Discrete Mathematics and Theoretical Computer Science DMTCS vol. **20**:2, 2018, #7

On locally irregular decompositions and the 1-2 Conjecture in digraphs

Olivier Baudon[1] Julien Bensmail[2] Jakub Przybyło[3] Mariusz Woźniak[3]

[1] *LaBRI, Université de Bordeaux, France*
[2] *I3S/INRIA, Université Nice-Sophia-Antipolis, France*
[3] *AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland*

received 30th Apr. 2018, revised 24th Aug. 2018, 11th Sep. 2018, accepted 21st Sep. 2018.

The 1-2 Conjecture raised by Przybyło and Woźniak in 2010 asserts that every undirected graph admits a 2-total-weighting such that the sums of weights "incident" to the vertices yield a proper vertex-colouring. Following several recent works bringing related problems and notions (such as the well-known 1-2-3 Conjecture, and the notion of locally irregular decompositions) to digraphs, we here introduce and study several variants of the 1-2 Conjecture for digraphs. For every such variant, we raise conjectures concerning the number of weights necessary to obtain a desired total-weighting in any digraph. We verify some of these conjectures, while we obtain close results towards the ones that are still open.

Keywords: 1-2 Conjecture, locally irregular decompositions, digraphs

1 Introduction

An edge-weighting $w : E(G) \to \mathbb{N}^*$ of an undirected graph G is called *sum-colouring* if the sums of weights "incident" to the vertices yield a proper vertex-colouring of G. More precisely, for each vertex v of G one can compute

$$\sigma^e(v) := \sum_{u \in N(v)} w(vu)$$

and we require σ^e to be a proper vertex-colouring. The smallest $k \geq 1$ such that G admits a sum-colouring k-edge-weighting (if any) is denoted $\chi^{e}_{\sigma}(G)$.(i)

The **1-2-3 Conjecture**, posed by Karonski, Łuczak and Thomason (9) in 2004, reads as follows (where an *isolated edge* refers to a connected component isomorphic to K_2).

Conjecture 1.1 (Karonski, Łuczak, Thomason (9))**.** *For every graph G with no isolated edge, we have $\chi^{e}_{\sigma}(G) \leq 3$.*

Since its introduction, the 1-2-3 Conjecture has been attracting growing attention, resulting in many research works considering either the conjecture itself or variants of it. As a best result towards it, it was proved by Kalkowski, Karoński and Pfender that $\chi^{e}_{\sigma}(G) \leq 5$ holds for every graph G with no isolated edge (8). For more information, we refer the interested reader to (14) for a survey by Seamone on this wide topic.

In this paper, we mainly focus on two notions related to the 1-2-3 Conjecture. The first one is the total version of the 1-2-3 Conjecture, called the **1-2 Conjecture**, which was introduced by Przybyło and Woźniak in (12). Quite similarly as in the context of weighting edges only, we say that a total-weighting $w : V(G) \cup E(G) \to \mathbb{N}^*$ of G is *sum-colouring* if the vertex-colouring σ^t defined as

$$\sigma^t(v) := w(v) + \sum_{u \in N(v)} w(vu)$$

(i) This notation and its variants should be understood as follows throughout: χ is a chromatic parameter; the superscript refers to the elements to be weighted or coloured; the subscript refers to the aggregate, computed from the weighting or colouring, to be distinguished on the adjacent vertices.

ISSN 1365–8050

for every vertex v is a proper vertex-colouring. We then denote by $\chi^t_\sigma(G)$ the least $k \geq 1$ such that G admits a sum-colouring k-total-weighing. It is believed that being granted the possibility to "locally" modify the sums of weights incident to the vertices (by altering the vertex weights) should, compared to the original edge version, reduce the number of needed weights.

Conjecture 1.2 (Przybyło, Woźniak (12))**.** *For every graph G, we have $\chi^t_\sigma(G) \leq 2$.*

The 1-2 Conjecture is known to hold for several families of graphs, such as 3-colourable graphs, complete graphs, and 4-regular graphs (12). As for upper bounds on χ^t_σ, the best known one is due to Kalkowski (7), who proved that $\chi^t_\sigma(G) \leq 3$ holds for every graph G. Actually, Kalkowski even proved that stronger sum-colouring 3-total-weightings, assigning weights in $\{1, 2\}$ to the vertices and in $\{1, 2, 3\}$ to the edges, exist for all graphs.

It is worthwhile mentioning that both the 1-2-3 Conjecture and the 1-2 Conjecture were considered in the more general list context. In that context, instead of weighting all the graph elements (edges and possibly vertices) with values from the same list ($\{1, 2, 3\}$ for the 1-2-3 Conjecture, $\{1, 2\}$ for the 1-2 Conjecture), each element now has a private list from which a weight must be chosen. Given a list assignment to the elements we want to weight, the goal is then to design a sum-colouring weighting where the weights are picked from the assigned lists. The list version of the 1-2-3 Conjecture, posed by Bartnicki, Grytczuk and Niwcyk (1), asserts that such sum-colouring edge-weightings can be constructed for every assignment of lists of size 3 to the edges. The list version of the 1-2 Conjecture, posed by Przybyło and Woźniak (13), asserts that such sum-colouring total-weightings can be constructed for every assignment of lists of size 2 to the vertices and edges. Again, we refer the interested reader to (14) for more details on these two list variants. A remarkable result we should mention, however, is one due to Wong and Zhu (15), who proved that sum-colouring total-weightings can be constructed for every assignment of lists of size 3 to the edges and lists of size 2 to the vertices. In contrast, it is still not known whether there is a constant $k \geq 3$ such that sum-colouring edge-weightings can be constructed for every assignment of lists of size k to the edges.

