subpath of length ℓ of $v_0 v_1 \dots v_{z-1}$ cannot hit too many vertices of the i-pattern. If ℓ is at least $3i$, it could take the whole beginning (i vertices) and a fourth of the rest. This amounts to $i + \lceil \frac{\ell+1-3i}{4} \rceil$ which equals $\lceil \frac{\ell+1+i}{4} \rceil$. Whenever ℓ is strictly less than $3i$, it would take at most $\lceil \frac{\ell+1}{3} \rceil$ vertices. But in that case,

$$\begin{aligned}
\left\lceil \frac{\ell+1}{3} \right\rceil &\leq \left\lceil \frac{4\ell+4}{12} \right\rceil \\
&\leq \left\lceil \frac{3\ell+3+\ell+1}{12} \right\rceil \\
&\leq \left\lceil \frac{3\ell+3+3i}{12} \right\rceil && \text{(since } \ell+1 \leq 3i\text{)} \\
&\leq \left\lceil \frac{\ell+1+i}{4} \right\rceil .
\end{aligned}$$

In the end, we may state that

$$\text{a subpath of length } \ell \text{ hits at most } \left\lceil \frac{\ell+1+i}{4} \right\rceil \text{ vertices on a } i\text{-pattern.} \tag{2}$$

2.3 Large grids

As said before, small grids require some extra-care. In this part, we only consider grids with dimensions at least 8 in both directions. Fix n and m two even integers greater than or equal to 8. We let $k = n/2$

and $k' = m/2$. We view each side of the grid as a path from which we remove the last three vertices (see Figure 2). In these paths, we pack an adequate number of vertices using a specific i-pattern. Finally, we will estimate an upper bound on the number of such vertices in a ball of size r. This will cover most of the radii but the last few ones will be treated using some tailor-made arguments.

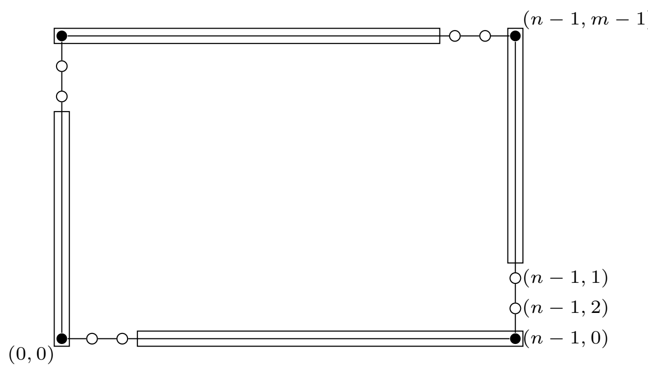

Fig. 2: General sketch, packing on the perimeter.

We use the i-patterns (where $i = 0, 1,$ or 2) to select vertices on "horizontal" and "vertical" sides. The packing on the horizontal sides is as follows (with the vertical sides being similar). Our choice of i depends on the parity of k. In all cases, $2k - 4 \geq 3i$. If k is even, then we use a 1-pattern on the top $(0, m-1), (3, m-1), (7, m-1), \ldots, (n-5, m-1)$ and a 1-pattern on the bottom $(n-1, 0), (n-4, 0), (n-8, 0), \ldots, (4, 0)$. In this case we shall write $i_t = i_b = 1$. If k is odd, we use a 2-pattern on the top $(0, m-1), (3, m-1), (6, m-1), (10, m-1), \ldots, (n-4, m-1)$ and a 0-pattern on the bottom $(n-1, 0), (n-5, 0), \ldots, (5, 0)$. In this case we write $i_t = 2$ and $i_b = 0$. Using (1), we see there are exactly $\frac{k}{2}$ vertices selected on a (horizontal) side, when k is even. When k is odd $\lfloor \frac{k}{2} \rfloor$ are selected on the bottom while $\lceil \frac{k}{2} \rceil$ are selected on the top. In all cases $\frac{n}{2} = k$ vertices are selected. These selections are depicted on Figure 3 for $n = 16$ and $n = 18$ (only the top and bottom sides of the grid are drawn). We call H the set of vertices selected on the horizontal paths. Similarly we select a total of k' vertices on the vertical sides and let V denote these vertices. After this process, we have a set P of $k + k'$ vertices. We shall prove that it is a valid multipacking.