The second notion considered in this paper is the one of *locally irregular decompositions*. We say that a graph G is *locally irregular* if every two of its adjacent vertices have distinct degrees. A *locally irregular edge-colouring* of G is then an edge-colouring where each colour induces a locally irregular subgraph. We denote by $\chi^e_{\mathrm{irr}}(G)$ the least number of colours in a locally irregular edge-colouring of G (if any). Intuitively, the parameter χ^e_{irr} can be seen as a measure of how "far" from (locally) irregular a graph is. This parameter was introduced and studied by the current authors in (2) mainly because of its link to the 1-2-3 Conjecture and some of its variants. In particular, let us mention that in very particular settings, such as when dealing with regular graphs and only two colours, finding a sum-colouring edge-weighting is equivalent to finding a locally irregular edge-colouring.

Since their introduction, locally irregular edge-colourings gave birth to several investigations. An important first result to have in hand is the exhaustive list of those exceptional graphs for which the parameter χ^e_{irr} is not defined at all. In (2), it was proved that these exceptional graphs are exactly those in the family $\mathcal{P} \cup \mathcal{C} \cup \mathcal{T}$, where $\mathcal{P}$ is the class of all odd-length paths, $\mathcal{C}$ is the class of all odd-length cycles, and $\mathcal{T}$ is a peculiar class of graphs obtained by joining, via disjoint paths with particular lengths, disjoint triangles in a tree-like fashion (refer to (2) for the exact definition). For graphs not in $\mathcal{P} \cup \mathcal{C} \cup \mathcal{T}$, hence when χ^e_{irr} is defined, the following conjecture was raised.

Conjecture 1.3 (Baudon, Bensmail, Przybyło, Woźniak (2))**.** *For every graph G not in $\mathcal{P} \cup \mathcal{C} \cup \mathcal{T}$, we have $\chi^e_{\mathrm{irr}}(G) \leq 3$.*

Conjecture 1.3 has been mainly verified for several families of graphs, including regular graphs of large degree (2) and graphs of large minimum degree (11). It should be noted that it is NP-complete in general to compute the exact value of $\chi^e_{\mathrm{irr}}(G)$ for a given graph G, as shown in (3) by Baudon, Bensmail and Sopena. In a recent work (5), Bensmail, Merker and Thomassen provided the first constant upper bound on χ^e_{irr}, showing that $\chi^e_{\mathrm{irr}}(G) \leq 328$ holds for every graph G admitting locally irregular decompositions. Later on (10), Lužar, Przybyło and Soták proved that $\chi^e_{\mathrm{irr}}(G) \leq 220$ always holds.

This paper is mainly inspired by two papers, namely (4) (authored by Baudon, Bensmail and Sopena) and (6) (authored by Bensmail and Renault), which brought Conjectures 1.1 and 1.3 in the context of digraphs in the particular setting where all notions of "incident weights" and "locally irregular graphs" are with respect to the *outdegree* parameter. So that we avoid any confusion, we omit the formal definitions and statements here and will rather recall them in the corresponding upcoming sections. Let us just mention that the directed version of Conjecture 1.1 from (4) was completely verified in the same paper, while, towards the directed version of Conjecture 1.3 from (6), only partial results, proved in that same paper, are known to date. To the best of our knowledge, nothing was known for total variants of these problems.

Section 2 is dedicated to sum-colouring edge-weightings and total-weightings in digraphs, while Section 3 is devoted to irregular decompositions in digraphs. The three series of results from these sections are comparable, and should hence be regarded in parallel. We start, in Section 2, by filling in the space showing that the directed version of the 1-2 Conjecture in the setting of (4) is false in a strong sense, and introduce a holding variant. In Section 3, we start by improving the main result of (6), going one step closer to the main conjecture in that paper. We then investigate two total versions of the same problem inspired by the 1-2 Conjecture. For these two versions, we provide bounds which are close to what we conjecture to be optimal. Some conclusions are gathered in Section 4.

Notation and terminology: Throughout this paper, we focus on *simple digraphs*, i.e. loopless digraphs with no two arcs directed in the same direction between any pair of distinct vertices. Note that this definition allows our digraphs to have digons, i.e. directed cycles of length 2. Any arc (u, v) of a digraph D will be denoted $\overrightarrow{uv}$ to lighten the notation and make the arc's direction apparent. The *outdegree* (resp. *indegree*) of a vertex v of D is its number $d^+_D(v)$ (resp. $d^-_D(v)$) of outgoing (resp. incoming) incident arcs. In case no ambiguity is possible, the subscript in this notation will be freely omitted. The *maximum outdegree* (resp. *maximum indegree*) of D, denoted $\Delta^+(D)$ (resp. $\Delta^-(D)$) refers to the maximum outdegree (resp. indegree) over the vertices of D.

2 Sum-colouring arc- and total-weightings in digraphs

In this section, we investigate how the results from (4), which concern a directed version of Conjecture 1.1, can be extended to the total context (i.e. when also the vertices are weighted). We start in Section 2.1 by recalling the investigations from (4). Then we consider, in Sections 2.2 and 2.3, two directed analogues of the 1-2 Conjecture derived from the problem considered in that paper. The first such variant is shown to be false, even in a strong sense, while the second one is shown to hold.

2.1 Outsum-colouring arc-weightings

Let D be a digraph, and w an arc-weighting of D. From w, one can compute, for every vertex v, the sum (*outsum*) $\sigma^e_+(v)$ of "outgoing weights", formally defined as

$$\sigma^e_+(v) := \sum_{u \in N^+(v)} w(\overrightarrow{vu}).$$

In case σ^e_+ is a proper vertex-colouring, we call w *outsum-colouring*. The least number $k \geq 1$ of weights needed to obtain an outsum-colouring k-arc-weighting of D is denoted $\chi^e_{\sigma+}(D)$. Using a very simple argument, Baudon, Bensmail and Sopena showed in (4) that the tightest upper bound on $\chi^e_{\sigma+}$ is 3, which cannot be improved as deciding whether $\chi^e_{\sigma+}(D) \leq 2$ holds for a given digraph D is NP-complete in general. Since this upper bound will be of some use in the upcoming sections, we state it here.

Theorem 2.1 (Baudon, Bensmail, Sopena (4))**.** *For every digraph D, we have $\chi^e_{\sigma+}(D) \leq 3$.*

2.2 Outsum-colouring total-weightings

We now consider the natural directed variant of the 1-2 Conjecture, where the terminology we use is inspired by that introduced in Section 2.1. Assume w is a total-weighting of a digraph D. To every vertex v, we associate the colour $\sigma^t_+(v)$, where

$$\sigma^t_+(v) := w(v) + \sum_{u \in N^+(v)} w(\overrightarrow{vu}).$$

We say that w is *outsum-colouring* if σ^t_+ is a proper vertex-colouring. Again, the least number $k \geq 1$ of weights needed to deduce an outsum-colouring k-total-weighting of D is denoted $\chi^t_{\sigma+}(D)$.

Due to Theorem 2.1, clearly we have $\chi^t_{\sigma+}(D) \leq 3$ for every digraph D (start from an outsum-colouring $\chi^e_{\sigma+}(D)$-arc-weighting, and put weight 1 on all vertices). As a straight directed analogue of the 1-2 Conjecture, one could naturally wonder about the following question.

Question 2.2. *For every digraph D, do we have $\chi^t_{\sigma+}(D) \leq 2$?*

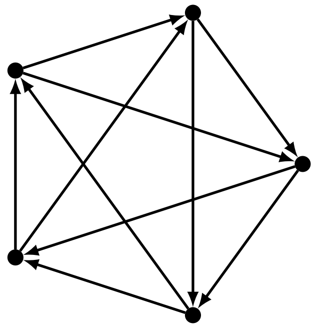

Figure 1: A digraph with no outsum-colouring 2-total-weighting.

Unfortunately, easy counterexamples to Question 2.2 can be exhibited, showing that 3 is actually the best general upper bound on $\chi^t_{\sigma+}$. It can even be proved that Question 2.2 is far from being true, in the sense that there exists no constant $k \geq 3$ such that every digraph admits an outsum-colouring $(k,2)$-*total-weighting*, i.e. an outsum-colouring total-weighting using weights among $\{1,...,k\}$ on the vertices and among $\{1,2\}$ on the arcs.

Proposition 2.3. *For every $k \geq 1$, there exist digraphs admitting no outsum-colouring $(k,2)$-total-weighting.*

Proof: Given k, choose any odd integer $n \geq 5$ such that $k < \lceil \frac{n}{2} \rceil$, and let $\overrightarrow{T_n}$ be the tournament on n vertices defined as follows. Denote $0,1,...,n-1$ the vertices of $\overrightarrow{T_n}$, and, for every vertex i of $\overrightarrow{T_n}$, add the arcs $\overrightarrow{(i,i+1)}, \overrightarrow{(i,i+2)}, ..., \overrightarrow{(i,i+\lfloor \frac{n}{2} \rfloor)}$, where the indexes are taken modulo n. By construction, every vertex of $\overrightarrow{T_n}$ has outdegree precisely $\lfloor \frac{n}{2} \rfloor$. For this reason, for any vertex v, the possible values for $\sigma^t_+(v)$ by a $(k,2)$-total-weighting w of $\overrightarrow{T_n}$ are those among the set

$$\mathcal{S} := \left\{ \left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor + 2, ..., 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor + k \right\},$$

which includes $\lfloor \frac{n}{2} \rfloor + k$ values. By our choice of n, we have $|\mathcal{S}| < \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$. From this, we deduce that there has to be at least two vertices of $\overrightarrow{T_n}$ having the same outsums by w. Since $\overrightarrow{T_n}$ is a tournament, this implies that w is not outsum-colouring. $\square$

Due to Proposition 2.3, digraphs may not admit outsum-colouring $(k,2)$-total-weightings with k being any fixed constant. Hence, to design outsum-colouring $(k,2)$-total-weightings, in general we should rather consider values of k depending on the given digraph. This is illustrated in the following result, which is actually tight.

Proposition 2.4. *Every digraph D admits an outsum-colouring $(\Delta^+(D)+1,2)$-total-weighting. Furthermore, there exist digraphs for which we cannot decrease the number of vertex weights.*

Proof: Let w be an outsum-colouring 3-arc-weighting of D. Such exists according to Theorem 2.1. Now for every vertex $v \in V(D)$, define

$$n_3(v) := |\{\overrightarrow{vu} \in A(D) : w(\overrightarrow{vu}) = 3\}|,$$

the number of arcs outgoing from v weighted 3 by w. Clearly, we have $n_3(v) \leq \Delta^+(D)$.

Now consider the $(\Delta^+(D)+1,2)$-total-weighting w' of D defined as

$$\begin{cases} w'(v) &= n_3(v)+1 \text{ for every } v \in V(D), \text{ and} \\ w'(\overrightarrow{uv}) &= \min\{2, w(\overrightarrow{uv})\} \text{ for every } \overrightarrow{uv} \in A(D). \end{cases}$$

By the way w' is defined, the value $\sigma^t_+(v)$ induced by w' is exactly $1+\sigma^e_+(v)$, for the value of $\sigma^e_+(v)$ induced by w. Since w is outsum-colouring, then w' is also outsum-colouring.

To conclude the proof, we just note that the construction from the proof of Proposition 2.3 confirms the last part of the statement, as every considered tournament $\overrightarrow{T_n}$ verifies $\Delta^+(\overrightarrow{T_n})+1 = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$. $\square$

We end this section by mentioning that Proposition 2.3 remains true even if one requires the adjacent "incident outmultisets" (rather than the "incident outsums") to be different. This will justify our investigations in Section 3.2. Let w be an arc-weighting (resp. total-weighting) of a digraph D. Here, by the *outmultiset* of any vertex v of D, we mean the multiset $\{\{w(a_1), ..., w(a_{d^+(v)})\}\}$ (resp. $\{\{w(a_1), ..., w(a_{d^+(v)}), w(v)\}\}$ of weights "outgoing" from v, where $a_1, ..., a_{d^+(v)}$ denote the arcs outgoing from v. Note that, for any two adjacent vertices u, v, the outmultisets of u and v differ whenever their outsums differ (but the contrary is not always true). Hence, distinguishing vertices via their outmultisets is easier than distinguishing vertices via their outsums.