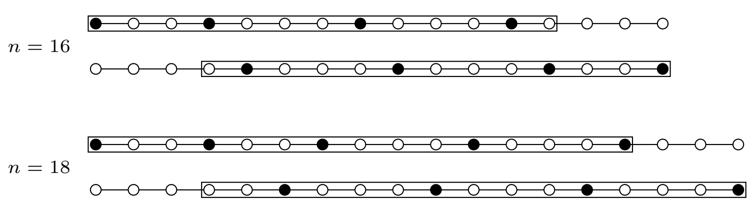

Fig. 3: Selection of vertices on horizontal paths

Most balls are valid. Let r be some integer between 1 and $k+k'-5$, and let B be a ball of radius r.

- If B does not intersect any side of the grid, then its intersection with P is empty and $|B \cap P| \leq r$ trivially.
- Suppose now that B intersects only one side of the grid, or two consecutive sides. Then its intersection with P lies on an isometric path of the grid where selected vertices are at distance at least 3 from each other. Thus the cardinality of $B \cap P$ is bounded above by $\lceil \frac{2r+1}{3} \rceil$ which is at most r since r is a positive integer (see Lemma 3).
- Now if B intersects two opposite sides of the grid (let them be top and bottom), let y denote the ordinate of the center of B. Recall that bottom has ordinate 0 while top has ordinate $m-1$. Now observe that the metric induced by the grid is similar to ℓ_1 metric. Thus B intersects the bottom side on a subpath of length at most $2(r-y)$ and the top side on a subpath of length at most $2(r-2k'+1+y)$. We claim that in most cases $|B \cap H| \leq r-k'+2$. Only in very specific cases can $|B \cap H|$ be equal to $r-k'+3$.

  Let x_b be the difference between $2(r-y)$ and the actual length of the intersection between B and the bottom part of H (recall that the bottom part of H does not include the last three vertices as depicted on Figures 2 and 3). Similarly, we define x_t for the top part of H. Then B intersect the bottom part of H on a subpath of length $2(r-y)-x_b$. We now use (2):

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r-y)-x_b+1+i_b}{4} \right\rceil + \left\lceil \frac{2(r-2k'+1+y)-x_t+1+i_t}{4} \right\rceil \\ &\leq \left\lceil \frac{2(r-y)+1+i_b-x_b}{4} \right\rceil + \left\lceil \frac{2(r+y)+3+i_t-x_t}{4} \right\rceil - k'. \end{aligned}$$

  Notice that $r-y$ and $r+y$ have same parity. First suppose they are both odd. Then $2(r-y)-2$ and $2(r+y)-2$ are multiples of 4. We can rewrite our bound.

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r-y)-2+3+i_b-x_b}{4} \right\rceil + \left\lceil \frac{2(r+y)-2+5+i_t-x_t}{4} \right\rceil - k' \\ &\leq \left\lceil \frac{3+i_b-x_b}{4} \right\rceil + \left\lceil \frac{5+i_t-x_t}{4} \right\rceil + r-1-k'. \end{aligned}$$

  Our pattern choice is either $i_b = i_t = 1$ or $i_b = 0$ and $i_t = 2$. In both cases, the ceilings add up to at most 3. So $|B \cap H| \leq r-k'+2$ when $r-y$ is odd. Now suppose that $r-y$ is even. Then the rewriting is straightforward.

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r-y)+1+i_b-x_b}{4} \right\rceil + \left\lceil \frac{2(r+y)+3+i_t-x_t}{4} \right\rceil - k' \\ &\leq \left\lceil \frac{1+i_b-x_b}{4} \right\rceil + \left\lceil \frac{3+i_t-x_t}{4} \right\rceil + r-k'. \end{aligned}$$

  When the pattern is $i_b = i_t = 1$, ceilings add up to at most 2 and once again $|B \cap H| \leq r-k'+2$. When $i_b = 0$ and $i_t = 2$, ceilings can unfortunately sum up to 3. But for this, both x_b and x_t must be 0. Let us be more precise. It means that both bottom and top intersections must be full (subpaths of length $2(r-y)$ and $2(r-2k'+1+y)$). Moreover, $B \cap H$ must use the corner vertex in the top part since otherwise, it would be intersecting a path which is nothing but a 1-pattern (or a 0-pattern). As a consequence, the center of B must be at distance exactly r from the top left corner. Moreover, B cannot reach the top right corner of the grid (otherwise, x_t would be strictly positive). Similarly, B cannot reach the bottom left corner since it is out of the bottom part of H and it would require x_b to be strictly positive.