Remark 2.5. *For every $k \geq 1$, there exist digraphs admitting no $(k, 2)$-total-weighting distinguishing the adjacent vertices by their outmultisets.*

2.3 Pair-colouring total-weightings

As pointed out in the previous section, the directed analogue of the 1-2 Conjecture in the setting of (4) is false in a strong sense (recall Proposition 2.3 and Remark 2.5). In this section, we show that by modifying the aggregate to be distinguished on the adjacent vertices, we get another directed variant of the 1-2 Conjecture, which, here, is verified.

Let w be a total-weighting of a digraph D. From w, we compute at every vertex v the value

$$\rho_+^t(v) := \left(w(v), \sum_{u \in N^+(v)} w(\overrightarrow{vu}) \right).$$

In case the vertex-colouring ρ_+^t is proper, we call w *pair-colouring*. The least number $k \geq 1$ of weights needed to obtain a pair-colouring k-total-weighting of D is denoted $\chi_{\rho+}^t(D)$.

We now prove the analogue of the 1-2 Conjecture for pair-colouring total-weighting.

Theorem 2.6. *For every digraph D, we have $\chi_{\rho+}^t(D) \leq 2$.*

Proof: Set $n := |V(D)|$. Let $v_1, ..., v_n$ be the ordering of $V(D)$ defined in the following way. Let v_n be a vertex of D satisfying $d_D^-(v_n) \leq d_D^+(v_n)$. Such a vertex exists since

$$\sum_{v \in V(D)} d_D^-(v) = \sum_{v \in V(D)} d_D^+(v).$$

Now consider the digraph $D - \{v_n\}$, and denote v_{n-1} one vertex of $V(D) \backslash \{v_n\}$ satisfying $d_{D-\{v_n\}}^-(v_{n-1}) \leq d_{D-\{v_n\}}^+(v_{n-1})$. Repeat the same procedure until all vertices of D are labelled. Namely, assuming that the vertices $v_{n-i+1}, ..., v_n$ have been defined, choose v_{n-i} as a vertex of $D - \{v_{n-i+1}, ..., v_n\}$ satisfying

$$d_{D-\{v_{n-i+1},...,v_n\}}^-(v_{n-i}) \leq d_{D-\{v_{n-i+1},...,v_n\}}^+(v_{n-i}),$$

which again exists according to the same argument as above.

We construct a pair-colouring 2-total-weighting w of D by considering the vertices $v_1, ..., v_n$ from "left" to "right", i.e. in increasing order of their indexes. Assume $v_1, ..., v_{i-1}$ have already been correctly treated, i.e. $\rho_+^t(v_1)$, ..., $\rho_+^t(v_{i-1})$ are defined (these vertices and their outgoing arcs have been each assigned a weight) and $\rho_+^t(v_j) \neq \rho_+^t(v_{j'})$ for every $j, j' \in \{1, ..., i-1\}$ such that v_j and $v_{j'}$ are adjacent. Let $D_i := D - \{v_{i+1}, ..., v_n\}$. We now assign a weight to v_i and its outgoing arcs by w in such a way that no conflict arises. An important thing to keep in mind is that weighting an arc $\overrightarrow{v_i v_j}$ does not alter $\rho_+^t(v_j)$. Note further that $\rho_+^t(v_i) \neq \rho_+^t(v_j)$ whenever $w(v_i) \neq w(v_j)$.

For $\alpha = 1, 2$, let

$$n_\alpha := \left| \left\{ v_j \in N_{D_i}^-(v_i) \cup N_{D_i}^+(v_i) : j < i \text{ and } w(v_j) = \alpha \right\} \right|$$

be the number of already treated adjacent vertices which have been assigned weight α. There has to be a value of $\alpha \in \{1, 2\}$ for which

$$n_\alpha \leq \left\lfloor \frac{d_{D_i}^+(v_i) + d_{D_i}^-(v_i)}{2} \right\rfloor.$$

Let us assume $\alpha = 1$ in what follows.

Set $w(v_i) = 1$. Then v_i is already distinguished from all its already treated adjacent vertices which received weight 2 by w. Now what remains to do is to weight the arcs outgoing from v_i in D so that v_i

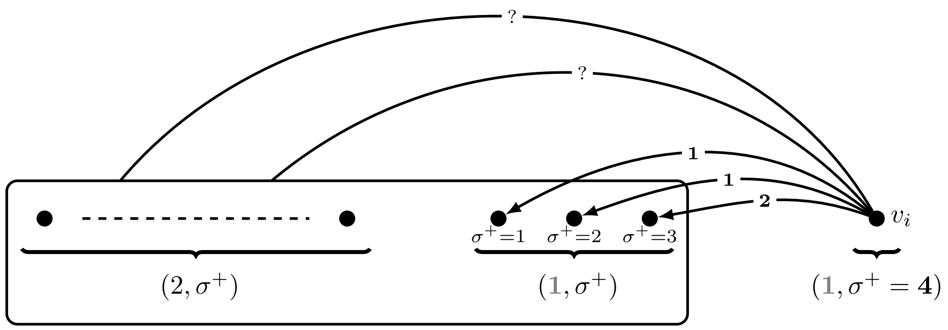

Figure 2: Illustration of the proof of Theorem 2.6.

is distinguished by its outsum from all its already treated adjacent vertices which received weight 1 by w (refer to Figure 2 for an illustration). The possible outsums for v_i in D by w are those among

$$\mathcal{S} := \left\{d_D^+(v_i), d_D^+(v_i)+1, ..., 2d_D^+(v_i)\right\},$$

forming a set with cardinality $d_D^+(v_i)+1$. But the outsum of v_i has to be different from the outsums of its n_1 previously treated adjacent vertices which also received weight 1 by w. By the ordering of the vertices of D, we have

$$d_{D_i}^+(v_i) + d_{D_i}^-(v_i) \le 2d_{D_i}^+(v_i),$$

yielding

$$n_1 \le d_{D_i}^+(v_i) < |\mathcal{S}|.$$

There is thus at least one value s among $\mathcal{S}$ which does not appear as the outsum by w of any vertex v_j with $j < i$ neighbouring v_i which received weight 1. Then just weight the arcs outgoing from v_i so that the outsum of v_i is s. Now v_i also gets distinguished from its previously considered neighbours weighted 1.

By repeating the above procedure until v_1 is treated, we eventually get the claimed pair-colouring 2-total-weighting w, concluding the proof. $\square$

3 Irregular arc- and total-decompositions in digraphs

We now focus on irregular decompositions of digraphs, where the notion of irregularity is with respect to the one introduced in (6) by Bensmail and Renault. We start by recalling the needed terminology and notation in Section 3.1. In the same section, we then improve the main result from (6) by showing that every digraph is decomposable into at most five locally irregular digraphs (while six is proved there). Total counterparts for irregular decompositions of the notions we have introduced in Sections 2.2 and 2.3, are then studied in Sections 3.2 and 3.3.

3.1 Locally irregular arc-colourings

A digraph D is called *locally irregular* if its adjacent vertices have distinct outdegrees. An arc-colouring of D is called *locally irregular* if its every colour class induces a locally irregular subdigraph. The smallest number of colours in a locally irregular arc-colouring of D is denoted by $\chi_{\mathrm{irr}+}^e(D)$.

The main conjecture stated in (6) is the following.

Conjecture 3.1 (Bensmail, Renault (6))**.** *For every digraph D, we have* $\chi_{\mathrm{irr}+}^e(D) \le 3$.

It is important to mention, as pointed out in (6), that the upper bound of 3 in Conjecture 3.1 cannot be reduced to 2 in general. For an easy illustration, consider the directed cycle $\overrightarrow{C_3}$ of length 3 (i.e. the digraph with vertex set $\{v_0, v_1, v_2\}$ and arc set $\{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \overrightarrow{v_2v_0}\}$), which clearly verifies $\chi_{\mathrm{irr}+}^e(\overrightarrow{C_3}) = 3$. It turns out that determining whether $\chi_{\mathrm{irr}+}^e(D) \le 2$ holds for a given digraph D is NP-complete in general (6); hence, many more digraphs needing three colours exist. To date, we are not aware of any digraph D verifying $\chi_{\mathrm{irr}+}^e(D) > 3$.

The originators of Conjecture 3.1 proved its following weakening.

Theorem 3.2 (Bensmail, Renault (6))**.** *For every digraph D, we have $\chi^e_{\mathrm{irr}+}(D) \leq 6$.*

The proof of Theorem 3.2 consists in first arc-decomposing any D into two acyclic digraphs, i.e. two digraphs with no directed cycles. The claimed upper bound then follows by showing that Conjecture 3.1 holds for acyclic digraphs (as first proved in (6)). We formally state this result as it will be used in some of our upcoming proofs.

Lemma 3.3 (Bensmail, Renault (6))**.** *For every acyclic digraph D, we have $\chi^e_{\mathrm{irr}+}(D) \leq 3$.*

The NP-completeness result mentioned earlier actually also holds for acyclic digraphs (6). Namely, deciding whether $\chi^e_{\mathrm{irr}+}(D) \leq 2$ is an NP-complete problem, even when restricted to acyclic digraphs D. This implies that there exist infinitely many acyclic digraphs that cannot be arc-coloured with only two colours, hence that the bound in Lemma 3.3 cannot be improved in general.

Our improvement on the bound in Theorem 3.2 from 6 down to 5, consists in showing that every digraph admits an arc-decomposition into one acyclic digraph and one *degree-decreasing acyclic digraph*, which we define as an acyclic digraph admitting an ordering $v_1, ..., v_n$ of its vertices such that

1. all arcs are directed "to the right" (i.e. for every two adjacent vertices v_i, v_j with $i < j$, the arc is $\overrightarrow{v_iv_j}$), and
2. $d^+(v_i) \geq d^+(v_j)$ whenever $i < j$.

Since acyclic digraphs D verify $\chi^e_{\mathrm{irr}+}(D) \leq 3$ (according to Lemma 3.3) and degree-decreasing acyclic digraphs D verify $\chi^e_{\mathrm{irr}+}(D) \leq 2$ (which we show in Lemma 3.6 below), our result follows.

Theorem 3.4. *For every digraph D, we have $\chi^e_{\mathrm{irr}+}(D) \leq 5$.*

Towards Theorem 3.4, as a first step we start by pointing out that every digraph indeed admits an arc-decomposition into one acyclic digraph and one degree-decreasing acyclic digraph.

Lemma 3.5. *Every digraph D admits an arc-decomposition into one acyclic digraph and one degree-decreasing acyclic digraph.*

Proof: Consider the following ordering $v_1, ..., v_n$ over the vertices of D. Start with v_1 being one vertex of D with largest outdegree (if there are several choices as v_1, pick any of them). Now remove v_1 from D and choose v_2 to be one vertex of $D - \{v_1\}$ with the largest outdegree. Then remove v_2 from $D - \{v_1\}$ and continue the procedure until all vertices are labelled. Basically, if we just read the vertices from "left" (i.e. v_1) to "right" (i.e. v_n) we get that for every two vertices v_i and v_j with $i < j$, vertex v_i has at least as many outneighbours as v_j towards the right.