- If B intersects only the top and bottom part of H, then $|B \cap H| \leq r - k' + 3 \leq r$ since k' is at least 4.
- If B intersects also exactly one vertical side. This one can contribute at most for $\lceil \frac{k'}{2} \rceil$ (by our choice of P). Thus, in most cases

$$\begin{aligned} |B \cap P| &\leq r - k' + 2 + \left\lceil \frac{k'}{2} \right\rceil \\ &\leq r - k' + 2 + \frac{k'}{2} + \frac{1}{2} \\ &\leq r - \frac{1}{2}(k' - 5) \end{aligned}$$

  which is at most r since k' is not less than 4 (when $k' = 4$ we observe $|B \cap P| \leq r + \frac{1}{2}$ implies $|B \cap P| \leq r$ since $|B \cap P|$ is an integer). In the special case when $|B \cap H|$ is $r - k' + 3$, recall that the vertical side cannot use the corner so it contributes at most for $\lceil \frac{k'}{2} \rceil - 1$ and the same conclusion holds.
- Finally, if B intersects all four sides, we may use the corner observation to state that at most one of the directions (vertical or horizontal) can contribute for $r - k' + 3$ (or $r - k + 3$). The other direction contributes at most for $r - k + 2$ (or $r - k' + 2$) so that

$$\begin{aligned} |B \cap P| &= |B \cap H| + |B \cap V| \\ &\leq 2r - (k + k') + 5. \end{aligned}$$

  This quantity is less than or equal to r whenever r is $k + k' - 5$ or less.

Balls with a big radius. To finish our proof, we only need to verify that balls with a radius r between $k + k' - 4$ and $k + k' - 1$ verify our constraint.

Let us treat the maximum radius $k + k' - 1$. Note that since n and m are both even, this grid, if seen as a chequerboard, has two diagonally opposite white corners and two diagonally opposite black corners. Suppose a ball of radius $k + k' - 1$ contains all the vertices of P. Then it must contain the four corners of the grid. Since opposite corners are at distance $2k + 2k' - 2$ it means that the centre of the ball is the middle vertex of a shortest path between opposite corners. But this middle vertex must be white for one pair of corners and must be black for the other pair, which is impossible. Thus every ball of radius $k + k' - 1$ misses at least one corner.

Now consider a ball of radius $k+k'-2$. Since both pairs of opposite corners are at distance $2k+2k'-2$, at most one corner of each pair can be in a ball of such radius. Thus, such a ball misses at least two corners.

Concerning radius, $k+k'-4$, we can match each corner vertex with the second selected vertex from the opposite side. The distance between them is $2k+2k'-5$ (corner to a 2-pattern or to a 1-pattern) or $2k+2k'-6$ (corner to a 0-pattern) depending on the chosen pattern. In any case, since vertices in the ball cannot have distance more $2k+2k'-8$, such a ball misses at least 4 vertices from the total and is valid.

Finally, we are left with balls of radius $k+k'-3$. We may again consider the same matching. If k or k' is even, we have at least one direction with two 1-patterns and so at least three of the pairs are at distance $2k+2k'-5$. So the ball misses at least three vertices and is valid. The last case is when both k and k' are odd. In that case, our matching has two pairs at distance $2k+2k'-5$ (from which the ball misses at least two vertices) and two pairs at distance $2k+2k'-6$. As for radius $k+k'-1$, both last pairs are on two different colors of the chequerboard (black and white) so that at least one of the four concerned vertices is missed. In the end, the ball misses at least three vertices and is valid.

This concludes the proof for grids with sizes at least 8 in both directions.

2.4 Long grids

The previous discussion leaves out all grids with one of their dimensions either 4 or 6. In this section, we provide a way of tackling long grids (for which $k \geq 3k'-\ell$ where ℓ depends on the parity of $k+k'$). In the end, there will only remain four cases to study.