Now let A_2 be the subset of arcs of D containing all arcs of the form $\overrightarrow{v_iv_j}$ with $i < j$ (i.e. the arcs going to the right). Clearly $D[A_2]$ cannot have a directed cycle. Besides, due to the ordering of the vertices, $D[A_2]$ is degree-decreasing. Now let A_1 be the subset of the remaining arcs, i.e. those going to the left. For the same reason as previously, $D[A_1]$ is acyclic (but clearly it may not be degree-decreasing). Then A_1 and A_2 yield the desired arc-decomposition. $\square$

We now prove the second ingredient of our proof of Theorem 3.4, namely that degree-decreasing acyclic digraphs D verify $\chi^e_{\mathrm{irr}+}(D) \leq 2$. The proof is algorithmic and the result is also of interest as, as mentioned earlier, there are acyclic digraphs D verifying $\chi^e_{\mathrm{irr}+}(D) = 3$. So our result provides a new class of acyclic digraphs for which two colours are sufficient.

Lemma 3.6. *For every degree-decreasing acyclic digraph D, we have $\chi'_{\mathrm{irr}}(D) \leq 2$.*

Proof: We prove the claim by induction on the order n of D. As it can easily be proved by hand for small values of n, we proceed to the induction step. Let $v_1, ..., v_n$ be an ordering of $V(D)$ such that all arcs are directed to the right (i.e. for every two adjacent vertices v_i, v_j with $i < j$, the arc joining them is $\overrightarrow{v_iv_j}$), and verifying $d^+(v_1) \geq ... \geq d^+(v_n)$. Set $v := v_1$, and note that $D - \{v\}$ is a degree-decreasing acyclic digraph, which hence admits a locally irregular 2-arc-colouring, say with colour red and blue, according to the induction hypothesis. Our goal is to extend this arc-colouring to the arcs outgoing from v, without creating any outsum conflict.

Let $u_1, ..., u_k$ denote the outneighbours of v in D; recall that $d^+(v) \geq d^+(u_i)$ for every i. Note that colouring any arc outgoing from v with red or blue does not affect the red and blue outdegrees of the u_i's. Thus, when extending the colouring to the arcs outgoing from v, we just have to make sure that its red

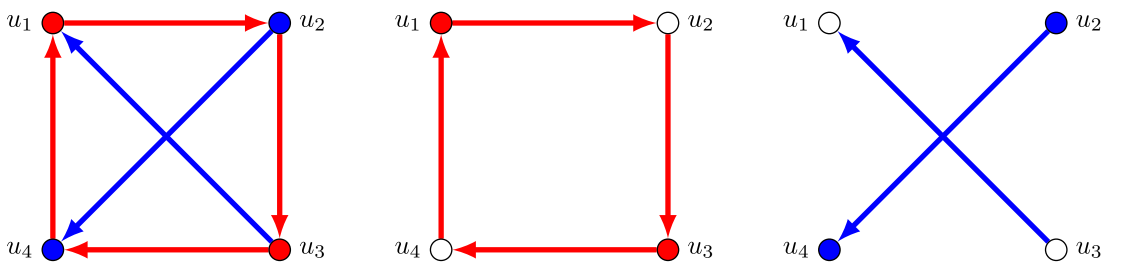

Figure 3: A locally irregular 2-total-colouring of a digraph with colours red and blue (left), the resulting red total subdigraph (middle), and the resulting blue total subdigraph (right). In the middle and right pictures, vertices filled in white are hollow while the other ones are solid.

outdegree does not meet that of any of the u_i's, and similarly for its blue outdegree. In the following, we say that one of the u_i's is a (d_1, d_2)-*vertex* if it has red outdegree d_1 and blue outdegree d_2.

Now consider the following procedure for colouring the arcs outgoing from v. As **Step 1**, we start by colouring all arcs outgoing from v red. If this extension of the arc-colouring is not locally irregular (otherwise we are done), then it means that at least one outneighbour of v, say u_1, has red outdegree k. Actually, by the ordering of the vertices of D, we even get that u_1 is a $(k, 0)$-vertex. As **Step 2**, we then colour the arc $\overrightarrow{vu_1}$ blue, and all other arcs outgoing from v red. If this does not result in a locally irregular 2-arc-colouring, then, since u_1 has blue outdegree 0, this means that v has at least one outneighbour, say u_2, which is a $(k-1, \leq 1)$-vertex different from u_1. As **Step 3**, we then colour the arcs $\overrightarrow{vu_1}$ and $\overrightarrow{vu_2}$ blue, and all other arcs red. This time, if an outsum conflict arises, then this means that at least one third outneighbour of v, say, u_3, different from u_1 and u_2, is a $(k-2, \leq 2)$-vertex.

We carry on this procedure step by step: at **Step** i, we colour the arcs $\overrightarrow{vu_1}, ..., \overrightarrow{vu_{i-1}}$ blue, and the remaining arcs outgoing from v red. This cannot cause any blue outsum conflict because the vertices $u_1, ..., u_{i-1}$ where revealed, in the previous steps, to have blue outdegree strictly less than $i-1$. So if a conflict arises, this is because at least one new outneighbour of v, denoted u_i, different from $u_1, ..., u_{i-1}$, is revealed to have red outsum $k-i+1$. More precisely, u_i is revealed to be a $(k-i+1, \leq i-1)$-vertex. For these reasons, once **Step** k is achieved, if no locally irregular 2-arc-colouring has been obtained at any point, then it means that the $u_i's$ are precisely $(k, \leq 0)$-, $(k-1, \leq 1)$-, $(k-2, \leq 2)$-, ..., $(1, \leq k-1)$-vertices. Then colouring all arcs outgoing from v blue results in a correct colouring. □

3.2 Locally irregular total-colourings

Let us now discuss a total counterpart of locally irregular decompositions, which is inspired from (2), where the authors introduced similar concepts for undirected graphs, as an attempt for binding the 1-2 Conjecture and locally irregular decompositions (the latter concept being originally motivated by applications to the 1-2-3 Conjecture).

By a *total digraph*, we mean a triplet $D^t := (V_0, V_1; A)$ with vertex set $V_0 \cup V_1$ ($V_0 \cap V_1 = \emptyset$) and arc set A. The vertices in V_0 are called *hollow*, while those in V_1 are said *solid*. The main difference between total digraphs and usual digraphs lies in the definition of vertex degrees. Namely, for a total digraph D^t, the *total outdegree* (resp. *indegree*) of a vertex v, denoted $d_t^+(v)$ (resp. $d_t^-(v)$), is understood as only the number of arcs outgoing from (resp. incoming to) v if v is a hollow vertex, or this quantity plus 1 if v is solid. In other words, solid vertices have their indegrees and outdegrees being naturally altered by 1.

We call a total digraph $D^t := (V_0, V_1; A)$ *locally irregular* if $d_t^+(u) \neq d_t^+(v)$ for every arc $\overrightarrow{uv} \in A$. A *locally irregular* k*-total-colouring* of a usual digraph D is then a total-colouring c of D with k colours such that each colour class induces a locally irregular total subdigraph (where for any given colour i, the vertices of D coloured with i define V_1 for the corresponding total digraph, and the other vertices of D yield V_0). The least number of colours needed to colour D in this way is denoted by $\chi^t_{\text{irr+}}(D)$.

These concepts are illustrated in Figure 3, which depicts a 2-total-colouring of a given digraph (left). The resulting red total subdigraph (middle) is locally irregular since it verifies $d_t^+(u_1) = 2$, $d_t^+(u_2) = 1$, $d_t^+(u_3) = 2$, $d_t^+(u_4) = 1$, and its arcs are $\overrightarrow{u_1u_2}$, $\overrightarrow{u_2u_3}$, $\overrightarrow{u_3u_4}$, $\overrightarrow{u_4u_1}$. The resulting blue total subdigraph (right) is locally irregular since it verifies $d_t^+(u_1) = 0$, $d_t^+(u_2) = 2$, $d_t^+(u_3) = 1$, $d_t^+(u_4) = 1$, and its arcs are $\overrightarrow{u_3u_1}$, $\overrightarrow{u_2u_4}$. Thus, the 2-total-colouring in Figure 3 (left) is locally irregular.

As for a general upper bound on $\chi^t_{\text{irr}+}$, it is worth observing that if $\chi^t_{\text{irr}+}(D) \leq 2$ held for every digraph D, then every digraph would admit a 2-total-weighting distinguishing the adjacent vertices by their out-multisets, contradicting Remark 2.5. We however believe that the following conjecture should be the right direction.

Conjecture 3.7. *For every digraph D, we have* $\chi^t_{\text{irr}+}(D) \leq 3$.

Below, we get quite close to Conjecture 3.7 by proving that 4 bounds $\chi^t_{\text{irr}+}$ above. The proof is reminiscent of that of Theorem 3.4; namely, it mainly follows from decomposing digraphs into acyclic digraphs which we then decompose independently. We first need to show that, for acyclic digraphs D, actually even $\chi^t_{\text{irr}+}(D) \leq 2$ holds.

Lemma 3.8. *For every acyclic digraph D, we have* $\chi^t_{\text{irr}+}(D) \leq 2$.

Proof: We prove the claim by induction on the order n of D. As it can easily be proved true for small values of n, we focus on proving the induction step. Let $v_1, ..., v_n$ be an ordering of $V(D)$ such that all arcs go to the right. Set $v := v_1$; clearly $D - \{v\}$ is acyclic, and $D - \{v\}$ hence admits a locally irregular 2-total-colouring with colours red and blue. As in the proof of Lemma 3.6, our goal is to extend this colouring to v and its outgoing arcs without creating any conflict.

Let $u_1, ..., u_k$ denote the outneighbours of v in D. For every u_i, we denote by d^i_r and d^i_b the total outdegrees of u_i in the total subdigraphs induced (thus far) by colours red and blue, respectively. By the pigeonhole principle, for at least one of the k ordered pairs $(1, k)$, $(2, k-1)$, ..., $(k, 1)$, say $(r, k+1-r)$ (where $r \in \{1, ..., k\}$), we must have

$$\left|\{i : d^i_r = r\}\right| + \left|\{i : d^i_b = k+1-r\}\right| \leq 1,$$

as otherwise none of the u_i's can have $d^i_r = k+1$ nor $d^i_b = k+1$, and we could just colour all the arcs outgoing from v and v itself red. Suppose then that v has at most one outneighbour, say u_i (if any), with $d^i_r = r$, and no outneighbour u_j with $d^j_b = k+1-r$ for an $r \in \{1, ..., k\}$. Then we can colour exactly $k+1-r$ arcs outgoing from v, including vu_i (if it exists), blue, and the remaining outgoing arcs of v and v itself red, thus creating no conflict. The construction follows the same pattern in the symmetrical case, i.e. when v has at most one outneighbour, say u_i (if any), with $d^i_b = k+1-r$ and no outneighbour u_j with $d^j_r = r$. □

Theorem 3.9. *For every digraph D, we have* $\chi^t_{\text{irr}+}(D) \leq 4$.

Proof: By Lemma 3.5, digraph D admits an arc-decomposition into a degree-decreasing acyclic digraph D_1 and an acyclic digraph D_2. Note that decomposing a digraph into two locally irregular total subdigraphs with hollow vertices only is similar to decomposing it into two locally irregular subgraphs; thus, by Lemma 3.6, we get that D_1 can be further decomposed into two locally irregular total subdigraphs where all vertices are hollow. By Lemma 3.8, we get that D_2 can be further decomposed into two locally irregular total subdigraphs. We thus obtain a decomposition of D into four locally irregular total subdigraphs. □

3.3 Locally pair-irregular total-colourings

As mentioned in previous Section 3.2, it is not true that every digraph admits a locally irregular 2-total-colouring. Thus, to fit with the spirit of the 1-2 Conjecture, we here introduce other distinguishing concepts which, although more artificial, allow us to come up with a conjecture quite similar to Theorem 2.6.