We shall pack vertices only on the top and bottom sides of the grid. We consider the whole sides (not the $2k-3$ first vertices as in Subsection 2.3). Recall to pack an i-pattern on a horizontal side requires $3i \leq n-1$. If k and k' have same parity, we use a $(2k'-3)$-pattern on both top and bottom sides. This requires $3(2k'-3) \leq n-1$ or $3k'-4 \leq k$. If k and k' have different parities, we use a $(2k'-5)$-pattern on one side (say bottom) and a $(2k'-1)$-pattern on the other. This requires $3k'-1 \leq k$. By (1), this process selects

$$\left\lceil \frac{2k+2k'-3}{4} \right\rceil + \left\lceil \frac{2k+2k'-3}{4} \right\rceil \text{ or } \left\lceil \frac{2k+2k'-1}{4} \right\rceil + \left\lceil \frac{2k+2k'-5}{4} \right\rceil$$

vertices. In both cases, this can be simplified as $k+k'$ (in the first case, $k+k'$ is even, while it is odd in the latter).

Now, if a ball B of radius r intersects only one horizontal side of the grid, this intersection lies on an isometric path from which we selected at most every third vertex. Then by Lemma 3, it cannot contain strictly more than r vertices. Suppose that the ball B intersects both paths. Like in the previous subsection, if this ball has its centre on a vertex with ordinate y (0 being the bottom and $m-1$ being the top), then it intersects the bottom on a path of length at most $2(r-y)$ and the top on a path of length at most $2(r-2k'+1+y)$. Then we use (2). If both sides are packed with $(2k'-3)$-patterns,

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r-y)+2k'-2}{4} \right\rceil + \left\lceil \frac{2(r-2k'+1+y)+2k'-2}{4} \right\rceil \\ &\leq \left\lceil \frac{2(r-y+k')-2}{4} \right\rceil + \left\lceil \frac{2(r-k'+y)}{4} \right\rceil . \end{aligned}$$

And since $r-y+k'$ and $r+y-k'$ have same parity, one of the ceilings adds $\frac{1}{2}$, and

$$\begin{aligned} |B \cap H| &\leq \frac{2(r-y+k')-2}{4} + \frac{2(r-k'+y)}{4} + \frac{1}{2} \\ &\leq r. \end{aligned}$$

Similarly, if we use the $(2k'-5)$-pattern on bottom and the $(2k'-1)$-pattern on the top, we have

$$|B \cap H| \leq \left\lceil \frac{2(r-y)+2k'-4}{4} \right\rceil + \left\lceil \frac{2(r-2k'+1+y)+2k'}{4} \right\rceil$$
$$\leq \left\lceil \frac{2(r-y+k')-4}{4} \right\rceil + \left\lceil \frac{2(r-k'+y)+2}{4} \right\rceil.$$

Once again, the rounding adds at most $\frac{1}{2}$ and

$$|B \cap H| \leq r.$$

When $k'=2$ and k is even, we use a $2k'-3=1$ pattern. Thus the previous paragraph is valid for all even $k \geq 2$. When k is odd we use a 0-pattern and a 3-pattern. This requires $k \geq 5$. In particular, we have valid multipackings for $4 \times n$ for any even $n \geq 4$ and $n \neq 6$. In the same manner the previous paragraph gives a valid multipacking of order $k+k'$ when $k'=3$ provided $k \geq 8$ for even k and $k \geq 5$ for odd k. Consequently we have packings of grids with dimensions $6 \times n$ for even $n \neq 6, 8, 12$. This concludes the proof for long grids. (We remark the above arguments show for a fixed k' and sufficiently large k, there is an optimal multipacking selecting vertices only on the horizontal sides.)

2.5 Remaining cases

Subsection 2.3 covers large grids ($4 \leq k \leq k'$), and Subsection 2.4 covers long grids ($2 \leq k' \leq (k+\ell)/3$). There are four remaining cases that can be checked by hand, and have been verified using SageMath. Three are depicted on Figure 4.

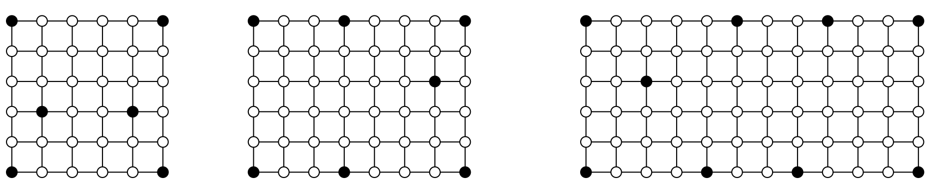

Fig. 4: Multipacking for 6×6, 8×6, and 12×6 grids.

Finally, the 6×4 grid is the only grid with dimensions at least 4 and multipacking number strictly smaller than expected. It is 4 while its broadcast domination number is 5. This completes the proof of Theorem 1.

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