Let $D^t := (V_0, V_1; A)$ be a total digraph. To every vertex v of D^t, we associate its *pair-degree* being the pair $(1, d^+(v))$ if v is solid, and $(0, d^+(v))$ if v is hollow. We call D^t *locally pair-irregular* if the pair-degrees of u and v are distinct for every arc $\overrightarrow{uv} \in A(D)$. In turn, a total-colouring of a usual digraph D is called *locally pair-irregular* if every colour class yield a locally pair-irregular subdigraph (where a vertex v assigned colour, say, red is regarded solid only in the red total subdigraph). The least number of colours in a locally pair-irregular total-colouring of D is denoted $\chi^t_{\rho,\text{irr}+}(D)$.

For these notions, we wonder, in the same spirit as our investigations in Section 2.3, whether every digraph can be total-coloured with at most two colours. We believe that this is so.

Conjecture 3.10. *For every digraph D, we have* $\chi^t_{\rho,\text{irr}+}(D) \leq 2$.

Since a locally irregular total digraph is also locally pair-irregular, we note that Theorem 3.9 implies the following as a first step towards Conjecture 3.10.

Remark 3.11. *For every digraph D, we have* $\chi^t_{\rho,\text{irr}+}(D) \leq 4$.

4 Conclusions

In this paper, we have considered several problems related to directed versions of the 1-2 Conjecture and locally irregular decompositions. Although some of our results are best possible, there is still a gap to fill in concerning some of the variants. In particular, though we have improved the upper bound on $\chi^e_{\mathrm{irr}+}$ from (6) from 6 down to 5 (recall Theorem 3.4), the conjectured upper bound 3 is still open. Unfortunately, we do not believe that our approach, which is already an improvement of the one used in (6) (consisting of independently colouring two arc-disjoint subdigraphs), could be improved to decrease the upper bound to 3 or even only 4. Concerning locally irregular total-colourings, our upper bound of 4 on $\chi^t_{\mathrm{irr}+}$ given in Theorem 3.9 is close to what we believe to be the optimal value, namely 3 (recall Conjecture 3.7). Here again, we doubt our proof scheme could be improved to lower the bound further. The situation is similar in the case of Conjecture 3.10. One should hence design new tools and techniques to tackle these three holding conjectures.

Another remaining (algorithmic) open question we have is related to Question 2.2:

Question 4.1. *What is the complexity of determining $\chi^t_{\sigma+}(D)$ for a given digraph D?*

Recall that the analogous problem of determining $\chi^e_{\sigma+}(D)$ for a given digraph D is NP-complete, as proved in (4). Furthermore, we are not aware of any NP-completeness result regarding variants of the 1-2 Conjecture. Settling Question 4.1 would thus be an interesting task.

Acknowledgements

The authors would like to thank an anonymous referee for his numerous judicious comments and suggestions. The first, third and fourth authors were supported by CNRS-PICS Project no. 6367 “GraphPar”. The second author was supported by ANR grant no. ANR-13-BS02-0007 “STINT”, ERC Advanced Grant GRACOL, project no. 320812, and PEPS grant POCODIS. The third author was supported by the National Science Centre, Poland, grant no. 2014/13/B/ST1/01855. The third and fourth authors were partly supported by the Polish Ministry of Science and Higher Education. The fourth author was supported by the National Science Centre, Poland, grant no. DEC- 2013/09/B/ST1/01772.

References

[1] T. Bartnicki, J. Grytczuk, and S. Niwczyk. Weight choosability of graphs. *Journal of Graph Theory*, 60:242–256, 2009.

[2] O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak. On decomposing regular graphs into locally irregular subgraphs. *European Journal of Combinatorics*, 49:90–104, 2015.

[3] O. Baudon, J. Bensmail, and E. Sopena. On the complexity of determining the irregular chromatic index of a graph. *Journal of Discrete Algorithms*, 30:113–127, 2015.

[4] O. Baudon, J. Bensmail, and E. Sopena. An oriented version of the 1-2-3 conjecture. *Discussiones Mathematicae Graph Theory*, 35:141–156, 2015.

[5] J. Bensmail, M. Merker, and C. Thomassen. Decomposing graphs into a constant number of locally irregular subgraphs. *European Journal of Combinatorics*, 60:124–134, 2017.

[6] J. Bensmail and G. Renault. Decomposing oriented graphs into six locally irregular oriented graphs. *Graphs and Combinatorics*, 32:1707–1721, 2016.

[7] M. Kalkowski. A note on 1,2-conjecture. *In Ph.D. thesis*, 2009.

[8] M. Kalkowski, M. Karoński, and F. Pfender. Vertex-coloring edge-weightings: towards the 1-2-3 conjecture. *Journal of Combinatorial Theory, Series B*, 100:347–349, 2010.

[9] M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91:151–157, 2004.

[10] B. Lužar, J. Przybyło, and R. Soták. New bounds for locally irregular chromatic index of bipartite and subcubic graphs. *arXiv:1611.02341*, 2016.

[11] J. Przybyło. On decomposing graphs of large minimum degree into locally irregular subgraphs. *Electronic Journal of Combinatorics*, 23:#P2.31, 2016.

[12] J. Przybyło and M. Woźniak. On a 1,2 conjecture. *Discrete Mathematics and Theoretical Computer Science*, 12:101–108, 2010.

[13] J. Przybyło and M. Woźniak. Total weight choosability of graphs. *Electronic Journal of Combinatorics*, 18:#P112, 2011.

[14] B. Seamone. The 1-2-3 conjecture and related problems: a survey. *arXiv:1211.5122*, 2012.

[15] T.-L.Wong and X. Zhu. Every graph is $(2, 3)$-choosable. *Combinatorica*, 36:121–127, 2016.